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A linearizing transformation for the Korteweg–de Vries equation; generalizations to higher-dimensional nonlinear partial differential equations

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It is shown that the Korteweg–de Vries (KdV) equation can be transformed into an ordinary linear partial differential equation in the wave number domain. Explicit solutions of the KdV equation can be obtained by subsequently solving this linear differential equation and by applying a cascade of (nonlinear) transformations to the solution of the linear differential equation. It is also shown that similar concepts apply to the nonlinear Schrödinger equation. The role of symmetry is discussed. Finally, the procedure which is followed in the one-dimensional cases is successfully applied to find special solutions of higher-dimensional nonlinear partial differential equations. © *1998 American Institute of Physics.*
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I. INTRODUCTION

Inverse scattering transformations (IST) form a powerful tool to solve certain classes of nonlinear partial differential equations (NPDEs). However, the success of the applicability of the IST is, modulo a few exceptions, limited to one-dimensional NPDEs only. This important limitation is caused by the fact that the IST uses the inverse problem of the Schrödinger equation to generate the solutions of the differential equations which have to be solved. The inverse problem of the Schrödinger equation is a well-studied problem in one dimension. Higher-dimensional NPDEs are rarely solved using inverse scattering techniques. There are a few reasons for the restricted applicability of inverse scattering methods in dimensions higher than one. The first reason is that higher-dimensional inverse scattering algorithms, like, for example, the Newton–Marchenko method (the inverse scattering problem of the Schrödinger equation in three-dimensions), are so complicated that is nearly impossible to apply these methods to real data. An alternative to the three-dimensional inverse scattering problem is given by the so-called $\bar{\partial}$ approach, which is successfully generalized to N dimensions (for an overview of the applications we refer to the book by Ablowitz and Clarkson[1]). But in applying the $\bar{\partial}$ approach, we readily face a second important restriction to the application of higher-dimensional inverse scattering methods. This restriction deals with the fact that for higher-dimensional inverse scattering methods the existence of the obtained solutions is difficult to prove. In one-dimensional cases this problem does not occur, since both the scattering data and the potential function depend on one single coordinate. In the three-dimensional case, where the scattering data depend on a three-component wave vector measured at a unit sphere, the five-dimensional data are mapped onto a three-dimensional potential function. As a result of this, in the three-dimensional inverse scattering problem two variables are redundant. The redundancy problem puts strong constraints on the classes of potential functions to be reconstructed and introduces additional complications in the application of inverse scattering methods for solving higher-dimensional NPDEs.

Another interesting and powerful approach which was successfully applied to other classes of NPDEs was developed by Calogero and Eckhaus.[2,3] It was shown by these authors that large classes of NPDEs can be transformed into linear partial differential equations by applying a cascade of limiting procedures involving rescaling techniques and asymptotic expansions. From now on we take over the Calogero terminology by denoting NPDEs which can be linearized using

[a)]Electronic mail: H.J.S.Dorren@ele.tue.nl limiting procedures ''*C*-integrable NPDEs,'' whereas those NPDEs which can be solved using inverse scattering techniques are called ''*S* integrable.'' Calogero[2] suggested that because the limiting procedures mentioned above all preserve the integrability, perhaps one universal equation follows by limiting procedures form large classes (all?) NPDEs. Keeping this idea in mind, we can conclude that it is not clear how *S*-integrable NPDEs fit in this concept, since these equations can be transformed into a linear integral equation. There is another important point with respect to *C* integrability which should be mentioned. The concept of *C*-integrability is easily generalized to higher dimensions.[4] An implication of this generalization is that if it is possible to fit in the concept of *S* integrability into the concept of *C*-integrability, an effective method is obtained to find solutions of higher-dimensional NPDEs.

Without aiming to be general, it is shown in this paper that in one dimension *S*-integrable NPDEs are indeed *C*-integrable. We show how to find a cascade of transformations that transform *S*-integrable NPDEs into an ordinary linear differential equation. Moreover, it is shown that in one dimension, there exists a clear relationship between solutions of the linearized equation and the inverse scattering transformation. From the results obtained in this paper, it must be concluded that in one dimension *S* integrability is a special case of *C*-integrability. Moreover, it is shown that by generalizing the ordinary linear differential equation, large classes of other integrable NPDEs can be obtained. All the obtained NPDEs contain an amount of symmetry. This symmetry is also present in its linearization. The most simple example, the linearization of the KdV equation, is only invariant under Galileian transformations. The linearization of the nonlinear Schrödinger equation is also invariant under the SU(2) generators.

This paper has the following structure. In Sec. II we derive a linearization scheme for the KdV equation. We give explicitly the linearized partial differential equation and the cascade of transformation that leads to the KdV-equation. In Sec. III, we solve the linearized equations and derive explicit solutions of the KdV equation using the transformations discovered in Sec. II. As an explicit example, the soliton solutions are constructed. In Sec. IV, the relation of the linearization method and the IST is highlighted. It is shown that the IST can be regarded as a special case of the linearizing procedure described in Sec. II. In Sec. V, the relation between more general one-dimensional differential equations and a generalization of the linearization procedure of Sec. II is discussed. As an explicit example, the nonlinear Schrödinger equation is investigated. In Sec. VI the concepts of Sec. II are generalized to more dimensions. As an example the three-dimensional equivalent of the KdV equation is investigated. We conclude this paper with a discussion.

II. A LINEARIZING TRANSFORMATION FOR THE KORTEWEG-DE VRIES EQUATION

As a starting point we consider the KdV equation:

$$\begin{aligned} u_{xxx}+u_t&=6u_xu, \\ u(x,t=0)&=u_0(x). \end{aligned} \tag{1}$$

It is well known that the KdV equation can be transformed into a linear integral equation.[5,6] We can ask ourselves the question whether it is also possible to transform the KdV equation directly into a linear differential equation. Without loss of generality, we can decompose a solution $u(x,t)$ of Eq. (1) in an infinite series of functions:

$$u(x,t)=\sum_{n=1}^{\infty} f^{(n)}(x,t). \tag{2}$$

If it is possible to solve all the functions $f^{(n)}(x,t)$ which determine the solution $u(x,t)$ in Eq. (1), we have solved the KdV equation. In this section it is shown that all the functions $f^{(n)}(x,t)$ can be obtained by applying nonlinear transformations to the solution of a linear differential equation. In order to find this differential equation, we substitute Eq. (2) into Eq. (1). We then obtain the following result:

$$\sum_{n=1}^{\infty} f_{xxx}^{(n)}(x,t)+f_t^{(n)}(x,t)=6\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_x^{(m)}(x,t)f^{(n)}(x,t),$$
$$\sum_{n=1}^{\infty} f^{(n)}(x,t=0)=\sum_{n=1}^{\infty} f_0^{(n)}(x). \tag{3}$$

Before we continue, we first have to explain the main concept used throughout this paper. The left-hand side of Eq. (3) describes a linear partial differential operator. The difficulties in solving the KdV equation, however, are related to the nonlinear nature of the right-hand side of Eq. (1). The presence of the nonlinear term makes that the standard techniques of substitution of special (exponential) solutions breaks down. We can overcome this problem by performing a perturbation theory. Instead of substituting a regular perturbation series $u(x,t)=\Sigma_{n=1}^{\infty}\epsilon^{(n)}f^{(n)}(x,t)$ into Eq. (1), and equalizing the different orders of ϵ, we treat in this paper the order n in Eq. (2) as the perturbation parameter. We can then find solutions of the differential equation by equalizing the sum of the perturbation parameter on both sides of Eq. (3). To illustrate this idea we take as an example the following solution:

$$u(x,t)=\sum_{n=1}^{\infty} A_n e^{in(kx-\omega t)}. \tag{4}$$

If we substitute, for example, the $n=2$ term of Eq. (4) onto the left-hand side of Eq. (3), we find that this contribution can be compared to the $n=1$ contribution on the right-hand side. If we substitute the full perturbation series, Eq. (4) into Eq. (1), is clear that there are always two equal orders on the right-hand side and the left-hand side which can be compared.

To implement this simple idea in a more general fashion (which is not valid for exponential solutions only) we proceed by equalizing the different orders of $f^{(n)}(x,t)$ in Eq. (3). If we collect the terms $f^{(n)}(x,t)$ of equal order (the sum of the upper indices of the nonlinear terms equals the upper index on the left-hand side), we obtain the following equations:

$$f_{xxx}^{(1)}(x,t)+f_t^{(1)}(x,t)=0,$$
$$f_{xxx}^{(2)}(x,t)+f_t^{(2)}(x,t)=6f_x^{(1)}(x,t)f^{(1)}(x,t),$$
$$f_{xxx}^{(3)}(x,t)+f_t^{(3)}(x,t)=6f_x^{(1)}(x,t)f^{(2)}(x,t)+6f_x^{(2)}(x,t)f^{(1)}(x,t), \tag{5}$$
$$f_{xxx}^{(4)}(x,t)+f_t^{(4)}(x,t)=6f_x^{(1)}(x,t)f^{(3)}(x,t)+6f_x^{(2)}(x,t)f^{(2)}(x,t)+6f_x^{(3)}(x,t)f^{(1)}(x,t),$$
$$\vdots$$

It is clear that if all the orders on the left-hand side of Eq. (5) and the right-hand side of Eq. (5) are summed up, Eq. (3) is retained. To deal with the nonlinearity in Eq. (5), it is convenient to introduce the following ansatz:

$$6f_x^{(n)}(x,t)f^{(m)}(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,t|x',t')f^{(n+m)}(x',t')dx'dt'. \tag{6}$$

The ansatz given in Eq. (6) introduces a constraint on the space of all the possible functions $f^{(n)}(x,t)$. In the following section it will be shown that the solutions we obtain using the linearization method are more general than the solutions obtained using the IST. If we can construct a satisfactory kernel $G(x,t|x',t')$, we can solve all the equations in Eq. (6) simultaneously. If we substitute Eq. (6) into Eq. (5), we find that Eq. (5) transforms into

$$
\begin{aligned}
&f^{(1)}_{xxx}(x,t)+f^{(1)}_t(x,t)=0;\\
&f^{(2)}_{xxx}(x,t)+f^{(2)}_t(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,t|x',t')f^{(2)}(x',t')dx'dt';\\
&f^{(3)}_{xxx}(x,t)+f^{(3)}_t(x,t)=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,t|x',t')f^{(3)}(x',t')dx'dt';\\
&\vdots\\
&f^{(n)}_{xxx}(x,t)+f^{(n)}_t(x,t)=(n-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,t|x',t')f^{(n)}(x',t')dx'dt';\\
&\vdots
\end{aligned}
\tag{7}
$$

From this result we can conclude that we have transformed the KdV equation (1), by using the ansatz (6) and doing some bookkeeping, into an infinite series of equations which all have a similar structure. In order to solve the integration kernel $G(x,t|x',t')$, it is convenient to perform the remaining part of the analysis in the Fourier domain. The Fourier transform of $f^{(n)}(x,t)$ is given by

$$f^{(n)}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}^{(n)}(k)e^{ikz}dk,\quad z=x-\frac{\omega(k)}{k}\,t. \tag{8}$$

We first solve $f^{(1)}(x,t)$ because this is the solution of a linear partial differential equation. If Eq. (8) is applied on both sides of $f^{(1)}(x,t)$ in Eq. (7), we obtain the following relationship:

$$f^{(1)}_{xxx}(x,t)+f^{(1)}_t(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}i[\omega(k)-k^3]\tilde{f}^{(1)}(k)e^{ikz}dk=0. \tag{9}$$

It follows from Eq. (9) that nontrivial solutions $f^{(1)}(x,t)$ exist if the following dispersion relation is satisfied:

$$\omega(k)=-k^3. \tag{10}$$

It is remarkable to conclude that the function $f^{(1)}(x,t)$ constraints the dispersion relation of all the remaining equations given in Eq. (7). As a result of this, the Fourier transform of the left-hand side of all the equations is given in Eq. (7) can be computed simultaneously. We continue by computing the Fourier transform of the right-hand side of Eq. (7). To carry out this calculation, it is useful to assume that the integration kernel $G(x,t|x',t')$ is invariant under translations in time and space:

$$G(x,t|x',t')=G(x-x'|t-t'). \tag{11}$$

Our aim is to compute the Fourier transform $\mathscr{F}(\cdot)$ of the right-hand side of all the equations appearing in Eq. (7):

$$\mathscr{F}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x-x'|t-t')f^{(n)}(x',t')dx'dt'\right). \tag{12}$$

To perform this computation, we first define the Fourier transform $\tilde{G}(k)$ of the integration kernel $G(x-x'|t-t')$:

$$G(x-x'|t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)e^{ik[z-z']}dk. \tag{13}$$

The variable z in Eq. (13) is specified by Eq. (8). If Eq. (8) and Eq. (13) are substituted into Eq. (12) and if the integrations over z' and k' are carried out, we obtain the following result:

$$\mathscr{F}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-x'|t-t')f^{(n)}(x',t')dx'dt' \right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k)\tilde{f}^{(n)}(k)e^{ikz}dk. \tag{14}$$

Once the Fourier transform of both the left-hand side and the right-hand side of Eq. (7) are computed, we find by combining Eq. (7), Eq. (9), and Eq. (14) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} i[\omega(k)-k^3]\tilde{f}^{(n)}(k)e^{ikz}dk = (n-1)\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)\tilde{f}^{(n)}(k)e^{ikz}dk. \tag{15}$$

If we define

$$g^{(n)}(z,k) = \tilde{f}^{(n)}(k)e^{ikz}, \tag{16}$$

we find that a solution of $g^{(n)}(z,k)$ is given by the following differential equation:

$$\frac{d}{dz}g^{(n)}(z,k) + ikq^{(n)}(z,k) = (n-1)S(k)g^{(n)}(z,k), \quad S(k) = -\frac{\tilde{G}(k)}{k^2}. \tag{17}$$

From this final result, it follows that the cascade of equations given by Eq. (7) in the Fourier domain is equivalent with

$$\begin{aligned}
&\frac{d}{dz}g^{(1)}(z,k) + ikg^{(1)}(z,k) = 0;\\
&\frac{d}{dz}g^{(2)}(z,k) + ikg^{(2)}(z,k) = S(k)g^{(2)}(z,k);\\
&\frac{d}{dz}g^{(3)}(z,k) + ikg^{(3)}(z,k) = 2S(k)g^{(3)}(z,k);\\
&\qquad\vdots\\
&\frac{d}{dz}g^{(n)}(z,k) + ikg^{(n)}(z,k) = (n-1)S(k)g^{(n)}(z,k);\\
&\qquad\vdots
\end{aligned} \tag{18}$$

From this simple derivation we can conclude that in the Fourier domain the KdV equation can be linearized into an ordinary linear differential equation. If this differential equation is solved we can retain a solution of the KdV equation by using Eq. (6). Explicit solutions of the KdV equation can be computed by solving the following problem:

$$\begin{aligned}
&\frac{d}{dz}g^{(n)}(z,k) + ikg^{(n)}(z,k) = (n-1)S(k)g^{(n)}(z,k),\\
&6f_x^{(n)}(x,t)f^{(m)}(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x-x'|t-t')f^{(n+m)}(x',t')dx'dt'.
\end{aligned} \tag{19}$$

With this result we have shown that the KdV equation can be transformed into an ordinary linear differential equation. The second equation in Eq. (19) defines a constraint on the solutions $f^{(n)}(x,t)$.

In the following section the equations given above will be solved and solutions of the KdV equation will be computed. The solutions which are obtained by using the linearization method described above are more general than the solutions obtained using an inverse scattering approach, because the method also applies to nonexponential functions $f^{(n)}(x,t)$. It follows from Eq. (3) that the method introduced in this paper can also be applied if the functions $f^{(n)}(x,t)$ are polynomials. In the following sections it will be shown explicitly that solutions obtained by the IST form a special subclass of the solutions which can be obtained by the linearization method. An important conclusion that already can be drawn from the results of this section is that the linearization method can easily be generalized to higher dimensions. This will be demonstrated later in this paper. It can be concluded from the approach followed in this section that there are two important factors which determine the transformations from the linear ordinary differential equation to the nonlinear partial differential equation. The first factor is the dispersion relation. The dispersion relation defines the linear part of the nonlinear partial differential equation. The second factor is the ansatz given in Eq. (6) which deals with the nonlinearity. By modifying the dispersion relation and the ansatz (6), it is possible to construct other NPDEs which can be linearized into a similar form as presented in Eq. (19). It is an interesting puzzle to find out whether there are other equations (not in the KdV hierarchy) which can be linearized using a similar approach as presented in this paper. In general, one can expect that the method presented in this paper can be used to find solutions of NPDEs for which the nonlinear part consists of a power series. For investigating the integrability of these equations, it is not necessary to find a suitable Lax pair. Explicit examples are given in following sections in which it is shown that generalizations of Eq. (19) can be transformed into other S-integrable NPDEs. For example, it is shown that the nonlinear Schrödinger equation can be linearized into a similar form as given in Eq. (19). Apparently symmetry plays an important role in the linearization process of NPDEs. The linearization of the KdV equation is invariant under Galileian transformations. This invariance is already known for a long time[7] but can be used here to consider the concept of integrability from a different point of view. The Galileian invariance can be regarded as the most simple symmetry in the hierarchy of equations which will follow. This Galileian invariance also explains the presence of solitons. If we introduce more symmetry in the linearized differential equation, we obtain that the linear differential equation can be transformed into more complicated NPDEs.

III. SOLUTIONS OF THE KDV EQUATION

As shown in the previous section, the KdV equation can be linearized into the ordinary linear differential equation (17). In this section we will derive explicit solutions of this equation:

$$\frac{d}{dz} g^{(n)}(z,k)+ikg^{(n)}(z,k)=(n-1)S(k)g^{(n)}(z,k). \tag{20}$$

Equation (20) is an ordinary linear first-order differential equation. The general solution of Eq. (20) is given by

$$g^{(n)}(z,k)=C_1e^{M(k,n)z};\quad M(k,n)=(n-1)S(k)-ik. \tag{21}$$

To verify the validity of the ansatz (6), it is convenient to define $S(k)$ in the complex plane. This enables us to solve Eq. (20) using the Green's function technique. If we formulate solutions of Eq. (20) using a Green's function technique, we can find a relationship between the linearization method described in this paper and an inverse scattering method. In order to find a suitable Green's function we first solve the following problem:

$$\frac{d}{d\hat{z}} \mathcal{G}(\hat{z},k)+ik\mathcal{G}(\hat{z},k)=\delta(\hat{z}),\quad \hat{z}=z-z', \tag{22}$$

Eq. (22) can be solved easily by applying the following inverse Fourier transform:

$$\tilde{\mathcal{H}}(k',k)=\int_{-\infty}^{\infty}\mathcal{G}(\hat{z},k)e^{-ik'\hat{z}}d\hat{z}. \tag{23}$$

If the Fourier transform (23) is substituted into Eq. (22) we obtain the following result:

$$ik'\tilde{\mathscr{H}}(k',k)+ik\tilde{\mathscr{H}}(k',k)=1. \tag{24}$$

From Eq. (24), it follows that the solution of $\tilde{\mathscr{H}}(k',k)$ is given by

$$\tilde{\mathscr{H}}(k',k)=\frac{1}{i(k+k')}. \tag{25}$$

By applying the inverse transformation of Eq. (23) we can solve the Green's function $\mathscr{G}(\hat{z},k)$:

$$\mathscr{G}(\hat{z},k)=\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ik'\hat{z}}}{k+k'+i\epsilon}\,dk'. \tag{26}$$

Once the Green's function is solved, we can formulate the general solution of Eq. (20) in the form of an integral equation:

$$g^{(n)}(z,k)=\lim_{\epsilon\to 0}\frac{(n-1)}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{ik'[z-z']}}{k+k'+i\epsilon}\,S(k)g^{(n)}(z',k)dz'dk', \quad n\neq 1. \tag{27}$$

From the function $g^{(n)}(z,k)$ given in Eq. (27), we can solve, using Eq. (16) and the Fourier transformation given in Eq. (8), the function $f^{(n)}(x,t)$:

$$f^{(n)}=(x,t)=\lim_{\epsilon\to 0}\frac{(n-1)}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(k-k')}{k+k'+i\epsilon}\,S(k)g^{(n)}(z,k')dkdk', \quad n\neq 1. \tag{28}$$

Before formulating solutions of the KdV equation, we first have to assure that a kernel $G(x-x'|t-t')$ indeed exists. If we compute the left-hand-side of Eq. (6) using the representation presented in Eq. (28) of the solutions $f^{(n)}(x,t)$, we obtain

$$\begin{aligned}6f_x^{(n)}(x,t)f^{(m)}(x,t)=&-\lim_{\epsilon\to 0}\frac{(n-1)(m-1)}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\\ &\times\frac{6ik'S(k)S(k'')}{k+k'+i\epsilon}\,\frac{\delta(k-k')\delta(k''-k''')}{k''+k'''+i\epsilon}\,g^{(n)}(z,k')\\ &\times g^{(m)}(z,k''')dkdk'dk''dk''', \quad n,m\neq 1.\end{aligned} \tag{29}$$

The kernel $G(x-x'|t-t')$ can be represented in the complex plane according to the following Cauchy formula:

$$G(x-x'|t-t')=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k'^2e^{ik'[z-z']}}{k+k'+i\epsilon}\,S(k')dk'dk. \tag{30}$$

In Eq. (30), it is used that $S(k)=k^2\tilde{G}(k)$. The expression obtained in Eq. (29) is, according to Eq. (6), equal to:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x-x'|t-t')f^{(n+m)}(x',t')dx'dt'\\ &=-\lim_{\epsilon\to 0}\frac{[n+m-1]}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k'^2S(k')S(k'')}{k+k'+i\epsilon}\,\frac{\delta(k''-k''')\delta(k'''-k')}{k''+k'''+i\epsilon}\\ &\quad\times g^{(n+m)}(z,k''')dkdk'dk''dk'''.\end{aligned} \tag{31}$$

If follows from comparing Eq. (29) to Eq. (31), that the ansatz is satisfied if the following condition is fulfilled:

$$\frac{3i(n-1)(m-1)}{\pi[n+m-1]} S(k)\,\delta(k-k')g^{(n)}(z,k')g^{(m)}(z,k''')=k'S(k')\,\delta(k'''-k')g^{(n+m)}(z,k'''). \tag{32}$$

If Eq. (32) is satisfied, we can finally formulate solutions of the KdV equation:

$$u(x,t)=\sum_{n=1}^{\infty} f^{(n)}(x,t)$$
$$=Ae^{ikz}+\lim_{\epsilon\to 0}\sum_{n=2}^{\infty}\frac{(n-1)}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(k-k')}{k+k'+i\epsilon}S(k')g^{(n)}(z,k')dk'dk. \tag{33}$$

We can find explicit solutions of Eq. (33) by using the fact that Eq. (32) defines a recursion relationship for the functions $g^{(n)}(z,k)$. This recursion relation can be made explicit by verifying Eq. (6) if one of the functions $f^{(n)}(x,t)$ is equal to $f^{(1)}(x,t)$. If we choose for $f^{(1)}(x,t)$ the following representation:

$$f^{(1)}(x,t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ik'e^{ik'z}}{k+k'+i\epsilon}S(k')dk'dk, \tag{34}$$

the inverse scattering transformation can be retained. From comparing Eq. (28) to Eq. (34), it follows that by giving $f^{(1)}(x,t)$ the form (34), we have implicitly chosen that $\tilde{f}^{(1)}(k')=ik'$. If we compute the left-hand side of Eq. (6), we find that

$$6f_x^{(1)}(x,t)f^{(n-1)}(x,t)=-\lim_{\epsilon\to 0}\frac{(n-2)}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{6ik'S(k)(k'')}{k+k'+i\epsilon}\,\frac{\delta(k-k')\,\delta(k''-k''')}{k''+k'''+i\epsilon}$$
$$\times g^{(1)}(z,k')g^{(m)}(z,k''')dkdk'dk''dk'''. \tag{35}$$

According to the ansatz, Eq. (6), this has to be equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x-x'|t-t')f^{(n)}(x',t')dx'dt'$$
$$=-\lim_{\epsilon\to 0}\frac{[n-1]}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k'^2S(k')S(k'')}{k+k'+i\epsilon}\,\frac{\delta(k''-k''')\,\delta(k'''-k')}{k''+k'''+i\epsilon}$$
$$\times g^{(n)}(z,k''')dkdk'dk''dk'''. \tag{36}$$

By comparing Eq. (35) to Eq. (36) we find that equality is obtained if the following relationship is satisfied:

$$\frac{3i(n-2)}{\pi(n-1)}S(k)\,\delta(k-k')g^{(1)}(z,k')g^{(n-1)}(z,k''')=k'S(k')\,\delta(k'''-k')g^{(n)}(z,k'''). \tag{37}$$

Equation (37) is a recursion relation which makes it possible to compute $g^{(n)}(z,k)$ from $g^{(n-1)}(z,k)$. In order to relate $g^{(2)}(z,k)$ to $g^{(1)}(z,k)$, we have to check explicitly the ansatz (6) for the $6f_x^{(1)}(x,t)f^{(1)}(x,t)$ case. This leads to the following result:

$$\frac{3i}{2\pi}S(k)\,\delta(k-k')g^{(1)}(z,k')g^{(1)}(z,k''')=k'S(K')\,\delta(k''-k''')g^{(2)}(z,k'''). \tag{38}$$

The result we have obtained indicates that we can solve the KdV equation by using the recursion relationship defined by Eq. (37). If we continue the process of iteration, we obtain the following solutions $f^{(n)}(x,t)$ which determine the solution of the KdV equation:

$$f^{(1)}(x,t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ik'e^{ik'z}}{k+k'+i\epsilon}\hat{S}(k')dk'dk, \tag{39}$$

$$f^{(2)}(x,t)=\lim_{\epsilon\to 0}\frac{1}{2\pi(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hat{S}(k')e^{ik'z}}{k+k'+i\epsilon}\frac{\hat{S}(k'')e^{ik''z}}{k'+k''+i\epsilon}i(k'+k'')dk''dk'dk, \tag{40}$$

$$\vdots$$

where we have used that $\hat{S}(k)=2/3iS(k)$. If the infinite series of solutions of which the first two are given by Eq. (39) and Eq. (40) are substituted into Eq. (33), we obtain a solution of the KdV equation of which the first two terms are given by

$$\begin{aligned}u(x,t)=&\lim_{\epsilon\to 0}\frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ik'e^{ik'z}}{k+k'+i\epsilon}\hat{S}(k')dk'dk\\&+\lim_{\epsilon\to 0}\frac{1}{2\pi(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hat{S}(k')e^{ik'z}}{k+k'+i\epsilon}\frac{\hat{S}(k'')e^{ik''z}}{k'+k''+i\epsilon}i(k'+k'')dk''dk'dk+\cdots .\end{aligned} \tag{41}$$

In the following section it is shown that Eq. (41) describes a generalization of the inverse scattering solution of the KdV equation. From Eq. (41), it is easy to derive the soliton solutions of the KdV equation. This can be done by choosing the following representation of $S(k)$ (see Ref. 8):

$$\hat{S}(k)=\frac{id}{k+2i\beta},\quad \beta>0,\quad d\in\mathbb{R}. \tag{42}$$

We find that the solution (41) reduces to

$$u(x,t)=4d\beta e^{-2(\beta x-4\beta^3 t)}+16d^2\,e^{-4(\beta x-4\beta^3 t)}+24\frac{d^3}{\beta}\,e^{-6(\beta x-4\beta^3 t)}+\cdots . \tag{43}$$

By carrying out the summation in Eq. (43), we can reformulate the solution (43) more compactly:

$$u(x,t)=\frac{8d\beta e^{-2(\beta x-4\beta^3 t)}}{\left(1+\frac{d}{\beta}\,e^{-2(\beta x-4\beta^3 t)}\right)^2}. \tag{44}$$

Hence, if we set

$$\beta=\frac{1}{2}\sqrt{c},\quad x_0=-\frac{1}{\sqrt{c}}\log\left(-\frac{d}{\beta}\right),\quad d<0, \tag{45}$$

we can simplify Eq. (44) a step further to

$$u(x,t)=-\frac{c}{2}\,\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{c}(x-ct+x_0)\right\}. \tag{46}$$

Equation (46) describes the well-known KdV soliton.

The results obtained in this section make it clear that the approach suggested in this paper can be used to generate solutions of the KdV equation. For instance, the KdV soliton can be retained as a special case of Eq. (33). In the following section, it is shown that the solutions obtained in this section are, in fact, the inverse scattering solutions. The two-soliton solutions can be obtained by choosing a function $\hat{S}(k)$ which has a double pole. It will be shown in the next section that the solution (41) is somewhat more general than the inverse scattering solution because the function $S(k)$ is less restricted than the physical scattering data as used in the IST.

IV. RELATION TO THE INVERSE SCATTERING TRANSFORMATION

To show that there is a close relationship between the results obtained by the linearization method and the well-known inverse scattering results, we discuss the latter in this section in more detail. The IST uses the inverse problem of the Schrödinger equation to generate solutions of the KdV equation. The inverse problem of the Schrödinger equation is given by the Marchenko equation:

$$K(x,y,t)+A(x+y,t)+\int_x^{\infty} K(x,z,t)A(y+z,t)dz=0. \tag{47}$$

The Marchenko equation relates a data function $A(x,t)$ to an integration kernel $K(x,y,t)$. In Eq. (47), the data function $A(x,t)$ is given by

$$A(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)e^{ik[x+8k^2t]}dt+\sum_{p=1}^{N}\rho_p e^{-k_p[x+4k_p^2t]}. \tag{48}$$

In Eq. (48), the function $R(k)$ describes the physical reflection coefficient. The function $R(k)$ corresponds to the continuous part of the spectrum of the Schrödinger equation. The numbers ρ_p represent the discrete part of the spectrum of the Schrödinger equation corresponding to the discrete eigenvalues k_p. The discrete part of the spectrum of the Schrödinger equation is given by the residues of the transmission coefficient $T(k,t)$ on the positive imaginary axis. The solution $u(x,t)$ of the KdV equation can be solved from the integration kernel $K(x,y,t)$ by using the following relationship:

$$u(x,t)=-2\frac{d}{dx}K(x,x,t). \tag{49}$$

The relation between the linearization method and the IST becomes more transparent if the analysis is performed in the wave number domain. A convenient way to do this is to relate the kernel $K(x,y,t)$ to a function $F(x,k,t)$ by using the following Fourier transform:[9]

$$K(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(y-x)}(F(x,k,t)-1). \tag{50}$$

It is well known that the function $F(x,k,t)$ is related to the Jost solutions of the Schrödinger equation.[9] If we substitute Eq. (50) into Eq. (47), we obtain the following relationship:

$$F(x,k,t)=1+\int_{-\infty}^{\infty} C(k,k',z)F(x,k',t)dk'. \tag{51}$$

For reasons of simplicity, we do not take the contribution of the transmission coefficient into account in the analysis which follows. It can be shown that the transmission coefficient can be included in the computations in a conceptually similar manner.[9] In the special case of a negligible transmission coefficient, we find that the kernel $C(k,k',t)$ in Eq. (51) is given by

$$C(k,k',z)=\frac{1}{2\pi i}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}; \quad z=x-8k^2t. \tag{52}$$

If Eq. (52) is substituted into Eq. (51), we find the following relationship:

$$F(x,k,t)-1=+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(k')F(x,k',t)e^{2ikz}}{k+k'+i\epsilon}dk'. \tag{53}$$

To find a relationship between the linearization technique and the IST, we expand Eq. (52) in a series:

$$F(k,x,t)=1+\int_{-\infty}^{\infty} C(k,k',z)dk' + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(k,k',z)C(k',k'',z)dk'dk'' + \cdots . \tag{54}$$

If we substitute Eq. (52) into Eq. (54), this result becomes equal to

$$F(k,x,t)-1=\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}\,dk' + \lim_{\epsilon\to 0}\frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}\,\frac{R(k'')e^{2ik''z}}{k'+k''+i\epsilon}\,dk'dk'' + \cdots . \tag{55}$$

By taking the Fourier transform Eq. (50) of both sides of Eq. (55) we can compute the integration kernel $K(x,x,t)$:

$$K(x,x,t)=\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}\,dkdk' + \lim_{\epsilon\to 0^+}\frac{2}{2\pi(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}\,\frac{R(k'')e^{2ik''z}}{k'+k''+i\epsilon}\,dkdk'dk'' + \cdots . \tag{56}$$

From this result we obtain by setting $x=y$ and using the relationship (49) that solutions of the KdV equation are given by the following relationship:

$$u(x,t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2ik'e^{2ik'z}}{k+k'+i\epsilon}R(k')dk'dk + \lim_{\epsilon\to 0}\frac{1}{2\pi(2\pi i)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R(k')e^{2ik'z}}{k+k'+i\epsilon}\,\frac{R(k'')e^{2ik''z}}{k'+k''+i\epsilon}\,2i(k'+k'')dk''dk'dk + \cdots . \tag{57}$$

Modulo some trivial rescaling, the analytical structure of Eq. (58) is equal to the result reflected in Eq. (41). We can therefore conclude that the functions $g^{(n)}(k,z)$ are related to the Jost solution of an inverse scattering problem. This is not a great surprise if we realize that the kernel $K(x,y,t)$ in the Marchenko equation is a Green's function by itself.[9] The linearization method constructs the Green's function directly. However, we have to realize that the results obtained in Sec. III are far more general than the IST. It turns out that Eq. (41) is only a special case on of the linearization method, depending on the initial conditions. Yet, it should be realized that although Eq. (58) and Eq. (41) have a similar structure; Eq. (41) is more general. This is related to the precise definition of the scattering data in Eq. (58). In general, the scattering data given by the reflection coefficient $R(k,t)$ and the transmission coefficient $T(k,t)$ have to satisfy a number of restrictions to assure a unitary S matrix and uniqueness of the corresponding inverse problem. Moreover, additional conditions have to be satisfied so that the Deift–Trubowits condition is not violated (an overview of all the conditions can be found in, for instance, the book by Chadan and Sabatier[9]). These additional constraints do not have to be satisfied by the results obtained by the linearization method. We can therefore consider the results reflected in Eq. (41) to be a generalization of the IST.

V. THE NONLINEAR SCHRÖDINGER EQUATION

In Sec. II it is shown that the KdV equation can be be transformed into a linear partial differential equation. As concluded in Sec. II, this linear differential equation depends on the balances of the dispersion of the nonlinear problem to a function $S(k)$ representing the nonlinearity. We can ask ourselves the question whether there are more NPDEs which can be linearized into an equation having a similar structure as Eq. (20). In this section it is shown that this is indeed the case if we introduce new symmetries in the linearized equation. The most simple one-dimensional generalization of Eq. (20) is given by

$$c_1\left[\frac{d}{dz}\,g_j^{(n)}(z,k)\right]+ikc_2g_j^{(n)}(z,k)=c_3Q(n)S(k)g_j^{(n)}(z,k). \tag{58}$$

In Eq. (58), we have replaced the function $g^{(n)}(z,k)$ of Eq. (20) by an N-dimensional vector function $g_j^{(n)}(z,k)$. Furthermore, in Eq. (58) the $N\times N$ matrices c_i are introduced. The matrices c_i are assumed to have constant entries. Similarly, as in Sec. II, the variable z takes its value in a dispersion relation so that Eq. (58) is invariant under Galileian transformations. As argued in Sec. II, we can expect that Eq. (58) can be transformed into a wide variety of nonlinear equations which all have solitonic solutions. An interesting special case can be considered if we choose $N=2$ and if the matrices c_i take the following representations:

$$c_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad c_2=c_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{59}$$

We will show in this section that if we choose the dispersion relation to be equal to

$$\omega(k)=\pm k^2, \tag{60}$$

and if the function $Q(n)$ is equal to

$$Q(n)=\tfrac{1}{2}(n-1)(n-2), \tag{61}$$

the linear equation (58) can be transformed into the nonlinear Schrödinger equation. To find a similar cascade of transformations and rescaling which transforms the nonlinear Schrödinger equation into Eq. (58), we take as a starting point the nonlinear Schrödinger equation:

$$iu_t+u_{xx}\pm 2|u|^2u=0. \tag{62}$$

It is useful to transform the nonlinear Schrödinger equation into the following set of coupled differential equations:

$$iu_t+u_{xx}-2u^2v=0,\quad iv_t-v_{xx}+2v^2u=0. \tag{63}$$

The two coupled partial differential equations presented in Eq. (63) are equivalent to the nonlinear Schrödinger equation if we set $v=\mp u^*$. In order to simplify the bookkeeping, we formulate the set of coupled differential equations (63) as the following matrix equation:

$$\mathbf{I}\mathbf{w}_t-i\sigma_3\mathbf{w}_{xx}=-2i\mathbf{A}\mathbf{w}. \tag{64}$$

In Eq. (64), $\mathbf{I}$ represents the unity matrix and the vector $\mathbf{w}$ is given by $\mathbf{w}=(u,v)^T$. Furthermore, the Pauli spin matrices σ_i are introduced:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{65}$$

Finally, all the nonlinearities are contained in the matrix $\mathbf{A}$ which is defined by the following relationship:

$$\begin{pmatrix}uv & 0\\ 0 & -uv\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\mathbf{A}\mathbf{w}. \tag{66}$$

We carry on in a similar manner as in Sec. II by assuming that the functions $u(x,t)$ and $v(x,t)$ can be decomposed in the following infinite perturbation series:

$$\mathbf{w}(x,t)=\sum_{n=1}^{\infty}\mathbf{f}^{(n)}(x,t). \tag{67}$$

Every component $f_i^{(n)}(x,t)$ of $\mathbf{f}^{(n)}(x,t)$ expresses the expansion of the components u and v of $\mathbf{w}(x,t)$. Similarly, as in Sec. II, the nonlinear Schrödinger equation is solved if solutions of all the functions $f_i^{(n)}(x,t)$ are constructed. If Eq. (67) is substituted into Eq. (64), we obtain the following result:

$$\sum_{n=1}^{\infty}(\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(n)}(x,t)=-2i\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}s^{(klm)}(x,t), \tag{68}$$

where the vector function $\mathbf{s}^{(klm)}(x,t)$ is given by

$$\mathbf{s}^{(klm)}(x,t)=\begin{pmatrix} f^{(k)}(x,t)f_{(l)}(x,t)g^{(m)}(x,t)\\ -g^{(k)}(x,t)g^{(l)}(x,t)f^{(m)}(x,t)\end{pmatrix}. \tag{69}$$

We proceed, as in Sec. II, by collecting all the terms of which the sum of the orders on the left-hand side equals the sum of the orders on the right-hand side. In order to have a better view on the structure of the infinite number of equations which are then obtained, we write out the first four explicitly:

$$\begin{gathered}(\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(1)}(x,t)=0,\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(2)}(x,t)=0,\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(3)}(x,t)=\mathbf{s}^{(111)}(x,t),\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(4)}(x,t)=\mathbf{s}^{(211)}(x,t)+\mathbf{s}^{(121)}(x,t)+\mathbf{s}^{(112)}(x,t),\end{gathered} \tag{70}$$

We can linearize the set of equations given in Eq. (70) in the Fourier domain by introducing kernel $G(x',t'|x,t)$:

$$-2i\mathbf{s}^{(klm)}(x,t)=\sigma_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,x'|t,t')\mathbf{f}^{(k+l+m)}(x',t')dx'dt'. \tag{71}$$

If Eq. (71) is substituted into Eq. (70), we can reformulate Eq. (70) in the following way:

$$\begin{gathered}(\mathbf{I}\partial^t-i\sigma_3\partial_{xx})\mathbf{f}^{(1)}(x,t)=0,\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(2)}(x,t)=0,\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(3)}(x,t)=\sigma_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,x'|t,t')\mathbf{f}^3(x',t')dx'dt',\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(4)}(x,t)=3\sigma_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,x'|t,t')\mathbf{f}^{(4)}(x',t')dx'dt',\\ \vdots\\ (\mathbf{I}\partial_t-i\sigma_3\partial_{xx})\mathbf{f}^{(n)}(x,t)=\sigma_3Q(n)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,x'|t,t')\mathbf{f}^{(n)}(x',t')dx'dt'.\end{gathered} \tag{72}$$

Similar to Sec. II, we proceed, continuing our analysis in the wave number domain, by searching for the following wave-package solutions:

$$\mathbf{f}^{(n)}(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{f}}^{(n)}(k)e^{ikz}dk,\quad z=x-\frac{\omega(k)}{k}\,t. \tag{73}$$

If Eq. (73) is substituted into Eq. (70) and if we solve $\tilde{\mathbf{f}}^{(1)}(k)$, we find that dispersion relation given by Eq. (60) has to be satisfied. If we define, furthermore, the Fourier transform of the kernel $G(x-x'|t-t')$ to be equal to

$$G(x-x'|t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k)e^{ik[z-z']}dk, \tag{74}$$

and if we define $S(k)=\tilde{G}(k)/k$, we obtain that the set of equations (72) in the Fourier domain are equal to

$$\begin{aligned}
&\mathbf{I}\,\frac{d}{dz}\,\tilde{\mathbf{g}}^{(1)}(z,k)+ik\sigma_3\tilde{\mathbf{g}}^{(1)}(z,k)=0,\\
&\mathbf{I}\,\frac{d}{dz}\,\tilde{\mathbf{g}}^{(2)}(z,k)+ik\sigma_3\tilde{\mathbf{g}}^{(2)}(z,k)=0,\\
&\mathbf{I}\,\frac{d}{dz}\,\tilde{\mathbf{g}}^{(3)}(z,k)+ik\sigma_3\tilde{\mathbf{g}}^{(3)}(z,k)=\sigma_3S(k)\tilde{\mathbf{g}}^{(3)}(z,k),\\
&\qquad\vdots
\end{aligned} \tag{75}$$

In Eq. (75), we have defined

$$\tilde{\mathbf{g}}^{(n)}(z,k)=\tilde{\mathbf{f}}^{(n)}(k)e^{ikz}\quad S(k)=\frac{G(k)}{k}. \tag{76}$$

Resuming, we can conclude that the nonlinear Schrödinger equation can be linearized into

$$\mathbf{I}\,\frac{d}{dz}\,\tilde{\mathbf{g}}^{(n)}(z,k)+ik\sigma_3\tilde{\mathbf{g}}^{(n)}(z,k)=Q(n)\sigma_3S(k)\tilde{\mathbf{g}}^{(n)}(z,k). \tag{77}$$

In this section it is shown that the nonlinear Schrödinger equation can be linearized into a generalization of Eq. (20). It can be shown that the ansatz (71) leads to conditions on the solutions of the nonlinear Schrödinger equation.[10] Explicit solutions of the nonlinear Schrödinger equation computed by using the linearization method are given in this reference. It has to be remarked that the linearized equation is invariant under transformations of the SU(2) generators σ_i. This reflects that besides the Galileian invariance (which takes its value as the dispersion relation) another symmetry is present in the linearization of the nonlinear Schrödinger equation.

VI. GENERALIZATION TO HIGHER DIMENSIONS

Inspired by the results obtained in the previous sections, we show that a similar procedure can be applied to find solutions of higher-dimensional NPDEs. We choose as a starting point the most simple three-dimensional generalization of Eq. (20):

$$\sum_{i=1}^{3}\partial_{z_i}g^{(n)}(\mathbf{z},\mathbf{k})+i\sum_{i=1}^{3}k_ig^{(n)}(\mathbf{z},\mathbf{k})=(n-1)S(\mathbf{k})g^{(n)}(\mathbf{z},\mathbf{k}). \tag{78}$$

In Eq. (78), the solution $g^{(n)}(\mathbf{z},\mathbf{k})$ of the linearized problem depends now on a three-dimensional vector $\mathbf{z}$ having components $(z_1,z_2,z_3)^T$ and a three-dimensional wave number $\mathbf{k}$ with elements $(k_1,k_2,k_3)^T$. We derive a three-dimensional generalization of the KdV equation by following the reverse track of Sec. II. We first define a function $\hat{G}(\mathbf{k})$ so that

$$S(\mathbf{k})=\frac{\hat{G}(\mathbf{k})}{\mathbf{k}\cdot\mathbf{k}}. \tag{79}$$

If Eq. (79) is substituted into Eq. (78), we obtain the following result:

$$\mathbf{k}\cdot\mathbf{k}\sum_{i=1}^{3}\{\partial_{z_i}g^{(n)}(\mathbf{z},\mathbf{k})+ik_i g^{(n)}(\mathbf{z},\mathbf{k})\}=(n-1)\hat{G}(\mathbf{k})g^{(n)}(\mathbf{z},\mathbf{k}). \tag{80}$$

We have to take into account that Eq. (80) has to be solved for all n in the range from unity to infinity. Following the recipe derived in Sec. II, we find, if we write out the lowest orders of n in Eq. (80) explicitly, that

$$\begin{aligned}
&\mathbf{k}\cdot\mathbf{k}\sum_{i=1}^{3}\{\partial_{z_i}g^{(1)}(\mathbf{z},\mathbf{k})+ik_i g^{(1)}(\mathbf{z},\mathbf{k})\}=0,\\
&\mathbf{k}\cdot\mathbf{k}\sum_{i=1}^{3}\{\partial_{z_i}g^{(2)}(\mathbf{z},\mathbf{k})+ik_i g^{(2)}(\mathbf{z},\mathbf{k})\}=\hat{G}(\mathbf{k})g^{(2)}(\mathbf{z},\mathbf{k}),\\
&\mathbf{k}\cdot\mathbf{k}\sum_{i=1}^{3}\{\partial_{z_i}g^{(3)}(\mathbf{z},\mathbf{k})+ik_i g^{(3)}(\mathbf{z},\mathbf{k})\}=2\hat{G}(\mathbf{k})g^{(3)}(\mathbf{z},\mathbf{k}),
\end{aligned} \tag{81}$$

We proceed by defining a function $f^{(n)}(\mathbf{x},t)$. The functions $g^{(n)}(\mathbf{z},\mathbf{k})$ in Eq. (81) are defined to be related to functions $f^{(n)}(\mathbf{x},t)$ by the following relationship:

$$f^{(n)}(\mathbf{x},t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\tilde{f}^{(n)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{z}}d^3k;\quad g^{(n)}(\mathbf{z},\mathbf{k})=\tilde{f}^{(n)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{z}};\quad z_i=x_i-\frac{\omega(\mathbf{k})}{k_i}\,t. \tag{82}$$

If we carry out the inverse Fourier transform of Eq. (82), we find that the set of equations given in Eq. (81) is transformed into the following set of equations:

$$\begin{aligned}
&\sum_{i=1}^{3}\partial_{x_i}^3 f^{(1)}(\mathbf{x},t)+\partial_t f^{(1)}(\mathbf{x},t)=0,\\
&\sum_{i=1}^{3}\partial_{x_i}^3 f^{(2)}(\mathbf{x},t)+\partial_t f^{(2)}(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathbf{x},t|\mathbf{x}',t')f^{(2)}(\mathbf{x}',t')d\mathbf{x}'dt',\\
&\sum_{i=1}^{3}\partial_{x_i}^3 f^{(3)}(\mathbf{x},t)+\partial_t f^{(3)}(\mathbf{x},t)=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathbf{x},t|\mathbf{x}',t')f^{(3)}(\mathbf{x}',t')d\mathbf{x}'dt',\\
&\vdots
\end{aligned} \tag{83}$$

The kernel $G(\mathbf{x},t|\mathbf{x}',t')$ in Eq. (83) is related to the function $\hat{G}(\mathbf{k})$ by means of the following Fourier transform:

$$G(\mathbf{x},t|\mathbf{x}',t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\hat{G}(\mathbf{k})e^{i\mathbf{k}\cdot[\mathbf{z}-\mathbf{z}']}d^3k. \tag{84}$$

We continue by defining the three-dimensional generalization of the ansatz (6) which relates the functions $f^{(n)}(\mathbf{x},t)$ to the integration kernel $G(\mathbf{x},t|\mathbf{x}',t')$:

$$6\left[\sum_{i=1}^{3}\partial_{x_i}f^{(n)}(\mathbf{x},t)\right]f^{(m)}(\mathbf{x},t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(\mathbf{x},t'|\mathbf{x}',t')f^{(n+m)}(\mathbf{x}',t')d\mathbf{x}'dt'. \tag{85}$$

If we substitute Eq. (85) into Eq. (83), we obtain that the set of Eqs. (83) can be uniquely decomposed as

$$\sum_{i=1}^{3} \partial_{x_i}^3 f^{(1)}(\mathbf{x},t) + \partial_t f^{(1)}(\mathbf{x},t) = 0,$$

$$\sum_{i=1}^{3} \partial_{x_i}^3 f^{(2)}(\mathbf{x},t) + \partial_t f^{(2)}(\mathbf{x},t) = 6\left[\sum_{i=1}^{3} \partial_{x_i} f^{(1)}(\mathbf{x},t)\right] f^{(1)}(\mathbf{x},t),$$

$$\sum_{i=1}^{3} \partial_{x_i}^3 f^{(3)}(\mathbf{x},t) + \partial_t f^{(3)}(\mathbf{x},t) = 6\sum_{i=1}^{3} \left([\partial_{x_i} f^{(1)}(\mathbf{x},t)] f^{(2)}(\mathbf{x},t) + [\partial_{x_i} f^{(2)}(\mathbf{x},t)] f^{(1)}(\mathbf{x}t)\right), \quad (86)$$

$$\sum_{i=1}^{3} \partial_{x_i}^3 f^{(4)}(\mathbf{x},t) + \partial_t f^{(4)}(\mathbf{x},t) = 6\sum_{i=1}^{3} \left([\partial_{x_i} f^{(1)}(\mathbf{x},t)] f^{(3)}(\mathbf{x},t) + [\partial_{x_i} f^{(2)}(x,t)] f^{(2)}(\mathbf{x},t) + [\partial_{x_i} f^{(3)}(\mathbf{x},t)] f^{(1)}(\mathbf{x},t)\right),$$

If we take the sums of both the left-hand sides and the right-hand sides of all the equations (86), we obtain, after some bookkeeping, the following result:

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{3} \partial_{x_i}^3 f^{(n)}(\mathbf{x},t)\right] + \sum_{i=1}^{\infty} \partial_t f^{(n)}(\mathbf{x},t) = 6\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\sum_{i=1}^{3} \partial_{x_i} f^{(n)}(\mathbf{x},t)\right] f^{(m)}(\mathbf{x},t). \quad (87)$$

Finally, if we define, for notational convenience,

$$u(\mathbf{x},t) = \sum_{n=1}^{\infty} f^{(n)}(\mathbf{x},t), \quad (88)$$

we find that Eq. (87) is equivalent with

$$\sum_{i=1}^{3} \partial_{x_i}^3 u(\mathbf{x},t) + \partial_t u(\mathbf{x},t) = 6\left[\sum_{i=1}^{3} \partial_{x_i} u(\mathbf{x},t)\right] u(\mathbf{x},t). \quad (89)$$

Equation (89) is a three-dimensional partial differential equation which can be linearized into Eq. (78) by applying a similar cascade of transformations and rescaling. Equation (89) is the three-dimensional generalization of the KdV equation. We can solve Eq. (89) by solving Eq. (78) and transforming the obtained solutions $g^{(n)}(\mathbf{x},\mathbf{k})$ into the functions $f^{(n)}(\mathbf{x},t)$. Before we can compute the solution, Eq. (78), we first have to find an adequate Green's function to solve the integration kernel $G(\mathbf{x},t|\mathbf{x}',t')$. We therefore first solve

$$\sum_{i=1}^{3} (\partial_{x_i} + ik_i)\mathscr{G}(\hat{\mathbf{z}},\mathbf{k}) = \delta(\hat{\mathbf{z}}), \quad \hat{\mathbf{z}} = \mathbf{z} - \mathbf{z}'. \quad (90)$$

The Fourier transform $\mathscr{H}(\mathbf{k}',\mathbf{k})$ of $\mathscr{G}(\hat{\mathbf{z}},\mathbf{k})$ is given by the following relationship:

$$\mathscr{H}(\mathbf{k}',\mathbf{k}) = \int_{-\infty}^{\infty} \mathscr{G}(\hat{\mathbf{z}},\mathbf{k}) e^{-i\mathbf{k}'\cdot\hat{\mathbf{z}}} d^3\hat{z}. \quad (91)$$

If Eq. (91) is substituted into Eq. (90), we obtain the following solution of $\mathscr{H}(\mathbf{k}',\mathbf{k})$:

$$\mathscr{H}(\mathbf{k}',\mathbf{k}) = \frac{1}{i\Sigma_{j=1}^{3}(p_j + k_j)}. \quad (92)$$

If we apply the inverse transform of Eq. (91) to Eq. (92), we find the following solution of the Green's function $\mathscr{G}(\hat{\mathbf{z}},\mathbf{k})$:

$$\mathscr{G}(\hat{\mathbf{z}},\mathbf{k})=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^3 i}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{k}'\cdot\hat{\mathbf{z}}}}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,d^3k'. \tag{93}$$

Once the Green's function is known, we can compute the general solution of the linear differential equation, Eq. (78):

$$g^{(n)}(\mathbf{z},\mathbf{k})=\lim_{\epsilon\to 0}\frac{(n-1)}{(2\pi)^3 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{k}'\cdot[\mathbf{z}-\mathbf{z}']}}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,S(\mathbf{k})g^{(n)}(\mathbf{z}',\mathbf{k})d^3k'd^3z', \quad n\neq 1. \tag{94}$$

If we apply the Fourier transform given in Eqs. (82)–(94), we find that solutions of the functions $f^{(n)}(\mathbf{x},t)$ are given by

$$f^{(n)}(\mathbf{x},t)=\lim_{\epsilon\to 0}\frac{(n-1)}{(2\pi)^6 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(\mathbf{k}-\mathbf{k}')}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,S(\mathbf{k})g^{(n)}(\mathbf{z},\mathbf{k}')d^3kd^3k', \quad n\neq 1. \tag{95}$$

In a similar manner as pointed out in Sec. III, it can be shown that if the integration kernel $G(\mathbf{x}-\mathbf{x}'|t-t')$ has the following structure:

$$G(\mathbf{x}-\mathbf{x}'|t-t')=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^6 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[\mathbf{k}'\cdot\mathbf{k}']e^{i\mathbf{k}'\cdot[\mathbf{z}-\mathbf{z}']}}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,S(\mathbf{k}')d^3kd^3k'. \tag{96}$$

A general form of the solutions $u(\mathbf{x},t)$ of the three-dimensional KdV equation is given by the following relationship:

$$u(\mathbf{x},t)=\sum_{n=1}^{\infty}f^{(n)}f^{(n)}(\mathbf{x},t)=Ae^{i\mathbf{k}\cdot\mathbf{x}}+\lim_{\epsilon\to 0}\sum_{n=2}^{\infty}\frac{(n-1)}{(2\pi)^6 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\delta(\mathbf{k}-\mathbf{k}')}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\times S(\mathbf{k}')g^{(n)}(\mathbf{z},\mathbf{k}')d^3kd^3k'. \tag{97}$$

It can be verified by redoing the computations which are carried out in Sec. III, that if the ansatz (85), is satisfied, a recursion relation for the solutions $f^{(n)}(x,t)$ can be obtained. The solutions $f^{(n)}(x,t)$ which lead to a solution of the three-dimensional KdV equation are given by

$$f^{(1)}(\mathbf{x},t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^6 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{k}'\cdot\mathbf{z}}}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,S(\mathbf{k}')\left[\sum_{m=1}^{3}ik_m'\right]d^3kd^3k', \tag{98}$$

$$f^{(2)}(\mathbf{x},t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^9 i^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{z}}S(\mathbf{k}')S(\mathbf{k}'')}{\Sigma_{l,m=1}^{3}(k_l+k_l'+i\epsilon)(k_m'+k_m''+\epsilon)}\times\left[i\sum_{m=1}^{3}(k_m'+k_m'')\right]d^3kd^3k'd^3k', \tag{99}$$

Once the functions $f^{(n)}(\mathbf{x},t)$ are computed, we construct the solution $u(\mathbf{x},t)$ of the three-dimensional KdV equation by summation of the functions $f^{(n)}(\mathbf{x},t)$:

$$u(\mathbf{x},t)=\lim_{\epsilon\to 0}\frac{1}{(2\pi)^6 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mathbf{k}'\cdot\mathbf{z}}}{\Sigma_{j=1}^{3}(k_j+k_j'+i\epsilon)}\,S(\mathbf{k}')\left[\sum_{m=1}^{3} ik_m'\right]d^3k d^3k'$$
$$+\lim_{\epsilon\to 0}\frac{1}{(2\pi)^9 i^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i(\mathbf{k}'+\mathbf{k}'')\cdot\mathbf{z}}S(\mathbf{k}')S(\mathbf{k}'')}{\Sigma_{l,m=1}^{3}(k_l+k_l'+i\epsilon)(k_m'+k_m''+i\epsilon)}$$
$$\times\left[i\sum_{m=1}^{3}(k_m'+k_m'')\right]d^3kd^3k'd^3k''+\cdots . \tag{100}$$

We can conclude that the approach followed in Sec. II for the one-dimensional KdV equation, can also be used to construct special solutions for the three-dimensional KdV equation. By generalizing Eq. (78) and by introducing more symmetry, we can, in principle, find special solutions of other higher-dimensional NPDEs. For proving the integrability, it would be necessary to prove that the set of solutions which are derived is sufficient for solving a reasonable Cauchy problem. It is also still unclear what the relationship between the procedure followed in this paper and the three-dimensional inverse scattering problem exactly is. In one dimension, this issue is clarified in Sec. IV, but in three dimensions the analysis is more complicated. It is not unthinkable that if this relationship is revealed, answers can be given with respect to the integrability of three-dimensional NPDEs.

VII. DISCUSSION

The following novel results are obtained in this paper. First, a new method is presented to obtain solutions of one-dimensional S-integrable differential equations. Whereas the IST solves S-integrable differential equations by using inversion techniques, the method presented in this paper solves these equations by using a direct method. The S-integrable nonlinear partial differential equation is linearized into an ordinary linear differential equation. There are several advantages of using a linearization method. The first advantage is that the method given is fairly general. In principle, the machinery applies to all NPDEs which have a nonlinear part consisting of a power series. Second, the method we have developed does not need Lax pairs for systematically investigating the integrability of NPDEs. The method can be generalized to higher-dimensional NPDEs. This is demonstrated in Sec. VI for the three-dimensional KdV equation. This result demonstrates that at least in one dimension, S-integrable NPDEs are also C-integrable.

As shown in Sec. II we can conclude that the linearized equation depends on two factors. The first factor is the dispersion relation which defines the space coordinate. The latter is invariant under Galileian transformations. The second factor is a generalization of the scattering data which is related by the ansatz used in this paper to the nonlinearity. By modifying the dispersion relation and the ansatz, Eq. (6), large classes of S-integrable NPDEs can be constructed and solved. It is remarked that the Galileian invariance of the space coordinate can be associated with a minimum symmetry property of the linearized differential equation. If the linearized equation, Eq. (17), is generalized, for instance, by making it invariant for other symmetries, more complex NPDEs can be constructed. It is shown in Sec. V that the nonlinear Schrödinger equation follows from a generalization of Eq. (17) which is invariant under the SU(2) generators. The mathematical reason for this invariance is related to the fact that the zero-curvature condition which is responsible for an infinite number of conservation laws, is invariant under gauge transformations, taking their value in a Lie group. As a result of this, the corresponding linearized problem is invariant under similar transformations in the corresponding Lie algebra.

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