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**Existence of general equilibrium
in a prototype L_1 economy**

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Abstract

It is often argued that in view of the second law of thermodynamics there is a limit to the total flow of commodities produced over time. If that is correct the economy's commodity space is a (proper) subspace of the space of Lebesgue integrable functions. This space poses serious problems in the literature on the existence of general equilibria in economies with infinite dimensional commodity spaces. It is shown in this paper that for a prototype model from the field of environmental/resource economics, these problems can successfully be tackled by the use of a limit argument on equilibria in the truncated economies

1 Introduction

Although there is no standard definition of sustainability as yet, it is clear that two issues play a crucial role in any definition: intergenerational fairness and substitutability. In their recent review and assessment of the literature Toman et al. (1994) elaborate on both issues and discuss the problems involved. As far as intergenerational fairness is concerned, one can distinguish between societal utilitarian objectives (maximizing the present value of total generational welfare), a Rawlsian approach or the objective of “opsustimality” (Asheim (1991), Pezzey (1989)), where it is required that utility is nondecreasing over time. The latter two objectives presuppose the feasibility of at least sustained utility, which may be at variance with the substitutability opportunities. This was shown long ago by, amongst others, Dasgupta and Heal (1974) and Stiglitz (1974) for the case of some specific aggregate production functions involving man-made capital and “natural” capital, in particular exhaustible resources. Moreover, there is growing awareness that physical laws, notably the laws of thermodynamics, may restrict substitutability (even in the presence of technical progress) in such a way that sustainability is not feasible (see e.g. Ayres and Kneese (1969) and Dasgupta and Heal (1974)). If we accept this view, total aggregate production and hence consumption in the global economy is limited over time.

The question we address in this paper is whether the economy then has a general competitive equilibrium. This question is relevant. First because it is known from the literature on economies with infinite dimensional commodity spaces that the existence of equilibria in economies where the commodity space is the space of Lebesgue integrable functions (L_1) poses a great problem. Such spaces have provided authors in this field with several examples of nonexistence of equilibria (see e.g. Zame (1987)). The main problem is that the norm interior of L_1 is empty so that existence theorems requiring initial endowments to lie in the interior of the commodity space (see e.g. Bewley (1972) and Zame (1987)) cannot be invoked. Moreover, theorems not using this property, make assumptions like uniform properness of the production set (Richard (1989)) or bounded marginal efficiency of production (Zame (1987)), which usually cannot be justified in resource economics. A second, fundamental, problem has to do with the definition itself of the commodity space. The commodity space can be described by saying that a commodity bundle belongs to it if and only if there are initial endowments which, together with the available technology, make this bundle feasible. Now,

if the commodity space is identified as L_1 for example, then it is natural to look for equilibrium prices in L_∞ , the dual of L_1 . However, if the commodity space is a proper subspace of L_1 (of which we shall provide examples), then it is far less straightforward what the dual (price) space looks like.

Clearly, the existing theory on infinite dimensional commodity spaces cannot be used in general to tackle these problems. In this paper we present an alternative approach. First, we sketch in Section 2 a prototype neoclassical economy with exhaustible resources and study the question of the “appropriate” commodity space. The second issue we address is existence. The model is in continuous time implying, formally speaking, that the commodity space is infinite dimensional, even when a finite horizon is considered. This problem is discarded by assuming that for every finite horizon economy there exists an equilibrium. It can be shown that this assumption is justified in most of the cases we consider, but this is not our main objective here. So, the central issue is whether or not the existence of a general equilibrium of the infinite horizon economy can be ascertained from the existence of equilibria in every truncated economy. The answer is in the affirmative if, among other things, the equilibrium prices in the truncated economies have bounded growth rates, which will be the case in a large class of models. This is shown in Section 3. Finally, Section 4 concludes.

2 A prototype model

In the economy under consideration there are $m(\geq 1)$ stocks of exhaustible resources and one composite commodity that serves as a capital stock in production and as a flow of a consumer good. So, a typical commodity bundle consists of $m + 2$ elements. The first element is the initial *stock* of the composite commodity. Elements 2 up to $m + 1$ are the *stocks* of the m exhaustible resources and the final element is a mapping from $[0, \infty)$ to the positive reals denoting the *flow* of the composite commodity.

There are $l(\geq 1)$ consumers or dynasties, infinitely living. Consumer h ($h = 1, 2, \dots, l$) has initial holding

$$\omega_h := (\bar{k}_h, \bar{s}_{h1}, \dots, \bar{s}_{hm}, 0) = (\bar{k}_h, \bar{s}_h, 0)$$

where \bar{k}_h denotes the initial holding of capital, \bar{s}_{hj} the initial holding of resource stock of type j ($j = 1, 2, \dots, m$) and “0” says that the flow of the consumer good accruing to the consumer as initial endowment (manna from heaven, now and in the future) is zero. We assume that $\bar{k}_h > 0$ for all h . We define the aggregate initial stocks k_0 and s_0 by

$$k_0 := \sum_h \bar{k}_h$$

$$s_0 := (s_{01}, s_{02}, \dots, s_{0m}) = \left(\sum_h \bar{s}_{h1}, \sum_h \bar{s}_{h2}, \dots, \sum_h \bar{s}_{hm} \right).$$

Consumer h is also entitled to a given constant proportion ϑ_h in the profits of the firm to be described below. The consumption set is denoted by X_h . Consumers only derive utility from the consumption of the composite commodity. Preferences of consumer h are represented by a utility functional of the following type:

$$U_h(x_h) = U_h(0, 0, c_h) = \int_0^\infty e^{-\rho_h s} u_h(c_h(s)) ds. \quad (2.1)$$

Here ρ_h is the given positive constant rate of time preference and u_h is the instantaneous utility function satisfying

Assumption 1.

$u_h : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly monotonic and strictly concave.

$u'_h(0) = \infty$; $u'_h(\infty) = 0$.

$\eta_h(c) := u''_h(c)c/u'_h(c) \leq \eta$ for all $c \geq 0$ and some $\eta < 0$.

This assumption is quite commonly made in the literature. It follows from $u'_h(0) = \infty$ that, along an equilibrium, the rate of consumption will always be positive. The final part of the assumption says that elasticity of marginal utility is bounded away from zero.

There are two types of firms. One type is engaged in the exploitation of the exhaustible resources. Such a firm buys resource stocks and depletes them with the aid of capital as a factor of production. Each deposit of a resource requires its own extraction technology: $G_j(v_j)$ of capital is needed to extract the amount v_j from resource j . The material once above the ground is homogeneous. The other type of firm converts it, together with capital, into a flow of the composite commodity according to a production function F .

If we assume convexity of the individual production sets we can aggregate over the firms and do as if there were only one firm. Without loss of generality it will be assumed that the firm buys all of the initial endowments (k_0, s_0) . If a commodity is not used, its price will be zero and we can do as if it is bought at a zero price. The technology is now given by the following

Definition 2.1

$y = (-k_0, -s_0, z)$ belongs to the production set Y if and only if there are absolutely continuous mappings k and $s = (s_1, \dots, s_m)$ and mappings $v = (v_1, \dots, v_m)$, belonging to the space of Lebesgue integrable functions, such that (omitting t when there is no danger of confusion)

$$\dot{k} = F(k - \sum_j G_j(v_j), \sum_j v_j) - \mu k - z, k(0) = k_0 \quad (2.2)$$

$$\dot{s}_j = -v_j, s_j(0) = s_{j0}, s_j \geq 0 \quad (j = 1, 2, \dots, m), \quad (2.3)$$

$$v_j \geq 0 \quad (j = 1, 2, \dots, m) \quad (2.4)$$

$$k - \sum_j G_j(v_j) \geq 0 \quad (2.5)$$

Here $\mu(> 0)$ is the rate of decay of capital. Differential equation (2.2) is to be interpreted as follows. Let the capital stock at some instant of time be k . If the firm wants to use an amount $\sum v_j$ of the raw material in the production of the composite commodity, $\sum G_j(v_j)$ of the capital stock is needed to extract. Therefore $k - \sum G_j(v_j)$ is available as capital in the production process of the composite commodity. Total net production of this commodity is then $F - \mu k$, which is used for net investments (\dot{k}) and consumption z .

About F and G_j ($j = 1, 2, \dots, m$) we make the following assumptions

Assumption 2

$F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuous, increasing, concave and linearly homogeneous.

$F(0, y) = F(x, 0) = 0$ for all $(x, y) \in \mathbb{R}_+^2$.

For all $x > 0$, $F_2(x, y) \rightarrow \infty$ as $y \rightarrow 0$.

Here $F_2(x, y)$ denotes the partial derivative of F with respect to its second argument. $F_1(x, y)$ is defined likewise.

Assumption 3

$G_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing and strictly convex.

$$G_j(0) = 0.$$

There exists $a > 0$ such that $G'_1(0) \geq G'_2(0) \dots \geq G'_m(0) \geq a$.

These assumptions will be maintained throughout. However, in the examples from the literature that we discuss below not all assumptions hold. This will be made clear wherever necessary. Concavity and monotonicity of F are innocuous assumptions. Linear homogeneity (or constant returns to scale) will play an important role to make the analysis not too complicated. The second line of assumption 2 stipulates the necessity of both inputs. The final part of assumption 2 requires that the resource is essential (see Dasgupta and Heal (1974)). The assumptions with respect to the extraction technologies are straightforward, except maybe the final one, which gives a positive lower bound on marginal extraction costs.

It follows from Assumptions 2 and 3 and the definition of the production set that total net production is bounded. This is stated in

Theorem 2.1

There exists $\bar{g} > 0$ such that

$$(i) \quad \int_0^{\infty} [F(k - \Sigma G_j(v_j), \Sigma v_j) - \mu k] dt \leq \bar{g} \Sigma s_{0j}$$

$$(ii) \quad k(t) + \int_0^t z(\tau) d\tau \leq k_0 + \bar{g} \Sigma s_{0j}$$

for any (k, s, v, z) with k and s absolutely continuous and v Lebesgue integrable such that (2.2)–(2.5) are satisfied.

Proof

Consider $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $g(k) = F(k, 1) - \mu k$. Clearly $g(0) = 0$ and $g(k) \leq 0$ for k sufficiently large, because $g(k) = k[F(1, 1/k) - \mu]$. It follows that $F(k, 1) - \mu k \leq \bar{g}$ for some $\bar{g} > 0$.

Take some $v > 0$. Then

$$\frac{1}{\Sigma v_j} [F(k - \sum_j G_j(v_j), \sum_j v_j) - \mu k] \leq$$

$$\frac{\Sigma v_j}{\Sigma v_j} [F(\frac{k}{\Sigma v_j}, 1) - \mu \frac{k}{\Sigma v_j}] \leq \bar{g}$$

Hence $\dot{k} + z = F(k - \Sigma G_j(v_j), \Sigma v_j) - \mu k \leq \bar{g} \Sigma v_j$.

Since $\int_0^\infty \sum_j v_j dt \leq \sum_j s_{0j}$, this proves part i) of the theorem. Part ii) of the theorem also follows. □

The spirit of the model stems from the field of resource economics. For the case of a single consumer, no extraction costs, no depreciation and a CES specification of F it coincides with the model extensively studied by Dasgupta and Heal (1974). Stiglitz (1974) uses a Cobb-Douglas F and allows for population growth and technological process. Chiarella (1980) deals with two consumers and Cobb-Douglas functional forms for F and the u_h 's. Elbers and Withagen (1984) also study the case of two consumers but they don't include production; on the other hand they introduce extraction costs. Heal (1976) and Kay and Mirrlees (1975) assume a linear extraction technology. The model also closely relates to van Geldrop et al. (1991) who employ a discrete time setting. We should also mention the work of Mitra (1978) and (1980) and Toman (1987) who study efficiency in a one consumer world. Finally, Zame (1987) uses this model with F identically zero to illustrate one of his existence theorems.

The question we address next is when the consumption profiles are in L_1 . We provide four examples from the literature to show that it is incorrect to choose L_1 as the commodity space a priori.

- a) The first example derives from Dasgupta and Heal (1974) and merely serves to show that the limited availability of the exhaustible resources is not necessarily insurmountable. Suppose that there is only one exhaustible resource which can costlessly be extracted ($m = 1, G_1 \equiv 0$ so that the final part of Assumption 3 does not hold). There is no depreciation ($\mu = 0$). The consumers have identical isoelastic utility functions ($\frac{u''(c)c}{u'(c)} = \eta < 0$). Suppose, finally, that F is CES with elasticity of substitution larger than unity. In that case there is ψ such that

$$F(x, 1)/x \rightarrow \psi \text{ as } x \rightarrow \infty$$

meaning that the average product of capital is bounded away from zero (so, Assumption 2 is not satisfied). Dasgupta and Heal (o.c) show that if $\psi > \rho > \psi(1 + \eta)$, there exists an equilibrium with an asymptotic growth rate of consumption equal to $(\rho - \psi)/\eta$, which is positive. Therefore, equilibrium consumption profiles are not in L_1 .

b) Here we maintain the assumptions of the first example except that we allow for depreciation and F is CES now with elasticity of substitution smaller than unity. We may then define $\varphi := \max_x F(1, x)$, the maximum product per unit of capital. Now, assume $\mu > \varphi$, implying that there exists some $\alpha > 0$ with $\mu > \varphi + \alpha$. Then it follows from (2.2) with $z(t) = e^{-\alpha t}$ inserted, that $\dot{k} \leq (\varphi - \mu)k - e^{-\alpha t}$ so that k becomes negative within finite time. Hence $e^{-\alpha t}$, although in L_1 , cannot be produced. Obviously, with extraction costs included this conclusion holds a fortiori.

c) A third example is the case studied by Zame (1987). F is identically zero. Then the development of the economy is described by

$$\dot{k} = -c - \mu k \text{ hence } \int_0^{\infty} e^{\mu t} c(t) dt < \infty$$

Zame (o.c) erroneously claims that L_1 is the commodity space and argues that his Theorem 1 is applicable, whereas no other hitherto known existence theorem is. Indeed, any feasible consumption pattern is in L_1 . But not every element of L_1 can be produced. Take for example $c(t) = e^{(\alpha - \mu)t}$ with $0 < \alpha < \mu$. As far as the existence of an equilibrium is concerned, there will exist an equilibrium but not with prices in the dual of L_1 . As a matter of fact the equilibrium price will grow at a rate μ .

d) A final example shows that with decreasing returns to scale aggregate net production over time might be unbounded even with a positive rate of depreciation and convex extraction costs. Suppose F is Cobb-Douglas and $G_j(v_j) = av_j$ for all j with a constant (so, virtually there is only one exhaustible resource). Then net production over a period of time of length T is given by

$$\int_0^T [(k - av)^{\alpha} v^{\beta} - \mu k] dt$$

where α and β are positive constants with $\alpha + \beta < 1$. Choose for $t \in [0, T]$

$$v(t) := s_0/T$$

$$k(t) := \left(\frac{\alpha}{\mu}\right)^{\frac{1}{1-\alpha}} v(t)^{\frac{\beta}{1-\alpha}} + av(t) .$$

Then the value of the integral given above is

$$(1 - \alpha) \left(\frac{\alpha}{\mu}\right)^{\frac{\alpha}{1-\alpha}} s_0^{\frac{\beta}{1-\alpha}} T^{\frac{1-\alpha-\beta}{1-\alpha}} - a\mu s_0 .$$

and this expression goes to infinity as T goes to infinity. Clearly, also with strictly convex extraction costs the same result can be obtained, as long as $G'(0)$ is bounded.

Of course, also examples can be given where L_1 is indeed the commodity space. Among the simplest ones is the following.

- e) There are no extraction costs and no depreciation. F satisfies: there exists ψ such that $F(k, v) \leq \psi v$ for all $(k, v) \geq 0$. Then, for any given (k_0, s_0) any feasible consumption trajectory is integrable since

$$\int_0^t c(\tau) d\tau \leq \int_0^t \psi v(\tau) d\tau + k_0 \leq \psi s_0 + k_0$$

Moreover, also the converse is true. Any consumption pattern in L_1 can be produced by a proper choice of k_0 .

A second example is more interesting.

- f) Theorem 2.1 shows that with homogeneity of F and a positive rate of depreciation μ total net output is bounded. Moreover, it is clear from the proof of the theorem that the capital stock is bounded. This implies that total consumption over time is bounded. Clearly, homogeneity and a positive μ are not sufficient to be able to produce any consumption pattern in L_1 , by a proper choice of the (initial) inputs. For that we need that the technology is productive enough. A set of sufficient conditions would be the following. The production function F displays constant returns to scale, there is j (say $j = 1$) and $a > 0$ such that $G_1(v) = av$ for all $v \geq 0$ and there exist $\lambda > 0$ and $\varphi > 0$ such that $F(1 - a\lambda, \lambda) - \mu = \varphi$. Take some $c \in L_1$ and take

$$k(t) = e^{\varphi t} \int_t^{\infty} e^{-\varphi \tau} c(\tau) d\tau, \quad t \geq 0$$

$$v_1(t) = \lambda k(t)$$

Then $\dot{k} = F(k - G_1(v_1), v_1) - \mu k - c$ and if

$$s_{10} = \frac{\lambda}{\varphi a} \int_0^{\infty} (1 - e^{-\varphi \tau}) c(\tau) d\tau$$

then

$$\int_0^{\infty} v_1(\tau) d\tau \leq s_{10}$$

since

$$\begin{aligned} \int_0^{\infty} k(s) ds &= \int_0^{\infty} d\tau \int_0^{\tau} e^{\varphi(t-s)} c(s) ds \\ &= \frac{1}{\varphi} \int_0^{\infty} c(\tau) (1 - e^{-\varphi \tau}) d\tau \end{aligned}$$

We conclude from these examples that although equilibrium consumption patterns will lie in L_1 under certain conditions, it may well be that equilibrium prices are not in the dual of L_1 .

3 Existence of a general equilibrium

We postulate the existence of general equilibria in all finite horizon economies. We introduce the following notation. x_h^T is a consumption bundle of consumer h in the economy with horizon $T \in \mathbb{N}$. x_h^T will always mean $(0, 0, c_h^T)$, where $c_h^T(t) = 0$ for $t > T$. The initial endowment vector of consumer h is just ω_h . A production vector is denoted by y^T and we say that $y^T = (-k_0, -s_0, z^T)$ belongs to the production set Y^T if $y^T \in Y$ (see definition 2.1) and $z^T(t) = 0$ for all $t > T$. A price vector is denoted by π^T , containing $m + 1$ constants and, as the final entry, a function describing the price of the composite commodity. The value of

a bundle, say z , at prices π^T is denoted by $\pi^T \cdot z$.

We now make

Assumption 4

There exist \bar{T} such that for all $T > \bar{T}$ there are

$$x_h^T = (0, \dots, 0, c_h^T) \in X_h^T$$

$$y^T = (-k^T(0), -s^T(0), z^T) \in Y^T$$

$$\pi^T = (1, q^T, p^T) = (1, q_1^T, \dots, q_m^T, p^T)$$

where q^T is a vector of constants and p^T is a mapping from $[0, \infty]$ to \mathbb{R}_+ with p^T continuous on $[0, T)$ and $p^T(t) = 0$ for $t \geq T$, with the following properties

$$\sum_h x_h^T \leq y^T + \sum_h \omega_h$$

$$\pi^T \cdot y^T \geq \pi^T \cdot z \text{ for all } z \in Y^T$$

$$\pi^T \cdot x_h^T \leq \pi^T \cdot \omega_h \text{ (all } h)$$

$$\pi^T \cdot z \leq \pi^T \cdot x_h^T \text{ and } z \in X_h^T \text{ implies } U_h(x_h^T) \geq U_h(z) \text{ (all } h) \quad \square$$

Assumption 4 thus states that there exists a general equilibrium in each finite horizon economy. It is also assumed that the price per unit of the capital stock equals one. This is no loss of generality because the composite commodity, which is perfectly malleable with capital, is always desirable. We can take that commodity at time zero as the numéraire. It follows that $p^T(0) = 1$ for all T . It is also assumed that the price of the consumer commodity is continuous over the relevant time domain. This can be justified by the fact that the strict concavity of all utility functions implies continuity over time of the consumption profiles which requires continuity of the price function. In view of the homogeneity of the production function F profits are zero, so that there appear no dividends in the budget constraints.

We first derive some properties of the finite horizon equilibria.

The problem of consumer h can be formulated as:

$$\begin{aligned} & \max_{c_h} \int_0^T e^{-\rho_h t} u_h(c_h(t)) dt \\ & \text{subject to } \int_0^T p^T(t) c_h(t) dt \leq \pi^T \cdot \omega_h \end{aligned}$$

It is straightforward to see that the following holds:

Lemma 3.1

There exists $b^T := (b_1^T, b_2^T, \dots, b_l^T)$, a vector of constants, such that for all $t \leq T$

$$e^{-\rho_h t} u'_h(c_h^T(t)) = b_h^T p^T(t) \quad (\text{all } h) .$$

Proof

This follows from the application of Pontryagin's maximum principle to the consumer's problem stated above. □

The problem of the producer is:

$$\max_z \int_0^T p^T(t) z(t) dt - k_0 - q^T \cdot s_0$$

subject to (2.2)–(2.5).

According to Pontryagin's maximum principle there exists an absolutely continuous λ^T such that for all $t \leq T$

$$p^T(t) = \lambda^T(t) \tag{3.1}$$

$$F_1(k^T(t) - \Sigma G_j(v_j^T(t)), \Sigma v_j^T(t)) - \mu = -\dot{\lambda}^T(t) / \lambda^T(t) \tag{3.2}$$

and

$$\lambda^T(t) F(k^T(t) - \Sigma G_j(v_j), \Sigma v_j) - q^T \cdot v \tag{3.3}$$

is maximized with respect to v subject to (2.4)–(2.5).

We can now prove

Lemma 3.2

There exists $\beta > 0$ such that for all T and all $t < T$

$$\mu - \beta \leq \dot{p}^T(t)/p^T(t) \leq \mu .$$

Proof

Since $F_1 \geq 0$ it is clear from (3.1) and (3.2) that $\dot{p}^T(t)/p^T(t) \leq \mu$.

Consider, for $x \geq 0$, $h(x) := F(x, 1)/(a + x)$, where a is defined in Assumption 3. For all $x \geq 0$ we have $h(x) \geq 0$. Also $h(0) = 0$ since $F(0, 1) = 0$ and $h(x) \rightarrow 0$ as $x \rightarrow \infty$ because $\frac{1}{x}F(x, 1) = F(1, 1/x) \rightarrow 0$ as $x \rightarrow \infty$. F is continuously differentiable, whence h reaches a maximum for $x = \bar{x}$, defined by $F(\bar{x}, 1) - (a + \bar{x})F_1(\bar{x}, 1) = 0$. Define $\beta := F_1(\bar{x}, 1)$. Note that

$$F(x, 1) - (a + x)F_1(x, 1) < 0 \text{ for all } x < \bar{x} \tag{*}$$

Now suppose that $\dot{p}^T(t)/p^T(t) < \mu - \beta$ for some T and some $t < T$. It follows from (3.1) and (3.2) that (omitting T and t)

$$F_1(k - \Sigma G_j(v_j), \Sigma v_j) > \beta$$

and hence $x := (k - \Sigma G_j(v_j))/\Sigma v_j < \bar{x}$.

We must have $v_j > 0$ for at least one j . Then (3.3) implies

$$q_j = \lambda[F_2(k - \Sigma G_j(v_j), \Sigma v_j) - G'_j(v_j)F_1(k - \Sigma G_j(v_j), \Sigma v_j)] = \lambda[F(x, 1) - (G'_j + x)F_1(x, 1)]$$

But the term between brackets is negative since $G'_j \geq a$ and $x < \bar{x}$ (see (*)), which contradicts $q_j \geq 0$. □

Lemma 3.3

There exist \bar{T} , $\gamma > 0$ and $\delta > 0$ such that for all $T > \bar{T}$ and all $t < T$

i) $0 \leq k^T(t) \leq \gamma$

$$\text{ii) } -\delta \leq \dot{k}^T(t) \leq \delta$$

where $k^T(t)$ is the equilibrium capital stock at t in the economy with horizon T .

Proof

ad i) Obviously $k^T(t) \geq 0$. That $k^T(t)$ is uniformly bounded from above follows from theorem 2.1 part ii).

ad ii) $\dot{k}^T(t)$ is bounded from above by the fact that $z^T(t)$ is positive, $k^T(t)$ is bounded and $\Sigma G_j(v_j^T(t)) \leq k^T(t)$. $\dot{k}^T(t)$ can be made arbitrarily small only if $z^T(t)$ is arbitrarily large. It follows from lemma 3.1 that

$$-\rho_h + \eta_h(c_h^T(t))\dot{c}_h^T(t)/c_h^T(t) = \dot{p}^T(t)/p^T(t)$$

Hence (omitting the index h)

$$\dot{c}^T/c^T = \frac{-\rho - \dot{p}^T/p^T}{-\eta(c^T)} \tag{*}$$

It follows from lemma 3.2 that $\rho + \dot{p}^T/p^T$ is bounded. Moreover, $\eta(c)$ is bounded away from zero. Therefore there is some $\gamma > 0$ and some \bar{T} such that $|\dot{c}^T(t)/c^T(t)| \leq \gamma$ for all $T > \bar{T}$ and all $t \leq T$. Hence $e^{\gamma t}c^T(t)$ is increasing and $e^{-\gamma t}c^T(t)$ is decreasing. Take some T and some $t_0 \in [0, T]$. Omitting T as an index we have

$$c(t) \geq e^{\gamma(t-t_0)}c(t_0) \quad \text{for } 0 \leq t \leq t_0$$

$$c(t) \geq e^{\gamma(t_0-t)}c(t_0) \quad \text{for } t_0 \leq t \leq T$$

Hence,

$$\int_0^T c(t)dt \geq c(t_0) \int_0^{t_0} e^{\gamma(t-t_0)}dt + c(t_0) \int_{t_0}^T e^{\gamma(t_0-t)}dt = c(t_0) \left[\int_0^{t_0} e^{-\gamma s}ds + \int_0^{T-t_0} e^{-\gamma s}ds \right]$$

$$=: c(t_0)\varphi(t_0) .$$

Clearly $\varphi'(0) > 0$, $\varphi'(T) < 0$ and $\varphi''(t_0) < 0$. Hence

$$\varphi(t_0) \geq \varphi(0) = \varphi(T) = \int_0^T e^{-\gamma s} ds \geq \int_0^1 e^{-\gamma s} ds \text{ for } T \geq 1 .$$

It follows from theorem 2.1 part ii that $c(t_0) \leq M$ for some $M > 0$. □

We now proceed to the existence of a general equilibrium in the infinite horizon economy by means of a sequence of lemmata. Formally, the statements made in these lemmata apply to subsequences but in order to avoid abundance of notation, we will refer to the sequence indexed by T .

Lemma 3.4

There exists $b_h > 0$ ($h = 1, 2, \dots, l$) such that

$$b_h^T \rightarrow b_h \text{ as } T \rightarrow \infty$$

Proof

It will be shown that b_h^T is uniformly bounded from above and from below (away from zero). Since u_h is concave (all h) there exist A and B such that $u_h(c) \leq A + Bc$ for all $c \geq 0$. So, from the boundedness of c_h^T established in the previous lemma, there is $M > 0$ such that for all $T > \bar{T}$

$$\begin{aligned} M &\geq \int_0^T e^{-\rho_h s} (A + Bc_h^T(s)) ds \geq \int_0^T e^{-\rho_h s} u'_h(c_h^T(s)) c_h^T(s) ds \\ &= \int_0^T b_h^T p^T(s) c_h^T(s) ds = b_h^T \pi^T \cdot \omega_h \tag{*} \\ &\geq b_h^T \bar{k}_h \end{aligned}$$

Since $\bar{k}_h > 0$ (all h), b_h^T is uniformly bounded from above. Since $u'(c_h^T(0)) = b_h^T$ ($p^T(0) = 1$) and $c_h^T(0) \leq \bar{c}$ for some \bar{c} we have that b_h^T is bounded away from zero. □

Lemma 3.5

There exists $\pi = (1, q, p)$ with $p(0) = 1$ and p absolutely continuous on $[0, \infty)$ such that

- i) $q^T \rightarrow q$ as $T \rightarrow \infty$ (pointwise)

ii) $p^T \rightarrow p$ as $T \rightarrow \infty$ (pointwise)

Proof

i) Summation of the inequalities of the previous lemma over h yields

$$lM \geq \sum_h b_h^T \pi^T \cdot \omega_h$$

Recall that $\pi^T = (1, q_1^T, q_2^T, \dots, q_m^T, p^T)$. Since b_h^T is bounded away from zero and $\Sigma \omega_h > 0$, q_j^T is uniformly bounded from above. Of course $q_j^T \geq 0$ for all j so that the sequence q^T has a convergent subsequence.

ii) Define $\hat{p}^T(t) := e^{-\mu t} p^T(t)$. In view of (3.2) \hat{p}^T and $\dot{\hat{p}}^T$ can on $[0, \infty)$ be considered as uniformly bounded L_∞ functions. By Alaoglu's theorem there exists $f \in L_\infty$ and a subsequence (again denoted by T) such that $\dot{\hat{p}}^T$ converges to f "weak star". Choose $t_0 \in [0, \infty)$. Then, for $T > t_0$, we have

$$\hat{p}^T(t_0) = 1 + \int_0^{t_0} \dot{\hat{p}}^T(s) ds$$

from which it follows that

$$\hat{p}(t_0) := \lim \hat{p}^T(t_0) = 1 + \int_0^{t_0} f(s) ds$$

Hence $\hat{p}^T \rightarrow \hat{p}$ and

$$\hat{p}(t) = 1 + \int_0^t f(s) ds$$

so that $\hat{p}(0) = 1, \dot{\hat{p}} = f$ (a.e.) and \hat{p} and hence p is absolutely continuous. □

Lemma 3.6

There exist $c_h \in L_\infty$ (all h) and $z \in L_\infty$ such that

i) $c_h^T \rightarrow c_h$ as $T \rightarrow \infty$ (point-wise) (all h)

ii) $z^T \rightarrow z$ as $T \rightarrow \infty$ (point-wise)

Proof

This is clear from the fact that

$$e^{-\rho t} u'(c_h^T(t)) = b_h^T p^T(t)$$

and the convergence of b_h^T and $p^T(t)$. Finally, note that $z^T = \Sigma c_h^T$. \square

It remains to be shown that $(\{x_h\}_h, y, \pi)$, with $y := (k_0, s_0, z)$, constitutes a general equilibrium for the infinite horizon economy.

Clearly $c_h(t) \geq 0$ for all h and all t ; therefore $x_h = (0, 0, c_h)$ is in the consumption set of consumer h .

The question whether $y \in Y$ or not is more difficult to handle. In the literature on infinite dimensional commodity spaces it is generally *assumed* that the production sets are closed in the topology of the commodity space, but this need not be the topology of point-wise convergence. So, it might be expected that there arises a problem here. In the case at hand this problem can be dealt with by means of the maximum theorem.

Lemma 3.7

- i) $x_h \in X_h$ (all h)
- ii) $y \in Y$

Proof

i) This is evident.

ii) Fix T and $t < T$. Let $k^T(t) (> 0)$, $\lambda^T(t)$ and q^T be given. Consider the problem of maximizing (3.3) subject to (2.4) and (2.5). The constraint set, given by (2.4) and (2.5) is non-empty, compact valued (because $k^T(t)$ is bounded) and continuous. The maximand is continuous and has a compact range, the latter because $\lambda^T(t)$ and $k^T(t)$ are bounded. Since G_j is strictly convex for all j , the solution $v^T(t)$ of the problem is unique. It follows from the maximum theorem (see e.g. Berge (1963)) that $v^T(t)$ is a continuous function of the parameters $(k^T(t), \lambda^T(t), q^T)$.

It follows from the previous lemmata that (k^T, λ^T, q^T) converges point-wise to (k, λ, q) , and hence $v^T \rightarrow v$ point-wise, by the maximum theorem. We also have $z^T \rightarrow z$ point-wise. Since F is continuous, this proves that $(-k_0, -s_0, z) \in Y$. \square

Lemma 3.8

$0 = \pi \cdot y \geq \pi \cdot \bar{y}$ for all $\bar{y} \in Y$.

Proof

Suppose $\bar{y} = (-k_0, -s_0, \bar{z}) \in Y$ and $\pi \cdot \bar{y} > 0$. Hence

$$\int_0^{\infty} p(t)\bar{z}(t)dt > k_0 + q \cdot s_0$$

Then there exists $\bar{T} > 0$ such that

$$\lim_{T \rightarrow \infty} \int_0^{\bar{T}} p^T(t)\bar{z}(t)dt = \int_0^{\bar{T}} p(t)\bar{z}(t)dt > \lim_{T \rightarrow \infty} (k_0 + q^T \cdot s_0)$$

For T large enough and $T > \bar{T}$ we have

$$\int_0^{\bar{T}} p^T(t)\bar{z}(t)dt > k_0 + q^T \cdot s_0$$

and hence

$$\int_0^T p^T(t)\bar{z}(t)dt > k_0 + q^T \cdot s_0$$

contradicting the zero profit condition in the finite horizon economy. The proof of $\pi \cdot y = 0$ is straightforward and is omitted here. \square

Lemma 3.9

- i) $\pi \cdot x_h = \pi \cdot \omega_h$ (all h)
- ii) $\pi \cdot z \leq \pi \cdot \omega_h$ with $z \in X_h \Rightarrow U_h(x_h) \geq U_h(z)$ (all h)

Proof

Take some fixed $t \geq 0$ and h . Then $p^T(t)c_h^T(t) - p(t)c_h(t) = p^T(t)(c_h^T(t) - c_h(t)) + (p^T(t) - p(t))c_h(t) \rightarrow 0$ as $T \rightarrow \infty$ in view of point-wise convergence. Furthermore $p^T(t)c_h^T(t)$ is bounded from above, uniformly with respect to t and T , because $\pi^T \cdot \omega_h$ is uniformly bounded. By Lebesgue dominated convergence theorem we have $\pi^T \cdot \omega_h = \pi^T \cdot x_h^T \rightarrow \pi \cdot x_h$. Now take $\bar{x} \in X_h$ such that $\pi \cdot \bar{x} \leq \pi \cdot x_h$. In view of the concavity of u_h we have

$$U_h(x_h) - U_h(z) \geq \int_0^{\infty} b_h p(t)(c_h(t) - \bar{c}(t))dt \geq 0 \quad \square$$

In view of the previous lemmata we can state.

Theorem 3.1

$(x_1, x_2, \dots, x_l, y, \pi)$ constitutes a general competitive equilibrium in the infinite horizon economy.

4 Conclusions

In the field of environmental and resource economics there is growing interest in economies which have the capacity to produce a limited amount of goods only. The question arises whether or not in such economies there are general competitive equilibria. It has been shown here that for a prototype model the answer is in the affirmative under a set of rather weak assumptions. The method employed, deriving the infinite horizon equilibrium from the sequence of finite horizon equilibria, may prove helpful in other existence problems as well.

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