Frequency response functions of linear parameter-varying systems

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Abstract: Frequency-domain system representations offer important system analysis, control design and simulation tools. While this representation concept is well developed for linear time-invariant systems, there has been no general theory established for the linear parameter-varying (LPV) system class. This paper overviews and compares the existing concepts of frozen and instantaneous frequency-domain LPV representations, pointing out the need for a representation form where the effect of the spectrum of a varying scheduling trajectory can be clearly understood and analyzed on the output system. For this purpose, the harmonic frequency response function (hFRF), known from the linear time-varying literature, is introduced for LPV systems. More specifically, the paper demonstrates how the hFRF can be computed directly in the frequency domain for discrete-time affine input-output LPV systems.

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1. INTRODUCTION

The linear parameter-varying (LPV) framework has developed into one of the important control design frameworks for nonlinear dynamical systems [Mohammadpour and Scherer, 2012]. This resulted in the development of various system representations in the time domain such as the input-output (IO), linear fractional representation and state-space (SS) representations [Tóth, 2010, dos Santos et al., 2011]. It is often assumed that a given time-domain LPV representation of the underlying system has an affine and static dependency on the scheduling signal.

Frequency-domain system representations, such as the frequency response function (FRF), are often used for the identification and control design for linear time-invariant (LTI) systems [Steinbuch et al., 2010, Pintelon and Schoukens, 2012]. Their generalization in terms of transfer functions gave birth to fundamental theories which control engineering relies on. Such a well-developed frequency-domain framework is lacking for the LPV system class. The currently available concepts of frozen FRF (fFRF) and the instantaneous FRF (iFRF) are used as frequency-domain analysis tools for LPV systems [Tóth, 2010]. The iFRF is a local representation of the system, it is only valid under the assumption that the scheduling variable remains constant over time. As a result, it only has a limited use in gain scheduling control [Tóth, 2010]. The iFRF provides a representation form where the FRF of the system embeds the past of the scheduling signal implicitly. This makes it rather difficult to understand and to analyze the influence of the scheduling signal on the system output.

The main contribution of this paper is to review these existing concepts, to compare their properties, and to introduce a more useful representation form in terms of the harmonic FRF (hFRF) for LPV systems. The hFRF is a well-known concept in the (periodically) time-varying system literature [Louaroudi, 2014, Louaroudi et al., 2014], offering additional system analysis, control design, and simulation opportunities. More specifically this paper demonstrates how the hFRF can be computed for discrete-time single-input single-output (SISO) LPV-IO representations with affine dependence and how the hFRF represents the output spectrum for an arbitrary scheduling spectrum that is not directly embedded in the hFRF. This allows for analyzing the effect of the scheduling variable on the frequency transfer.

In the next section, the various FRF concepts for LPV representations are introduced and discussed, with a particular focus on the hFRF. Section 3 analyzes the considered FRF representations of LPV systems in more detail, while in Section 4, the hFRF, iFRF and fFRF concepts are compared on an illustrative simulation example.

2. FREQUENCY RESPONSE FUNCTIONS OF LPV SYSTEMS

2.1 Preliminaries

To derive various types of FRFs of LPV systems, first the so-called (im)pulse response representation of SISO discrete-time LPV systems is introduced [Tóth, 2010]:

\[ y(t) = \sum_{\tau=0}^{\infty} g(p(t), p(t-1), \ldots, p(t-\tau), \tau) u(t-\tau) \quad (1) \]

where \( u : \mathbb{Z} \to \mathbb{R} \) is the input, \( y : \mathbb{Z} \to \mathbb{R} \) is the output, \( p : \mathbb{Z} \to \mathbb{R} \) is the scheduling signal of the system and \( g(\cdot, \tau) \) is a series of bounded functions which are convergent in case of asymptotic stability of the
represented system. In terms of represented solutions by (1), we restrict the scope to \((y, p, u)\) trajectories with left compact support to avoid complications with initial condition effects. In (1), \(g_{p,t}(\tau)\) is shorthand notation for \(g(p(t), p(t-1), \ldots, p(t-\tau), \tau)\). Note that \(g_{p,t}(\tau)\) depends on \(\{p(t-\ell)\}_{\ell=0}^{\infty}\) which is called dynamic dependency. Analyzing (1) in the frequency domain gives a basic understanding of the spectral relationship of \(y, u\) and \(p\). The discrete-time Fourier transform (DTFT) and its inverse transform for a discrete-time signal \(x: \mathbb{Z} \to \mathbb{R}\) will be used extensively throughout this article:

\[
X(e^{j\omega}) = \mathcal{F}_x\{x(t)\} = \sum_{t=-\infty}^{\infty} x(t) e^{-j\omega t},
\]

\[
x(t) = \mathcal{F}_t^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega t} d\omega, \tag{2}
\]

where \(j = \sqrt{-1}\) is the imaginary, \(\omega \in \mathbb{R}\) is the frequency and \(\mathcal{F}_x\{x(t)\}\) denotes that the DTFT of \(x\) is taken over the argument \(t\) and with frequency variable \(\omega\). Similarly, \(\mathcal{F}^{-1}\) corresponds to the inverse transformation. The solution set \((y, p, u)\) of (1) is further restricted to trajectories for which (2) exists (see [Pintelon and Schoukens, 2012]).

### 2.2 LPV-IO representation

The LPV pulse response of a system can easily be obtained from its SS or IO representation. This is illustrated in the case of SISO LPV systems with IO representation:

\[
y(t) = \sum_{\tau=1}^{n_0} A_{p,t}(\tau) y(t-\tau) + \sum_{\tau=0}^{n_h} B_{p,t}(\tau) u(t-\tau) \tag{3}
\]

where \(A_{p,t}(\tau)\) and \(B_{p,t}(\tau)\) are bounded coefficient functions, dynamically dependent on \(p\). Under the assumption of asymptotic stability of (3), the equivalent pulse representation of (3) is given by

\[
g_{p,t}(\tau) = \begin{cases} 0, & \text{if } \tau < 0 \\ B_{p,t}(0), & \text{if } \tau = 0 \\ B_{p,t}(1) + A_{p,t}(1) B_{p,t}(0), & \text{if } \tau = 1 \\ B_{p,t}(2) + A_{p,t}(1) B_{p,t}(1) + A_{p,t}(2) B_{p,t}(0), & \text{if } \tau = 2 \\ \vdots 
\end{cases} \tag{4}
\]

Note that even in case \(A_{p,t}(\tau)\) and \(B_{p,t}(\tau)\) have static dependence (dependence on \(p(t)\) only), \(g_{p,t}(\tau)\) becomes dynamically dependent on \(p\) due to the time shifts of the coefficient functions in (4).

### 2.3 Frozen Frequency Response Function

For a constant scheduling signal \(p(t) \equiv p_0\), the scheduling-dependent coefficients in the considered LPV-IO representation (3) become time-invariant. Hence, also the pulse response in (4) becomes time-invariant, i.e., only dependent on the constant value of \(p_0\), denoted here as \(g_{p_0}(\tau)\), see [Tóth, 2010, Louaroudi, 2014]. This results in the following input-output relation:

\[
y(t) = \sum_{\tau=0}^{\infty} g_{p_0}(\tau) u(t-\tau). \tag{5}
\]

By applying DTFT on (5) we obtain:

\[
Y(e^{j\omega}) = G_{p_0}(e^{j\omega}) U(e^{j\omega}), \tag{6}
\]

where \(G_{p_0}(e^{j\omega})\), representing the fFRF for \(p(t) \equiv p_0\), is obtained as the DTFT of \(g_{p_0}(\tau)\):

\[
G_{p_0}(e^{j\omega}) = \mathcal{F}\{g_{p_0}(\tau)\} = \sum_{\tau=0}^{\infty} g_{p_0}(\tau) e^{-j\omega \tau}. \tag{7}
\]

Although the fFRF can offer quite some insight into the dynamics of a LPV system, such an analysis is only valid for a constant scheduling signal \((P(e^{j\omega}) = 0, \forall \nu \neq 0)\) as the fFRF cannot be used to compute the output \(y\) for (3) based on the spectrum of \(u\) and \(p\) unless the scheduling signal is constant. Even if fFRFs have been reported to give good approximation of the system behavior for sufficiently slow variations of \(p\) [Shamma and Athans, 1992], in practice their use is limited as the notion of “sufficiently slow variation” is illusive and largely dependent on the use case.

### 2.4 Instantaneous Frequency Response Function

The iFRF \(G_{p,t}(e^{j\nu t})\), in contrast with the fFRF, is based on the DTFT of time-dependent impulse response \(g_{p,t}:\)

\[
G_{p,t}(e^{j\nu t}) = \mathcal{F}_t\{g_{p,t}(\tau)\} = \sum_{\tau=0}^{\infty} g_{p,t}(\tau) e^{-j\nu \tau}. \tag{8}
\]

Note that \(G_{p,t}(e^{j\nu t})\), just like \(g_{p,t}(\tau)\), varies with time \(t\) and depends on the whole past of \(p(t)\) embedded in \(g_{p,t}(\tau)\). The output \(y\) can be computed based on the iFRF:

\[
y(t) = \frac{1}{2\pi} \int_{0}^{2\pi} G_{p,t}(e^{j \nu t}) U(e^{j \nu t}) e^{j \nu t} d\nu, \tag{9}
\]

where \(U(e^{j \nu t}) = \mathcal{F}_t\{u(t)\}\). Observe that the obtained iFRF is not easy to use for system analysis. It is not straightforward to extract how the output spectrum depends on the spectrum of the input and scheduling signal.

### 2.5 Harmonic Frequency Response Function

The iFRF offers a hybrid time-frequency domain representation via (8) by using the DTFT over the index variable \(\tau\). The hFRF concept considers the idea of taking a second DTFT over time \(t\), resulting in \(G_{p}(e^{j\omega}, e^{j\nu t})\) [Louaroudi, 2014]. This results in a frequency-domain only interpretation of the signal behavior of an LPV system:

\[
G_{p}(e^{j\omega}, e^{j\nu t}) = \mathcal{F}_t\{F_t\{g_{p,t}(\tau)\} \} = \sum_{t=-\infty}^{\infty} G_{p,t}(e^{j\omega}) e^{-j\nu \omega t}. \tag{10}
\]

with a resulting 2 dimensional FRF \(G_{p}(e^{j\omega}, e^{j\nu t})\). The so-called hFRFs \(G_{p,\omega}\) of the system are obtained by evaluating \(G_{p}(e^{j\omega}, e^{j\nu t})\) over the \(\omega\)-dimension:

\[
G_{p,\omega}(e^{j\nu t}) = G_{p}(e^{j\omega}, e^{j\nu t}) \tag{11}
\]

The spectrum of \(y\) is directly characterized by the convolution:

\[
Y(e^{j\omega}) = \frac{1}{2\pi} \int_{0}^{2\pi} G_{p,\omega}(e^{j\nu t}) U(e^{j\nu t}) d\nu, \tag{12}
\]

where the hFRFs can be interpreted as the harmonics of the frequency transfer.

By re-arranging \(G_{p}\), the Wereley-FRF \(W_p(e^{j\omega}, e^{j\nu t})\) [Wereley, 1991, Sandberg et al., 2005, Allen et al., 2011, Louaroudi, 2014], is obtained as \(W_p(e^{j\omega}, e^{j\nu t}) = G_{p}(e^{j\omega-e^{j\nu t}}, e^{j\omega})\). This reformulated FRF \(W_p(e^{j\omega}, e^{j\nu t})\) contains the hFRFs on its diagonals. The output spectrum \(Y(e^{j\omega})\) can now be computed as:

\[
Y(e^{j\omega}) = \frac{1}{2\pi} \int_{0}^{2\pi} W_p(e^{j\omega}, e^{j\nu t}) U(e^{j\nu t}) d\nu, \tag{13}
\]
which is $\mathcal{F}_{\omega}^{-1}\{W_p(e^{j\omega}, e^{j\nu})U(e^{j\nu})|\nu\}$, the inverse DTFT with respect to $\nu$. Hence, $W_p(e^{j\omega}, e^{j\nu})$ can be interpreted as the scheduling-dependent 2-dimensional FRF that describes how the signal content of the input at frequency $\nu$ contributes to the output at frequency $\omega$. Although now the frequency transfer can be analyzed in terms of the trajectory of $p$, it is still difficult to see how the spectrum of $p$ influences this transfer.

3. FREQUENCY RESPONSE ANALYSIS UNDER AFFINE DEPENDENCY

To develop a factorization of the Wereley-FRF where the contribution of the spectrum of $p$ explicitly appears, allowing analysis of the corresponding frequency transfer, we restrict the scope to a sub-class of LPV systems that has IO representation with affine dependency.

3.1 Frequency response under affine IO dependence

Let’s assume that for the considered LPV system, the IO representation (3) has affine coefficient functions given by:

$$A_{p,t}(\tau) = A_0^{[\tau]} + \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} p(t-\tau_p),$$

$$B_{p,t}(\tau) = B_0^{[\tau]} + \sum_{\tau_p=0}^{n_p} B_1^{[\tau,\tau_p]} p(t-\tau_p),$$

where $A_0^{[\tau]}, B_0^{[\tau]}, A_1^{[\tau,\tau_p]}$ and $B_1^{[\tau,\tau_p]}$ are real valued scalars. This results in the following IO relation:

$$y(t) + \sum_{\tau=1}^{n} A_0^{[\tau]} y(t-\tau) + \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} p(t-\tau_p) y(t-\tau)$$

$$= u(t) + \sum_{\tau=0}^{n} B_0^{[\tau]} u(t-\tau) + \sum_{\tau=0}^{n} \sum_{\tau_p=0}^{n_p} B_1^{[\tau,\tau_p]} p(t-\tau_p) u(t-\tau)$$

The frequency-domain counterpart of (14) is

$$(1 + A_0(e^{j\omega})) Y(e^{j\omega})$$

$$+ \int_0^{2\pi} A_1(e^{j(\omega-\nu)}, e^{j\nu}) Y(e^{j(\omega-\nu)}) P(e^{j\nu}) d\nu$$

$$= B_0(e^{j\omega}) U(e^{j\omega})$$

$$+ \int_0^{2\pi} B_1(e^{j(\omega-\nu)}, e^{j\nu}) U(e^{j(\omega-\nu)}) P(e^{j\nu}) d\nu$$

where

$$A_0(e^{j\omega}) = \sum_{\tau=1}^{n} A_0^{[\tau]} e^{-j\omega \tau},$$

$$B_0(e^{j\omega}) = \sum_{\tau=0}^{n} B_0^{[\tau]} e^{-j\omega \tau},$$

$$A_1(e^{j(\omega-\nu)}, e^{j\nu}) = \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} e^{-j(\omega-\nu) \tau_p} e^{-j\nu \tau_p},$$

$$B_1(e^{j(\omega-\nu)}, e^{j\nu}) = \sum_{\tau=0}^{n} \sum_{\tau_p=0}^{n_p} B_1^{[\tau,\tau_p]} e^{-j(\omega-\nu) \tau_p} e^{-j\nu \tau_p},$$

and $U(e^{j\omega}), Y(e^{j\omega}), P(e^{j\nu})$ are obtained by the DTFT of $u(t), y(t), p(t)$ respectively. Note that this results in a generalized form of the representation presented in [de Rozario and Oomen, 2018] since also dynamical dependencies on the scheduling signals are included.

The proof for the expressions of $A_0(e^{j\omega})$ and $B_0(e^{j\omega})$ is straightforward. How the expression for $A_1(e^{j(\omega-\nu)}, e^{j\nu})$ can be obtained is illustrated below (the reasoning for $B_1(e^{j(\omega-\nu)}, e^{j\nu})$ is similar). Taking the DTFT from the third term of the left-hand side of (15) results in:

$$\mathcal{F}_{\omega,\nu} \left\{ \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} p(t-\tau_p) y(t-\tau) \right\}$$

$$= \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} \mathcal{F}_{\omega,\nu} \{ p(t-\tau_p) y(t-\tau) \}$$

$$= \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} \int_0^{2\pi} P(e^{j\nu}) Y(e^{j(\omega-\nu)}) e^{-j\nu \tau_p} e^{-j(\omega-\nu) \tau} d\nu$$

$$= \int_0^{2\pi} \sum_{\tau=1}^{n} \sum_{\tau_p=0}^{n_p} A_1^{[\tau,\tau_p]} e^{-j\nu \tau_p} e^{-j(\omega-\nu) \tau} P(e^{j\nu}) Y(e^{j(\omega-\nu)}) d\nu$$

$$= \int_0^{2\pi} A_1(e^{j(\omega-\nu)}, e^{j\nu}) P(e^{j\nu}) Y(e^{j(\omega-\nu)}) d\nu,$$

where we made use of the linearity of the DTFT, summation and integral operations.

Equation (16) can be further simplified to:

$$Y(e^{j\omega}) + \int_0^{2\pi} \tilde{A}_1(e^{j(\omega-\nu)}, e^{j\nu}) Y(e^{j(\omega-\nu)}) P(e^{j\nu}) d\nu$$

$$= G_0(e^{j\omega}) U(e^{j\omega})$$

$$+ \int_0^{2\pi} \tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu}) U(e^{j(\omega-\nu)}) P(e^{j\nu}) d\nu,$$

where:

$$G_0(e^{j\omega}) = \frac{B_0(e^{j\omega})}{1 + A_0(e^{j\omega})},$$

$$\tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu}) = \frac{B_1(e^{j(\omega-\nu)}, e^{j\nu})}{1 + A_0(e^{j\omega})}.$$

The above given multi-dimensional frequency-domain responses $G_0(e^{j\omega}), \tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu})$ and $\tilde{A}_1(e^{j(\omega-\nu)}, e^{j\nu})$ describe the system dynamics without implicitly depending on the scheduling signal $p(t)$. The LTI part of the system response is represented by $G_0(e^{j\omega})$ while $\tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu})$ and $\tilde{A}_1(e^{j(\omega-\nu)}, e^{j\nu})$ represent the cross-spectral response between $u$ and $p$ and $y$ and $p$ respectively. In the next section, we show how $G_0(e^{j\omega}), \tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu})$ and $\tilde{A}_1(e^{j(\omega-\nu)}, e^{j\nu})$ are related to fFRF, iFRF and hFRF.

3.2 Harmonic Frequency Response Function

Note that the hFRFs of a system with an affine LPV IO representation can be directly computed starting from (19). Rewrite (19) as:

$$\int_0^{2\pi} (\delta_\nu + \tilde{A}_1(e^{j(\omega-\nu)}, e^{j\nu}) P(e^{j\nu})) Y(e^{j(\omega-\nu)}) d\nu$$

$$= \int_0^{2\pi} (\delta_\nu G_0(e^{j\omega}) + \tilde{B}_1(e^{j(\omega-\nu)}, e^{j\nu}) P(e^{j\nu})) U(e^{j(\omega-\nu)}) d\nu,$$

where $\delta_\nu = 1$ when $\nu = 0$ and 0 otherwise.
To obtain a factorized form of (21), let us restrict the signal trajectories to a finite length $N$, like in a measured data set. Eq. (21) can now be evaluated on a finite frequency grid corresponding to the discrete Fourier transform (DFT) of the signals. Hence, the IO relation provided in (21), while ignoring the transients effects, can be written in a form as:

$$(I_N + \tilde{A} \otimes T(P))Y = (\operatorname{diag}(G_0) + \tilde{B} \otimes T(P))U,$$  \hspace{1cm} (22)$$

where $\operatorname{diag}(G_0)$ results in a diagonal matrix with the elements of $G_0$ on its diagonal, $\otimes$ represent the element-wise Hadamard product. $U$, $Y$ and $P$ are vectors containing the DFT of the finite-length signals $u(t)$, $y(t)$ and $p(t)$ respectively:

$$X_k = \sum_{t=0}^{N-1} x(t)e^{-j\omega_k t}, \quad x(t) = \frac{1}{N} \sum_{k=0}^{N-1} X_ke^{j\omega_k t},$$  \hspace{1cm} (23)$$

where $\omega_k = 2\pi\frac{k}{N}$ and $X = [X_0 \; X_1 \; \ldots \; X_{N-1}]^\top$, with $\top$ denoting the transpose of a vector. $T(P)$ is a complex-valued circulant Toeplitz matrix formed by the vector $P$:

$$T(P) = \begin{bmatrix} P_0 & P_{N-1} & \cdots & P_1 \\ P_1 & P_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ P_{N-1} & P_{N-2} & \cdots & P_0 \end{bmatrix}. \hspace{1cm} (24)$$

The vector $G_0$ and the matrices $\tilde{B}$, $\tilde{A}$ are given by:

$$G_0 = \begin{bmatrix} G_0(e^{j\nu_0}) & G_0(e^{j\nu_1}) & \cdots & G_0(e^{j\nu_{N-1}}) \end{bmatrix}^\top,$$

$$\tilde{B} = \begin{bmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,N-1} \\ B_{1,0} & B_{1,1} & \cdots & B_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{N-1,0} & B_{N-1,1} & \cdots & B_{N-1,N-1} \end{bmatrix}, \hspace{1cm} (25)$$

$$\tilde{A} = \begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N-1,0} & A_{N-1,1} & \cdots & A_{N-1,N-1} \end{bmatrix},$$

$$B_{k,l} = B_1(e^{j\omega_k}, e^{j\omega_l}), \quad A_{k,l} = A_1(e^{j\omega_k}, e^{j\omega_l}).$$

Under the assumption that $(I_N + \tilde{A} \otimes T(P))$ is of full rank, (22) can now be rewritten as:

$$Y = (I_N + \tilde{A} \otimes T(P))^{-1}(\operatorname{diag}(G_0) + \tilde{B} \otimes T(P))U,$$  \hspace{1cm} (26)$$

where the Weireley matrix is given by:

$$W = (I_N + \tilde{A} \otimes T(P))^{-1}(\operatorname{diag}(G_0) + \tilde{B} \otimes T(P)). \hspace{1cm} (27)$$

Observe the clear separation between the role of the scheduling signal spectrum $P$ and the system dynamics $G_0$, $\tilde{B}$, $\tilde{A}$. This formulation opens the possibility for further system analysis, and for an easy frequency-domain computation of the system response. The hFRFs can be constructed by evaluating the diagonals of the Weireley matrix, as is shown in Section 2.5.

### 3.3 Instantaneous Frequency Response Function

Since the iFRF is a hybrid form having both time- and frequency-domain elements, it can either be constructed starting from the parameter-dependent impulse response as in Section 2.4, or it can be obtained starting from the hFRF in (26) in combination with the inverse DTFT:

$$G_{p,t}(e^{j\nu}) = \frac{1}{2\pi} \int_0^{2\pi} G_p(e^{j\omega}, e^{j\nu})e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} W_p(e^{j\nu}, e^{j(\nu-\omega)})e^{j\omega t} d\omega. \hspace{1cm} (28)$$

### 3.4 Frozen Frequency Response Function

The fFRF is quite straightforward to obtain for an LPV system with affine LPV-IO representation. Since the scheduling signal is limited to a constant value: $P(e^{j\nu}) = 0$, $\forall \nu \neq 0$ and $P(e^{j0}) = P_0$. This simplifies (19) to:

$$Y(e^{j\omega}) + \tilde{A}_1(e^{j\omega}, e^{j0})Y(e^{j\omega})P_0 = G_0(e^{j\omega})U(e^{j\omega}) + \tilde{B}_1(e^{j\omega}, e^{j0})U(e^{j\omega})P_0,$$  \hspace{1cm} (29)$$

resulting in

$$Y(e^{j\omega}) = \frac{G_0(e^{j\omega}) + \tilde{B}_1(e^{j\omega}, e^{j0})P_0}{1 + \tilde{A}_1(e^{j\omega}, e^{j0})} U(e^{j\omega}). \hspace{1cm} (30)$$

Note that in terms of (22), this is equivalent to reducing the circular Toeplitz matrix $T(P)$ in (24) to its diagonal, preventing any mixing of the spectrum of $u$. This clearly demonstrates how the fFRFs neglect all the dynamics related to variation of the scheduling signal.

### 4. SIMULATION EXAMPLE

To compare the various frequency response notions we introduce and to demonstrate the usefulness of the hFRF, a simulation example is provided in this section.

#### 4.1 LPV system

Consider the following 1st order discrete-time LPV system with varying local pole and zero locations as an example:

$$[1 - (0.5 - 0.45p(t))q^{-1}]y(t) = [1 - (0.5 + 0.45p(t))q^{-1}]u(t), \hspace{1cm} (31)$$

where $q^{-1}$ represents the backward shift operator. The scheduling signal $p(t)$ is bounded such that $|p(t)| < 1$. This results in a low-pass system behavior when $p(t) < 0$, a high-pass behavior when $p(t) > 0$ and an all-pass behavior for $p(t) = 0$. To verify the validity of the output spectrum calculations, the system is excited by a white noise signal with $u(t) \sim \mathcal{N}(0, 1)$ under various scheduling trajectories:

- Sine wave:
  $$p(t) = \sin \left(2\pi\frac{t}{N}\right), \quad t \in \{0, 1, \ldots, N-1\}.$$

- Periodic Gaussian white-noise process filtered by a 2nd order low-pass Butterworth filter with a normalized (w.r.t. the sampling frequency) cut-off frequency at 0.05. The low-pass filtered signal is normalized to have a maximum absolute value of 1.

The number of samples in the input signal is set to $N = 1024$.

#### 4.2 Frozen Frequency Response Function

The fFRFs of the system for scheduling variable values ranging from -1 to 1 are depicted in Figure 1 illustrating the variations of the local low-pass, high-pass and all-pass behaviors of the system.
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4.3 Harmonic and Instantaneous FRFs

Unlike the FRFs, the hFRFs and the iFRFs are dependent on the scheduling signal. Therefore, we illustrate these FRFs for the two considered scheduling signal types. The iFRFs are obtained as the inverse DFT of the combined hFRFs as is discussed in Section 3.3.

Sine wave
It can be observed that such a slow and smooth p results in a slowly varying, smooth iFRF surface as depicted in Figure 2. For the hFRFs, the frequency response in terms of ω is plotted for ν = 0 in Figure 2 together with the first 4 positive and negative harmonics in terms of the frequency of the sine wave. Note that the hFRFs do not satisfy the complex conjugate property for ±ν (i.e., G_{p,ω}(e^{jν}) ≠ G_{p,ω}(e^{-jν})) as in case of FRFs of real LTI systems.

Filtered Gaussian noise
The Gaussian scheduling signal is much less smooth. This reflects itself in a wild variation of the iFRF surface, as can be seen in Figure 3. For the hFRFs, the frequency response in terms of ω is plotted for ν = 0 in Figure 3. Besides of the lack of conjugate symmetry, one can also observe the introduction of higher-order dynamics in the hFRFs. These dynamics can be explained using the Floquet theory, as is discussed in [Louarroudi, 2014, Floquet, 1883].

4.4 Wereley Frequency Response

Sine wave
By computing the spectrum of p, the Wereley matrix W_p can be directly computed via (22) and the magnitude surface of the resulting response is shown in Figure 4. During calculation, it can be observed that for the chosen p, P has only one non-zero element, hence (24) rotates around the spectrum U in the computation. For the selected example system this means that y at a given frequency only depends on the input at that frequency and its close neighbors, as can be seen in Figure 4. The frequency dependency rolls off rapidly with an increasing distance between the input and output frequency. The system under test behaves dominantly linear time-invariant when the off-diagonal elements are very small compared to the main-diagonal elements of W_p.

Observe that the Wereley matrix offers besides a system analysis tool, also an interesting alternative for the simulation of the steady-state periodic response of stable LPV systems. Figure 5 illustrates that the frequency-domain
analyze how the signal content of the input at frequency \( \nu \) contributes to the output at frequency \( \omega \). It is also discussed how the frozen FRF and instantaneous FRF of this class of LPV systems can be constructed directly from the Wereley frequency response function.

REFERENCES


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5. CONCLUSION

This paper introduced the harmonic frequency response function (hFRF) and the 2-dimensional Wereley frequency response function for LPV systems. It was demonstrated that in case of LPV systems with affine scheduling dependence of their IO representation, the Wereley frequency response matrix gives an efficient way to describe and