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Citation for published version (APA):
Antunes, D. J., & Balaghi I, M. H. (2020). Consistent event-triggered control for discrete-time linear systems with partial state information. *IEEE Control Systems Letters*, *4*(1), 181-186. [8734856]. https://doi.org/10.1109/LCSYS.2019.2922187

DOI:
10.1109/LCSYS.2019.2922187

Document status and date:
Published: 01/01/2020

Document Version:
Accepted manuscript including changes made at the peer-review stage

Consistent event-triggered control for discrete-time linear systems with partial state information

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Abstract—**An event-triggered control strategy is consistent if it achieves a better closed-loop performance than that of traditional periodic control for the same average transmission rate and does not generate transmissions in the absence of disturbances. In this paper, we propose a consistent event-triggered control strategy for discrete-time linear systems with partial state information and Gaussian noise and disturbances when the performance is measured by an average quadratic cost, just as in the Linear Quadratic Gaussian (LQG) framework. This strategy incorporates a scheduler determining transmissions based on the error between two state estimates, which are provided by a stationary Kalman filter at the sensors/scheduler side and an estimator at the controller/actuators side relying on previously transmitted data. Through a numerical example, we show that the proposed strategy can achieve impressive performance gains with respect to periodic control for the same average transmission rate.**

Index Terms—**Markov processes, Optimal control, Stochastic optimal control, Observers for Linear systems, Networked control systems**

I. Introduction

Event-triggered control (ETC) is an alternative to periodic control proposed in recent years to reduce the communication burden in networked control systems. For concreteness, consider the networked control system depicted in Figure 1. The plant's sensors operate at a given, typically fast, rate and transmit data through a network with bandwidth limitations to a controller co-located with the actuators, operating at the same rate as the sensors. A scheduler, collocated with the sensors, decides when measurement/estimation data should be sent to the controller. The ETC problem is to design both the control input law (controller) and the transmission decision law (scheduler) to reduce transmissions while meeting desired performance and stability criteria. The resulting transmission sequence is in general not periodic.

While designing optimal event-triggered controllers in a given sense is typically hard [1]–[3], several researchers have proposed different strategies with stability guarantees and related closed-loop performance properties [4]–[10]; performance can be defined, for example, in terms of a quadratic cost [5]–[7], [9] and an ℓ_2 induced norm bound [9], [10]. In the present work, we follow [7], [8], where an ETC policy was proposed to guarantee two, so-called consistency, properties: (i) it achieves a better closed-loop performance than that of traditional periodic control for the same average transmission rate; and (ii) no transmissions are generated in the absence of disturbances; performance in [7], [8] is measured by an average quadratic cost just as in traditional Linear Quadratic Gaussian control. Other works have proposed ETC strategies, relying on different ideas, that can achieve a better performance than that of periodic control for the same average transmission rate (first consistency property) [5]–[7], [9], [11], [12]. However, all these works consider that full state-feedback is available, and, to the best of our knowledge, only [11] has been extended to output-feedback, see [13], [14] and also [15]. Still, the policy in [11] is not necessarily *strictly* consistent in the sense that it might only achieve the *same* (and not a better) performance when compared to periodic control, and, even if the full state-feedback is available, it is only applicable when the disturbances are Gaussian.

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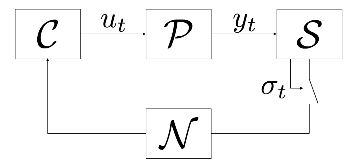

Fig. 1. Setup: an event-triggered controller consists of a controller $\mathcal{C}$ and a scheduler $\mathcal{S}$; $\mathcal{N}$ represents the network, $\mathcal{P}$ represents the plant, u_t, y_t, σ_t are the control input, output, and transmission decision, respectively, at time t. The scheduler sends measurement/estimation data to the controller.

In the present paper, we propose for the first time a *strictly* consistent ETC policy for any linear system with partial state-feedback (output-feedback) and Gaussian disturbances and noise, under standard stabilizability and detectability assumptions. This strategy incorporates a scheduler determining transmissions based on the error between two state estimates, provided by a stationary Kalman filter at the sensors/scheduler side and an estimator at the controller/actuators side relying on previously transmitted data. In particular, transmissions are triggered when a weighted (semi-)norm of this error exceeds a certain threshold; this thresholds may be constant or, in general, depend on the time elapsed since the last transmission. The controller follows a linear feedback law. While the strategy is inspired by the full state-feedback policy in [7], it is a static transmission policy rather than dynamic as in [7], in the sense that it requires no additional auxiliary state variables. In the case of full state-feedback, the results also hold for non-Gaussian disturbances, as opposed to the results in [11].

The remainder of the paper is organized as follows. Section II introduces the problem. Section III describes the proposed ETC method and provides the main results. Section IV discusses a numerical example and Section V provides concluding remarks.

II. Problem Formulation

The plant to be controlled is modelled by

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t, \\ y_t &= Cx_t + v_t \end{aligned} \tag{1}$$

where $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $x_t \in \mathbb{R}^{n_x}$ is the state, $y_t \in \mathbb{R}^{n_y}$ is the plant's output, and $u_t \in \mathbb{R}^{n_u}$ is the control input at time t; the pairs (A, B) and (A, C) are assumed to be stabilizable and detectable, respectively, and the initial condition x_0 is a random variable with mean $\check{x}_0$ and covariance $\check{\Phi}_0$; w_t, v_t, for $t \in \mathbb{N}_0$ are sequences of independent and identically distributed random variables with zero mean and finite covariance, $W := \mathbb{E}[w_t w_t^\intercal]$, $V := \mathbb{E}[v_t v_t^\intercal]$, for every $t \in \mathbb{N}_0$. Letting, $W = B_w B_w^\intercal$, we assume (A, B_w) is controllable.

The results presented in this paper apply to two important cases, differing on the assumptions on the measurements:

(FS) Full state is available; i.e., $y_t = x_t$, or, equivalently, $C = I$ and $v_t = 0$ for every $t \in \mathbb{N}_0$.

(PS) Partial state measurements, assuming that x_0 v_t , w_t for every $t \in \mathbb{N}_0$ follow Gaussian distributions and V is a positive-definite matrix.

The scheduler processes the past and the current output information up to time t, $\mathcal{I}_t^{sc} := \{y_r | r \in \{0, 1 \dots, t\}\}$, to decide whether or not to transmit new information to the controller at time t. Therefore, the decision at time t must be a function of $\mathcal{I}_t^{sc}$ and this function is denoted by $\sigma_t = \zeta_t(\mathcal{I}_t^{sc})$, where $\sigma_t = 1$ if there is a data transmission from sensors to the controller at time t and $\sigma_t = 0$, otherwise. By convention, there is a transmission at time $t = 0$, i.e., $\sigma_0 = 1$. The controller must decide u_t based on the information set

$$\mathcal{I}_t^c := \{y_r | r \in \{0, 1, \dots, \iota(t)\}\},$$

where $\iota(t) := \max\left\{\ell \in \{0, 1, \dots, t\} | \sigma_\ell = 1\right\}$ is the time of the last transmission from the initial up to the current time. We use μ_t to denote this function $u_t = \mu_t(\mathcal{I}_t^c)$. Conceptually, when a transmission occurs at time t the scheduler can send y_r for $r \in \{\iota(t-1)+1, \dots, t\}$ to the controller in a transmission packet, but, as we shall see shortly, for the policies considered, it will suffice to send an estimate of the state x_t computed by the scheduler given $\mathcal{I}_t^{sc}$. An event-triggered controller is then a set of functions $\pi = (\mu, \zeta)$ where $\mu = (\mu_0, \mu_1, \mu_2, \dots)$, $\zeta = (\zeta_0, \zeta_1, \zeta_2, \dots)$.

A crucial assumption is that the network is ideal (no packet drops and no delays), which is implicit in the definition of the information available to the controller, $\mathcal{I}_t^c$. Then, since u_k is a function of $\mathcal{I}_t^c$, which is a subset of $\mathcal{I}_t^{sc}$, we can assume the scheduler also knows $\{u_0, u_1, \dots, u_t\}$ at time t.

Performance is measured by the following cost[1]

$$J := \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}[\sum_{t=0}^{T-1} x_t^\intercal Q x_t + u_t^\intercal R u_t], \tag{2}$$

[1] As typically done in the literature, we define the average cost performance with $\limsup$ instead of $\lim$, since then we do not have to prove existence of the limit for the proposed ETC scheme. To show existence of the limit one may resort to ergodicity [16], see [6] in the context of ETC, but this is beyond the scope of the present paper.

for positive definite matrices R and Q. The average transmission rate associated with a given scheduler and controller policies is $\bar{R} = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}[\sum_{t=0}^{T-1} \sigma_t]$, and the average inter-transmission time is $\bar{\tau} := 1/\bar{R}$.

For periodic transmission, the scheduler is specified by

$$\sigma_t = \begin{cases} 1, & \text{if } t \text{ is zero or an integer multiple of } h, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where h coincides with the average inter-transmission time $\bar{\tau} = h$. As stated in Theorem 1 below, under assumptions (FS) or (PS), the optimal policy for the case of periodic transmission (3) is a multi-rate controller composed of: (i) an estimator at the sensor side (run by the scheduler) given by $\bar{x}_{t|t} = x_t$ if (FS) holds and by the stationary Kalman filter

$$\begin{aligned} \bar{x}_{t+1|t} &= A\bar{x}_{t|t} + Bu_t, \\ \bar{x}_{t|t} &= \bar{x}_{t|t-1} + L(y_t - C\bar{x}_{t|t-1}), \end{aligned} \tag{4}$$

if (PS) holds, where $t \in \mathbb{N}_0$, $\bar{x}_{0|-1} = \check{x}_0$, $L = \Phi C^\intercal (C\Phi C^\intercal + V)^{-1}$ and Φ is the unique positive semi-definite solution to

$$\begin{aligned} \Phi &= A\Theta A^\intercal + W. \\ \Theta &= \Phi - \Phi C^\intercal (V + C\Phi C^\intercal)^{-1} C\Phi \end{aligned} \tag{5}$$

(ii) an estimator at the actuator side (run by the controller)

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} + Bu_t, \\ \hat{x}_{t|t} &= \begin{cases} \bar{x}_{t|t}, & \text{if } \sigma_t = 1 \\ \hat{x}_{t|t-1}, & \text{otherwise}, \end{cases} \end{aligned} \tag{6}$$

where $t \in \mathbb{N}_0$, $\hat{x}_{0|-1} = \check{x}_0$, and (iii) a static control input law (run by the controller and the scheduler)

$$u_t = K\hat{x}_{t|t}, \tag{7}$$

where $K := -(R + B^\intercal P B)^{-1} B^\intercal P A$, and P is the unique positive definite (since Q is positive definite) solution to

$$P = A^\intercal P A + Q - A^\intercal P B (R + B^\intercal P B)^{-1} B^\intercal P A.$$

Note that, in fact, the control policy for u_t is a function of $\mathcal{I}_t^c$, and it suffices for the scheduler to send $\bar{x}_{t|t}$ to the controller at transmissions times $\{t | \sigma_t = 1\}$.

Theorem 1: Consider (1) with scheduling policy (3) and control policy (6), (7). Moreover, assume that either (FS) or (PS) hold. Then, the control policy minimizes (2), resulting in a cost (2) given by

$$J^*_{\text{per}}(h) = \text{tr}(PW) + \text{tr}(Z\Theta) + g(h),$$

where $g(1) = 0$ and, defining $Z := K^\intercal (R + B^\intercal P B) K$,

$$g(h) = \frac{1}{h} \text{tr}\big(Z(\sum_{s=1}^{h-1} Y(s))\big), \quad h \in \mathbb{N}_{\geq 2}, \tag{8}$$

where $Y(s) = A^s \Theta A^{\intercal s} + \sum_{r=0}^{s-1} A^r W A^{\intercal r} - \Theta$ for $s \in \mathbb{N}$. Moreover, $g(i) \leq g(j)$, if $i < j$, $i, j \in \mathbb{N}$. Furthermore, if

$$KX \neq 0, \quad X := \begin{cases} W & \text{if (FS) holds}, \\ W + A\Theta A^\intercal - \Theta & \text{if (PS) holds} \end{cases} \tag{9}$$

then $g(h) > 0$, for $h \in \mathbb{N}_{\geq 2}$. $\square$

The proof can be found in the appendix. Note that $J^*_{\text{per}}(h)$ is an increasing function of $h \in \mathbb{N}$ in the setting considered here, by which both scheduler and controller operate at a fixed sampling period, coinciding with one time step of the model (1), and the transmission period from the scheduler to the controller is a parameter $h \in \mathbb{N}$; often, (1) and (2) result from discretizing a continuous-time plant and cost at a fixed sampling period. However, in a different setting, where the continuous-time plant and cost are discretized with a sampling period τ that plays the role of a parameter, at which the controller and scheduler also operate, the cost associated with an optimal controller is in general not an increasing function of the sampling period τ, as shown in [17].

A. Problem definition

Let J_π and $\bar{\tau}_\pi$ denote the average cost (2) and the average inter-transmission time of an event-triggered control policy π. Note that, although for periodic control the average inter-transmission time can only take integer values, $h \in \mathbb{N}$, for a general π, $\bar{\tau}_\pi$ can take values on the real line. Therefore, to compare the performance of J_π with that of periodic control we consider a function $\bar{J}_{\text{per}}(a)$ defined for $a \in \mathbb{R} \cap [1, \infty)$ where for each subinterval $a \in [h, h+1)$, $h \in \mathbb{N}$,

$$\bar{J}_{\text{per}}(a) = (a-h)J^*_{\text{per}}(h+1) + (h+1-a)J^*_{\text{per}}(h). \quad (10)$$

Note that this function linearly interpolates between two values of the periodic controller cost at integer transmission periods and coincides with $J^*_{\text{per}}(a)$ when a is an integer. We say that an event-triggered policy π is *consistent* [7], [8] if:

(i) $J_\pi < \bar{J}_{\text{per}}(\bar{\tau}_\pi)$.

(ii) The following holds: when $(v_r, w_r) = 0$, for every $r \in \mathbb{N}_0$, and $\check{\Phi}_0 = 0$, if $\sigma_k = 1$ then $\sigma_{k+\ell} = 0$ for $\ell \in \{1, \ldots, \bar{h}-1\}$, where $\bar{h} \in \mathbb{N} \cup \{\infty\}$ is a fixed upper bound for the inter-transmission intervals.

These two properties are adaptations of the definitions in [7], [8] to the discrete-time and output-feedback case. The first consistency property can be interpreted as π achieves a better closed-loop performance than that of periodic control policy (6), (7) corresponding to the scheduler (3) for the same average transmission rate (see Fig 2(b) below). The second consistency property requires that π generates no transmissions when there are no disturbances, no noise, and no uncertainty on the initial state x_0; this is possible under assumption (FS), but has probability zero under assumption (PS). Still, this can be seen as a desired property for more general disturbances. Note that the definition of consistency depends on $\bar{h}$, which is typically a large constant. Hence, [7] uses the terminology $\bar{h}$-consistent, which is simplified here.

The problem considered in this paper is to find a consistent policy.

Remark 2: Note that we have considered the stationary Kalman filter (4) instead of the Kalman filter with, in general, time-dependent gains L_t (instead of L). These coincide if

$$\check{\Phi}_0 = \Phi \quad (11)$$

However, the gains of the Kalman filter converge to L under the assumptions considered here. Since (2) only depends on the asymptotic behaviour of the system, we can directly consider the stationary filter, or assume (11), which we will often do for simplicity.

III. Consistent ETC method and main results

The proposed ETC policy consists of a controller and a scheduler both building upon the ones for periodic control. The estimator and controller at the actuator side and the estimator at the sensor side are chosen to take the same form as those of the optimal control law for periodic control (4), (6), (7). In particular, it suffices for the scheduler to send $\bar{x}_{t|t}$ to the controller at transmissions times. However, the transmission policy for σ_k is not given by the periodic scheduler (3), but it is instead described as follows.

Assume that $g(h) > 0$ for $h \in \mathbb{N}_{\geq 2}$, which, as stated in Theorem 1, is the case if (9) holds. Let f_h, $h \in \mathcal{H} := \{1, 2, \ldots, \bar{h}\}$, where $\bar{h} \in \mathbb{N}$, be a non-negative non-decreasing sequence such that $f_1 = 0$ and

$$0 < f_h < g(h), \text{ for } h \in \mathcal{H} \setminus \{1\}, \quad (12)$$

and

$$f_{h+1} - f_h \leq f_h - f_{h-1}, \text{ for } h \in \mathcal{H} \setminus \{1, \bar{h}\}. \quad (13)$$

This latter condition ensures that the following function, defined as in (10), is concave

$$\bar{f}(a) = (a-h)f_{h+1} + (1+h-a)f_h,\ a \in [h, h+1), h \in \mathcal{H} \setminus \{\bar{h}\}. \quad (14)$$

Moreover, let $\phi_1 = 0$, $\phi_2 = f_2$, and, for $h \in \mathcal{H} \setminus \{1, 2\}$, let $\phi_h := f_h - f_{h-1}$. Since $g(h)$ is positive for $h \in \mathbb{N}_{\geq 2}$ and non-decreasing, it is always possible to find a sequence f_h satisfying (12), (13), and these conditions imply that ϕ_h is a non-negative non-increasing sequence. In particular, one can pick $f_h = \gamma(h-1)$ for sufficiently small $\gamma > 0$, in which case $\phi_h = \gamma$ is constant for $h \in \mathcal{H} \setminus \{1\}$.

Given f_h and ϕ_h, we define the following class of scheduling policies parametrized by a scalar $\alpha \in \mathbb{R}_{\geq 1}$,

$$\sigma_t = \begin{cases} 1, \text{ if } t = s_k \text{ for some } k \in \mathbb{N}_0, \\ 0, \text{ otherwise,} \end{cases}$$

where k indexes the data transmissions, s_k is the time at which the k-th transmission occurs, $s_0 := 0$ (by convention there is a transmission at time $t = 0$),

$$s_{k+1} = s_k + \tau_{k,\alpha}, \quad k \in \mathbb{N}_0,$$

and the function $\tau_{k,\alpha}$ is given by

$$\tau_{k,\alpha} = \begin{cases} \min \mathcal{B}_k, & \text{if } \mathcal{B}_k \text{ is non empty,} \\ \bar{h}, & \text{otherwise,} \end{cases} \quad (15)$$

where

$$\mathcal{B}_k := \{\beta \in \mathcal{H} \setminus \{\bar{h}\} | e^{\intercal}_{s_k+\beta} Z e_{s_k+\beta} \geq \alpha \phi_{\beta+1}\},$$
$$e_t := \bar{x}_{t|t} - \hat{x}_{t|t-1}.$$

Note that to compute e_t, the scheduler also needs to run the controller equations (6), (7). Moreover, when $\phi_h = \gamma$ for $h \in \mathcal{H} \setminus \{1\}$ and $\bar{h}$ is large, (15) is a standard state error threshold-based policy with a constant threshold, by which transmissions are triggered when a weighted (semi-)norm of the error e_t is larger than a threshold, and which is often considered in the literature (see, e.g., [18]). Our framework allows for more general sequences ϕ_h, $h \in \mathcal{H}\backslash\{1\}$, which results in state error threshold-based policies with thresholds dependent on the time elapsed since the last transmission.

The proposed event-triggered control policy is defined for α such that

$$1 \leq \alpha \leq \mathbb{E}[\tau_{k,\alpha}|\mathcal{I}^{sc}_{s_k}]. \tag{16}$$

Let $L(\alpha) := \mathbb{E}[\tau_{k,\alpha}|\mathcal{I}^{sc}_{s_k}]$, which does not depend on the transmission index k when (11) holds, since $e_{s_k+1}, e_{s_k+2}, \dots, e_{s_{k+1}}$ have the same probability distribution, when conditioned on $\mathcal{I}^{sc}_{s_k}$, for every $k \in \mathbb{N}_0$. In fact, letting $\tilde{e}_t := x_t - \bar{x}_{t|t-1}$, we have

$$e_{t+1} = \begin{cases} LC\tilde{e}_{t+1} + LCv_{t+1}, & \text{if } t = s_k, \\ Ae_t + LC\tilde{e}_{t+1} + LCv_{t+1}, & \text{if } t \in \{s_k+1, \dots, s_{k+1}-1\}, \end{cases}$$

where $\tilde{e}_{s_k+1}, \dots, \tilde{e}_{s_{k+1}}$, have the same probability distribution for every k, when conditioned on $\mathcal{I}^{sc}_{s_k}$. Moreover, since, for every $k \in \mathbb{N}_0$ and α satisfying (16), $1 \leq \tau_{k,\alpha} \leq \bar{h}$, we conclude that $L(1) \geq 1$ and $L(\bar{h}) \leq \bar{h}$. It is also clear that $L(\alpha)$ is a non-decreasing function of α. Thus, we can plot $L(\alpha)$ for a dense grid of α and check when $\alpha \leq L(\alpha)$ (see Figure 2(a)). To compute $L(\alpha)$ it suffices to perform Monte-Carlo simulations (see Section IV). More specifically, for each α, one can run several times (1), (4)-(7), (15) and compute the average inter-transmission time. While this may be computationally expensive for large dimensional systems, it can be computed offline.

The next theorem is the main result of the paper and states that this policy, with α satisfying (16), meets the first consistency property. For α such that $\alpha \leq L(\alpha)$, let $\xi := \frac{\alpha}{L(\alpha)} \leq 1$.

Theorem 3: Suppose that f_h is such that (12), (13) holds for $h \in \{1, 2, \dots, \bar{h}\}$, (9) and (11) hold, and either (FS) or (PS) hold. Let J_π be the performance of the proposed ETC policy (6), (7), and (15) for α such that (16) holds when applied to (1). Then,

$$J_\pi \leq \xi \bar{f}(\bar{\tau}_\pi) + \mathrm{tr}(PW) + \mathrm{tr}(Z\Theta) < \bar{J}_{\text{per}}(\bar{\tau}_\pi). \tag{17}$$

□

The proposed policy meets the second consistency property since if there are no disturbances, noise and no initial uncertainty on the state, the two state estimates $\bar{x}_{t|t}$ and $\hat{x}_{t|t-1}$ coincide with the state x_t, e_t is zero, and the inter-sampling times $\tau_{t,\alpha}$ are equal to $\bar{h}$, for every t. Therefore, Theorem 3 assures that the proposed policy is consistent.

The proof of Theorem 3 can be found in the appendix. It resorts to the optional sampling theorem [19, Th.11, Sec.12.4] and to the optional stopping theorem [19, Th.1, Sec.12.5] which hold when the stopping times are bounded. This is the main reason for assuming $\bar{h} < \infty$. To extend Theorem 3 to the case $\bar{h} = \infty$, less stringent conditions of these two theorems can be pursued.

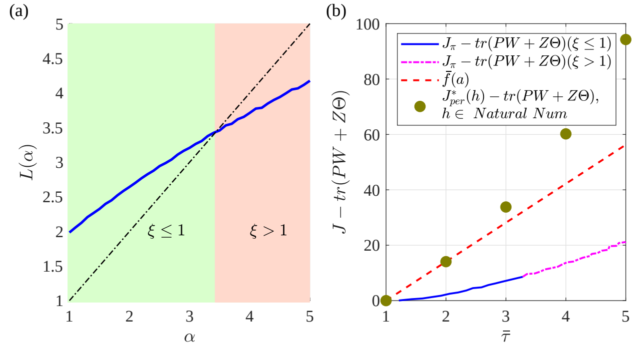

Fig. 2. (a) Result of the Monte Carlo simulations to find the range of α satisfying the consistency condition of the ETC, (b) Trade-off curve between the performance and average inter-transmission time for the event-triggered and periodic control. Note that for the same integer average inter-transmission time, the ETC policy results in a performance cost significantly below than the performance cost of periodic control.

IV. Numerical Example

Suppose that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, Q = I, R = 0.1.$$

and that $\{w_\ell\}_{\ell \in \mathbb{N}_0}$ and $\{v_\ell\}_{\ell \in \mathbb{N}_0}$ follow normal distributions with zero mean and covariance $W = 0.5I$ and $V = 0.1$, respectively. Considering these numerical values, one can compute $g(h)$ and check that $\bar{J}_{\text{per}}(a)$ is a convex function and $g(2) - g(1) = 14.0680$. Therefore, $f_h = \gamma(h-1)$ for $\gamma \in (0,\ 14.0680)$ satisfies the conditions in (12) and (13). We set $\bar{h} = 100$. In order to tune the value of α for a given value of γ, we run Monte Carlo simulations for a dense grid of α and find the value of $L(\alpha)$. In Fig. 2(a), we plot $L(\alpha)$ with respect to α when $\gamma = 14.0680$. Note that α must be selected in the region in which $\xi \leq 1$, i.e., $\alpha \in [0, 3.3]$ to guarantee the consistency condition of the proposed ETC. However, larger values can also be selected, although no guarantees of consistency are given. In Fig. 2(b), the values of $J^*_{\text{per}}(h) - \mathrm{tr}(PW + Z\Theta)$, for $h \in \{1, \dots, 5\}$, where $\mathrm{tr}(PW + Z\Theta) = 21.24$, are shown in dots which indicates the trade-off curve for periodic control. The dashed red line represents $\bar{f}(a)$ in (14) when $\gamma = 14.0680$. Moreover, by changing the value of α the proposed ETC policy results in different values of average inter-transmission time and average quadratic performance in the region where the policy is guaranteed to be consistent ($\alpha \in [0, 3.3]$, blue) and beyond that region (magenta). The comparison of these two trade-off curves not only shows the performance improvement of the proposed ETC policy, but it also conveys the message that we can achieve very large performance gains in comparison with periodic control by employing the proposed ETC policy.

V. Conclusions

We proposed a consistent policy for discrete-time event-triggered control system with partial information, extending [7], [8]. This policy provides a better performance than that of optimal periodic control for the same average transmission rate. Considering network delays, packet drops and unbounded inter-sampling times warrant future research.

Appendix
Proof of Theorem 1

Most of the statements follow from optimal control results [20]. Here we only prove the properties of g. We start by noticing that $Y(1) \geq 0$ due to (5) and, if we let $\bar{Y}(0) := \Theta$, $\bar{Y}(i) := Y(i) + \Theta$, for $i \in \mathbb{N}$, we have

$$\bar{Y}(i+1) = A\bar{Y}(i)A^\intercal + W, \quad i \in \mathbb{N}_0, \tag{18}$$

and $\bar{Y}(i) \leq \bar{Y}(i+1)$, $i \in \mathbb{N}_0$, by induction (which implies $Y(i) \leq Y(i+1)$, $i \in \mathbb{N}$): it holds for $i = 0$ due to (5) and assuming $\bar{Y}(i) \leq \bar{Y}(i+1)$ holds for a given i pre and postmultiplying this inequality by A and A^T, respectively, adding W to both sides and using (18) we conclude that it holds for $i+1$. The proof that $g(i) \leq g(j)$, if $i < j$ follows then from the fact that $g(h) = \frac{1}{h}\sum_{s=1}^{h-1} q(s)$ where the $q(s) := \mathrm{tr}\big(K^\intercal(R + B^\intercal PB)KY(s)\big)$ are non-negative, since $Y(s) \geq 0$ for $s \in \mathbb{N} \setminus \{1\}$, and non-decreasing, since $Y(i) \leq Y(i+1)$, $i \in \mathbb{N}$. Since $g(h)$ is the average value of a non-negative and non-decreasing sequence, it is also non-negative and non-decreasing. Since $R > 0$ then $(R + B^\intercal PB) > 0$. Moreover, for every $s \in \mathbb{N}$, $Y(s) \geq X$. Therefore, if $KX \neq 0$, $q(s) > 0$ for $s \in \mathbb{N}$ and $g(h) > 0$ for $h \in \mathbb{N} \setminus \{1\}$.

Proof of Theorem 3

Let $g(x_t, u_t) := x_t^\intercal Q x_t + u_t^\intercal R u_t$ and note that

$$\begin{aligned}
&\mathbb{E}[\sum_{t=0}^{T-1} g(x_t,u_t)] = \mathbb{E}[\sum_{t=0}^{T-1} \mathbb{E}[g(x_t,u_t)|\mathcal{I}_t^{sc}]] \\
&= \mathbb{E}[\sum_{k=0}^{N(T)}\sum_{t=s_k+1}^{s_{k+1}} \mathbb{E}[g(x_t,u_t)|\mathcal{I}_t^{sc}] + \epsilon] + \epsilon_0 \qquad (19)\\
&= \mathbb{E}[\sum_{k=0}^{N(T)} \mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} \mathbb{E}[g(x_t,u_t)|\mathcal{I}_t^{sc}]|\mathcal{I}_{s_k}^{sc}]] + \bar{\epsilon} + \epsilon_0
\end{aligned}$$

where $N(T) := \bar{N}(T) - 1$, $\bar{N}(T) := \min\{m \in \mathbb{N}_0 | s_m \geq T\}$, $\epsilon_0 := \mathbb{E}[g(x_0,u_0)]$, $\bar{\epsilon} := \mathbb{E}[\epsilon|\mathcal{I}_{s_{N(T)}}^{sc}]$, $\epsilon := -(\sum_{t=T}^{s_{\bar{N}(T)}} g(x_t,u_t))$; the first equality follows from the tower property of conditional expectations. To prove the last equality, let $\eta_r := \sum_{k=0}^{r} \rho(s_k, s_{k+1})$, for $r \in \mathbb{N}_0$, $\rho(a,a) := 0$, for $a \in \mathbb{N}_0$, and, for $b > a$,

$$\rho(a,b) := \sum_{t=a+1}^{b} \mathbb{E}[g(x_t,u_t)|\mathcal{I}_t^{sc}] - \mathbb{E}[\sum_{t=a+1}^{b} g(x_t,u_t)|\mathcal{I}_a^{sc}];$$

Then $\rho(0,b)$ is a martingale with respect to the filtration $\{\mathcal{I}_b^{sc}|b \in \mathbb{N}_0\}$, i.e., $\mathbb{E}[\rho(0,b+1)|\mathcal{I}_b^{sc}] = \rho(0,b)$, and $\mathbb{E}[\rho(0,b)] = 0$, for $b \in \mathbb{N}_0$. Moreover, s_1 is a stopping time with respect to (w.r.t) the same filtration, which is bounded since $\bar{h} < \infty$. Then, from the optional sampling theorem [19, Th.11, Sec.12.4], $\mathbb{E}[\rho(0,s_1)|\mathcal{I}_0^{sc}] = 0$. Using the same reasoning for the process started at s_ℓ, $\mathbb{E}[\rho(s_\ell, s_{\ell+1})|\mathcal{I}_{s_\ell}^{sc}] = 0$ for $\ell \in \mathbb{N}$. We can use this fact to conclude that (i) η_r is also a martingale but now with respect to the filtration $\{\mathcal{F}_r := \mathcal{I}_{s_{r+1}}^{sc} | r \in \mathbb{N}_0\}$, and $N(T)$ is a stopping time w.r.t. this filtration since $s_0, \ldots, s_{r+1}$ are measurable w.r.t. this filtration; and (ii) $\mathbb{E}[\eta_r] = 0$ for $r \in \mathbb{N}_0$. Note that $N(T) < T$. Then, the desired conclusion, equivalent to $\mathbb{E}[\eta_{N(T)}] = 0$, follows from the optional stopping theorem [19, Th.1, Sec.12.5].

We now show that the term inside the summation equals

$$\begin{aligned}
\mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} \mathbb{E}[g(x_t,u_t)|\mathcal{I}_t^{sc}]|\mathcal{I}_{s_k}^{sc}] &= \mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} \mathbb{E}[\hat{e}_t^\intercal Z \hat{e}_t|\mathcal{I}_t^{sc}]|\mathcal{I}_{s_k}^{sc}] \\
&\quad + \mathrm{tr}(PW)\mathbb{E}[s_{k+1} - s_k|\mathcal{I}_{s_k}^{sc}] \\
&\quad + \mathbb{E}[x_{s_k+1}^\intercal P x_{s_k+1} - x_{s_{k+1}+1}^\intercal P x_{s_{k+1}+1}|\mathcal{I}_{s_k}^{sc}] \qquad (20)
\end{aligned}$$

where $\hat{e}_t := x_t - \hat{x}_{t|t}$. To see this, let $\chi(0,0) := 0$ and

$$\begin{aligned}
\chi(t_0,r) \; &:= -\mathbb{E}[x_{r+1}^\intercal P x_{r+1}|\mathcal{I}_r^{sc}] + \mathbb{E}[x_{t_0+1}^\intercal P x_{t_0+1}|\mathcal{I}_{t_0+1}^{sc}] \\
&\quad + \textstyle\sum_{t=t_0+1}^{r} \mathbb{E}[x_{t+1}^\intercal P x_{t+1} - x_t^\intercal P x_t|\mathcal{I}_t^{sc}] \qquad (21)
\end{aligned}$$

for $r \in \mathbb{N}$. This is a martingale w.r.t the filtration $\{\mathcal{I}_r^{sc}|r \in \mathbb{N}_0\}$, $\mathbb{E}[\chi(t_0,r)] = 0$, for every $r \in \mathbb{N}_0$, and using the optional sampling theorem [19, Th.11, Sec.12.4] (which can be applied since $s_1 \leq \bar{h} < \infty$) we conclude that $\mathbb{E}[\chi(t_0,s_1)|\mathcal{I}_{t_0}^{sc}] = 0$; using the same reasoning for the process started at s_k,

$$\mathbb{E}[\chi(s_k,s_{k+1})|\mathcal{I}_{s_k}^{sc}] = 0, \text{for } k \in \mathbb{N}_0. \tag{22}$$

Moreover, from completion of squares,

$$\begin{aligned}
\mathbb{E}[x_{t+1}^\intercal P x_{t+1} - x_t^\intercal P x_t|\mathcal{I}_t^{sc}] &= -\mathbb{E}[x_t^\intercal Q x_t + u_t^\intercal R u_t|\mathcal{I}_t^{sc}] + \\
\mathrm{tr}(PW) + \mathbb{E}[(u_t - Kx_t)^\intercal &(R + B^\intercal PB)(u_t - Kx_t)|\mathcal{I}_t^{sc}].
\end{aligned}$$

Replacing this expression in (21), noticing that $u_t - Kx_t = -K\hat{e}_t$, and using the tower property of conditional expectations, we conclude that (22) is equivalent to (20).

Let $e_{t|t} := \bar{x}_{t|t} - \hat{x}_{t|t}$, i.e., $e_{t|t} = 0$ if $t = s_k$ for some k, $e_{t|t} = e_t$ otherwise. For the first term on the r.h.s of (20):

$$\begin{aligned}
\mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} \mathbb{E}[\hat{e}_t^\intercal Z \hat{e}_t|\mathcal{I}_t^{sc}]|\mathcal{I}_{s_k}^{sc}] &= \mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} e_{t|t}^\intercal Z e_{t|t} + \mathrm{tr}(Z\Theta)|\mathcal{I}_{s_k}^{sc}] \\
&= \mathbb{E}[\sum_{t=s_k+1}^{s_{k+1}} e_t^\intercal Z e_t|\mathcal{I}_{s_k}^{sc}] + \underbrace{\mathrm{tr}(Z\Theta)\mathbb{E}[s_{k+1} - s_k|\mathcal{I}_{s_k}^{sc}]}_{:=\nu} \\
&= \sum_{\ell=1}^{\bar{h}-1} \mathbb{E}[e_{s_k+\ell}^\intercal Z e_{s_k+\ell} \mathbf{1}_{\tau(\mathcal{I}_{s_k+\ell}^{sc}) > \ell}|\mathcal{I}_{s_k}^{sc}] + \nu \\
&\leq \sum_{\ell=1}^{\bar{h}-1} \alpha\phi_{\ell+1} \mathrm{Prob}[\tau_{k,\alpha}(\mathcal{I}_{s_k+\ell}^{sc}) > \ell|\mathcal{I}_{s_k}^{sc}] + \nu \\
&= \sum_{\ell=1}^{\bar{h}-1} \alpha\phi_{\ell+1} (\sum_{m=\ell+1}^{\bar{h}} \mathrm{Prob}[\tau_{k,\alpha}(\mathcal{I}_{s_k+\ell}^{sc}) = m|\mathcal{I}_{s_k}^{sc}]) + \nu \\
&= \alpha \sum_{m=1}^{\bar{h}} (\underbrace{\sum_{\ell=1}^{m} \phi_\ell}_{f_m}) \mathrm{Prob}[\tau_{k,\alpha}(\mathcal{I}_{s_k+\ell}^{sc}) = m|\mathcal{I}_{s_k}^{sc}]) + \nu \\
&\leq \alpha \bar{f}(\bar{\tau}_\pi) + \nu \qquad (23)
\end{aligned}$$

The first equality follows from the fact that if (PS) holds then x_t and $\hat{e}_t$ are Gaussian random variables when conditioned on $\mathcal{I}_t^{sc}$ with mean $\bar{x}_{t|t}$ and $e_{t|t}$, respectively, and both with covariance Θ. We can then apply [21, Ch. 8, Lemma 3.3] to compute the conditioned expected value of $\mathbb{E}[\hat{e}_t^\intercal Z \hat{e}_t]$, which is given by $e_{t|t}^\intercal Z e_{t|t} + \mathrm{tr}(Z\Theta)$; if (FS) holds instead of (PS) then $e_{t|t}$ is known when conditioned on $\mathcal{I}_t^{sc}$, the expected value boils down to the actual value and $\Theta = 0$. In the third equality we use the notation $\mathbf{1}_A = 1$ if the event A happens, $\mathbf{1}_A = 0$ if A does not happen. In the first inequality we used the fact that in the event $\tau_{k,\alpha} > \beta$, the following must hold $e_{s_k+\beta}^\mathsf{T} Z e_{s_k+\beta} < \alpha\phi_{\beta+1}$ due to (15). In the second inequality we used the fact that $\text{Prob}[\tau_{k,\alpha}(I_{s_k+\ell}) = m|\mathcal{I}_{s_k}^{sc}]$ are constant (do not depend on $\mathcal{I}_{s_k}^{sc}$, see the arguments below (16)), which in turn implies that the average inter-sampling time is $\bar{\tau}_\pi = \sum_{m=1}^{\bar{h}} \text{Prob}[\tau_{k,\alpha}(\mathcal{I}_{s_k+\ell}^{sc}) = m|\mathcal{I}_{s_k}^{sc}])$, and Jenssen's inequality building upon the fact that $\bar{f}$ is concave and f_m are the values of $\bar{f}$ at $m \in \mathcal{H}$.

Replacing (20) in (19) and using the bound (23) we obtain

$$\begin{aligned}&\mathbb{E}[\sum_{t=0}^{T-1} g(x_t, u_t)] \leq \qquad (24)\\ &+\mathbb{E}[\bar{N}(t)]\big((\text{tr}(PW) + \text{tr}(Z\Theta))\mathbb{E}[s_{k+1} - s_k|\mathcal{I}_{s_k}^{sc}] + \alpha\bar{f}(\bar{\tau}_\pi)\big)\\ &\quad +\bar{\epsilon} + \mathbb{E}[x_1^\mathsf{T} P x_1|\mathcal{I}_0^{sc}] - \mathbb{E}[x_{s_{\bar{N}(T)}+1}^\mathsf{T} P x_{s_{\bar{N}(T)}+1}|\mathcal{I}_{s_{\bar{N}(T)}}^{sc}]\end{aligned}$$

where $\mathbb{E}[x_{s_{\bar{N}(T)}+1}^\mathsf{T} P x_{s_{\bar{N}(T)}+1}|\mathcal{I}_{s_{\bar{N}(T)}}^{sc}]$ was added and subtracted to the right hand side and we used the fact that

$$\begin{aligned}\mathbb{E}[\sum_{k=0}^{N(T)} &\mathbb{E}[x_{s_{k+1}+1}^\mathsf{T} P x_{s_{k+1}+1}|\mathcal{I}_{s_{k+1}}^{sc}] \qquad (25)\\ &- \mathbb{E}[x_{s_{k+1}+1}^\mathsf{T} P x_{s_{k+1}+1}|\mathcal{I}_{s_k}^{sc}]] = 0,\end{aligned}$$

established as follows. Let $\lambda_0 := 0$, and, for $r \in \mathbb{N}$,

$$\begin{aligned}\lambda_r := \sum_{k=0}^{r} &\mathbb{E}[x_{s_{k+1}+1}^\mathsf{T} P x_{s_{k+1}+1}|\mathcal{I}_{s_k}^{sc}]\\ &-\mathbb{E}[x_{s_{k+1}+1}^\mathsf{T} P x_{s_{k+1}+1}|\mathcal{I}_{s_{k+1}}^{sc}].\end{aligned}$$

This is a martingale with respect to $\{\mathcal{F}_r | r \in \mathbb{N}\}$, $N(T)$ is a bounded stopping time w.r.t. this filtration as mentioned above, and $\mathbb{E}[\lambda_r] = 0$, for every $r \in \mathbb{N}_0$. Then we can use the optional stopping theorem [19, Th.1, Sec.12.5] to conclude that $\mathbb{E}[\lambda_{N(T)}] = 0$, which is equivalent to (25). Dividing both sides of (24) by T and taking the $\limsup_{T\to\infty}$, we conclude the desired result, i.e., (17) if we take into account the following facts: (i) $\mathbb{E}[\tau_{k,\alpha}(\mathcal{I}_{s_k+\ell})|\mathcal{I}_{s_k}^{sc}]$ is constant (does not depend neither on $\mathcal{I}_{s_k}^{sc}$ nor on k, see the arguments below (16));(ii) by the same token, $\{\tau_{k,\alpha} = s_{k+1} - s_k | k \in \mathbb{N}_0\}$ is a set of independent and identically distributed random variables, and from the key renewal theorem [22], $\lim_{T\to\infty} \frac{\mathbb{E}[\bar{N}(T)]}{T} = \frac{1}{\mathbb{E}[\tau_{k,\alpha}]} = \frac{1}{L(\alpha)}$ (iii) $\lim_{T\to\infty} \frac{T - s_{\bar{N}(T)}}{T} = 0$; (iv) and the facts that $\lim_{T\to\infty} \frac{\bar{\epsilon}}{T} \to 0$, $\lim_{T\to\infty} \frac{\mathbb{E}[x_{s_{\bar{N}(T)+1}}^\mathsf{T} P x_{s_{\bar{N}(T)+1}}|\mathcal{I}_{s_{\bar{N}(T)}}^{sc}]}{T} = 0$ which follows from mean square stability of the closed-loop system, i.e., if (4), (6), (7), and (15) is applied to (1), then $\sup\{\mathbb{E}[x_t^\mathsf{T} x_t] | t \in \mathbb{N}_0\} < c$ for a given $c \in \mathbb{R}$. Mean square stability can be proved by observing that, from (20),

$$\mathbb{E}[x_{s_{k+1}+1}^\mathsf{T} P x_{s_{k+1}+1}|\mathcal{I}_{s_k}^{sc}] - \mathbb{E}[x_{s_k+1}^\mathsf{T} P x_{s_k+1}|\mathcal{I}_{s_k}^{sc}] = -h(\mathcal{I}_{s_k}^{sc}) + c$$

with non-negative c and non-negative function $h(\mathcal{I}_{s_k}^{sc})$ and by following similar steps to the ones in [8], [11], [23].

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