# Selection of sensors and actuators based on a necessary condition for robust performance

# SELECTION OF SENSORS AND ACTUATORS BASED ON A NECESSARY CONDITION FOR ROBUST PERFORMANCE

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## Abstract

A recently presented method for actuator and sensor selection for linear control systems is applied and evaluated for an active suspension control problem. The aim is to eliminate actuator/sensor combinations, for which no controller exists that achieves a specified level of robust performance. Complete controller synthesis is avoided by using necessary conditions for robust performance. Due to this, it is not guaranteed that a stabilizing controller can be constructed for combinations passing the conditions. This is a major shortcoming of the method. The effectiveness and efficiency of the selection procedure are assessed and compared with $\mu$-synthesis.

**Keywords:** robust control, LMI, automotive.

## 1 Introduction

Preceding controller synthesis, it must be decided on an appropriate number, place and type of sensors ("outputs") and actuators ("inputs"). This process will be referred to as Input Output (IO) selection. Compared to other steps in control system design, IO selection has gained relatively little attention. Nevertheless, it is of crucial importance. First, the employed combination of actuators and sensors (IO set) may put fundamental limitations on the achievable performance. Second, the IO set determines control system complexity, hardware expenses, reliability, and maintenance effort. Since the number of candidate IO sets grows extremely rapidly with the number of candidate inputs and outputs, favorable candidates are easily overlooked. Hence, an effective and efficient IO selection method is desirable to complement the designer's physical understanding of the system. Though controller design and closed-loop evaluation for each IO set is the most effective approach, it is infeasible for a huge number of candidates.

Various IO selection methods are mentioned in [9]. Three limitations are commonly encountered. First, IO selection is often restricted to systems with an equal number of inputs and outputs. Second, the controlled and measured variables are not always treated separately; instead, it is frequently assumed that controlled variables can either directly be measured, or suitably be represented by measured variables. Third, if employed at all, quantitative performance specifications and uncertainty characterizations are usually restricted to one frequency or frequency range. To remedy these shortcomings, the standard formulation for robust control problems is employed, see, *e.g.*, [11]. In this respect, the goal for IO selection considered here is *to minimize the number of inputs and outputs, subject to the achievement of a desired Robust Performance (RP) level.* Thus, with an IO set it must be possible to construct a stabilizing controller meeting the performance specifications for a class of uncertainties. Such an IO set will be termed "viable." In [4], an IO selection method for *linear* control systems is proposed, which can be used for this goal.

Essentially, this IO selection method is based on *necessary* conditions for existence of a controller, achieving a desired RP level. This necessity is due to dropping the stabilizing property of the controller. Therefore, IO sets may be accepted for which a stabilizing controller achieving RP cannot be constructed. The main contribution of this paper is to evaluate the practical aspects of this method. An active suspension control problem is used for this. Contrary to the distillation column example in [4], dynamic effects (instead of steady-state effects only) and joint input and output selection (instead of output selection only) are considered to address the method's effectiveness. The computational effort (efficiency) is compared with $\mu$-synthesis.

The paper is organized as follows. To start with, the robust performance concept is discussed, followed by a summary of the IO selection conditions. After the control problem formulation, the IO selection method is applied and evaluated. Finally, conclusions are drawn and some suggestions for future research are given.

## 2 Robust Performance

The development of the IO selection method in [4] starts from the so-called structured singular value ($\mu$) theory, some aspects of which are mentioned in Section 2.1. The use of $\mu$ to address RP in control system analysis is shortly explained in Section 2.2.

### 2.1 Structured Singular Value

The structured singular value is defined as a matrix function operating on a complex matrix $M \in \mathbb{C}^{n \times m}$. Its value not only depends on $M$, but also on an underlying structure of block diagonal matrices $\mathbf{\Delta}$, defined as:

$$\begin{aligned} \mathbf{\Delta} \;&=\; \{\text{diag}(\delta_1 I_{r_1}, \ldots, \delta_k I_{r_k}, \Delta_1, \ldots, \Delta_l) : \\ &\quad\;\; \delta_i \in \mathbb{C};\;\; \Delta_i \in \mathbb{C}^{p_i \times q_i}\}\,, \\ \textstyle\sum_{j=1}^k r_j &+ \textstyle\sum_{i=1}^l p_i = m;\; \textstyle\sum_{j=1}^k r_j + \textstyle\sum_{i=1}^l q_i = n. \end{aligned} \tag{1}$$

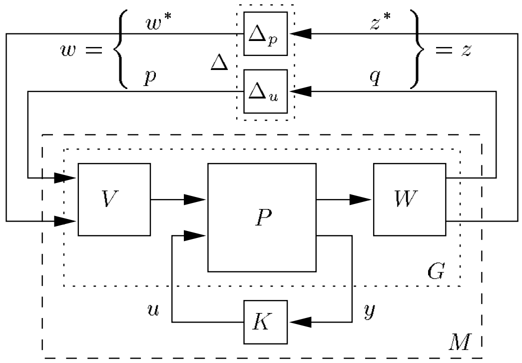


Figure 1: Standard control system set-up

Only *complex* blocks in $\boldsymbol{\Delta}$ are considered here, since the IO selection method is unable to deal with real uncertainties. Contrary to the treatment in [4], the $\Delta_i$ blocks are not restricted to be square. Now, $\mu$ is defined as follows [5]:

$$\mu_{\boldsymbol{\Delta}}(M) := \frac{1}{\min_{\Delta \in \boldsymbol{\Delta}}(\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0)} \quad (2)$$

and $\mu_{\boldsymbol{\Delta}}(M) := 0$ if no $\Delta \in \boldsymbol{\Delta}$ makes $I - M\Delta$ singular. Because $\mu$ cannot be computed exactly in an efficient way, upper and lower bounds are used [5]. The latter will not be paid attention to, since the development of the IO selection theory is based on the upper bound:

$$\mu_{\boldsymbol{\Delta}}(M) \leq \inf_{D \in \boldsymbol{D}} (D_z M D_w^{-1}). \quad (3)$$

Here, $D \in \boldsymbol{D}$ is shorthand for $D_z \in \boldsymbol{D_z}$, $D_w \in \boldsymbol{D_w}$ and $D_z$ and $D_w$ are the so-called $D$-scales in the sets:

$$\begin{aligned} \boldsymbol{D_z} &:= \{\text{diag}(D_1, \ldots, D_k, d_1 I_{q1}, \ldots, d_l I_{ql})\}, \\ \boldsymbol{D_w} &:= \{\text{diag}(D_1, \ldots, D_k, d_1 I_{p1}, \ldots, d_l I_{pl})\}, \\ D_j &\in \mathbb{C}^{r_j \times r_j},\ D_j = D_j^* > 0;\ d_i \in \mathbb{R},\ d_i > 0. \end{aligned} \quad (4)$$

Here, $\{\cdot\}^*$ denotes the complex conjugate transpose. $D_z$ and $D_w$ only differ in dimensions. Each full block in $\boldsymbol{\Delta}$ is accompanied by diagonal scaling matrices and each repeated block in $\boldsymbol{\Delta}$ by a full scaling matrix. Without loss of generality, the $D$-scales are normalized with respect to the last diagonal matrix, so $d_l = 1$. The minimization in (3) is convex and, in general, the $\mu$ upper bound is very close to the exact $\mu$ [5]. Therefore, the $\mu$ upper bound is often useful in practice.

### 2.2 Robust Performance Test Condition

While in the previous section $\mu$ was related with *constant* matrices, $\mu$ is now treated in the context of *dynamic* control systems and Transfer Function Matrices (TFMs), which are also complex matrices if considered at one frequency.

Consider the standard set-up for finite-dimensional, linear, time-invariant control systems in Fig. 1. The following signals play a role: the manipulated variables $u \in \mathbb{R}^{n_u}$ (inputs); the measured variables $y \in \mathbb{R}^{n_y}$ (outputs); the controlled variables $z^*$, which are ideally zero; the exogenous variables $w^*$, such as reference signals and disturbances; the signals $p$ and $q$ associated with $\Delta_u$. Finally, $w := \binom{p}{w^*} \in \mathbb{R}^{n_w}$ and $z := \binom{q}{z^*} \in \mathbb{R}^{n_z}$. The generalized plant $G$ incorporates both nominal plant data $P$ and design filters $V$ and $W$ reflecting performance specifications and uncertainty characterizations. $G$ is partitioned as:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \quad (5)$$

$K$ denotes the controller. It is assumed, that $G$ and $K$ are stabilizable and detectable [11, Chapter 16]. The generalized closed-loop $M$ is given by:

$$M(G, K) = G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21}. \quad (6)$$

The possibly structured uncertainty block $\Delta_u$ accounts for the plant uncertainties, while $\Delta_p$ is an unstructured *fictitious* block. As will become clear below, $\Delta_p$ is introduced to arrive at the RP condition.

In this paper, controller design and IO selection are restricted to *dynamic* uncertainties. This leaves out real parametric ones, which are instead covered with the larger class of dynamic uncertainties, potentially introducing conservatism. In the context of control systems, the uncertainty structure $\boldsymbol{\Delta_u}$ corresponding to $\Delta_u$ takes the same form as in (1), but with the diagonal blocks replaced by proper, real-rational, and stable TFMs, *i.e.*, $\Delta_u \in \mathcal{RH}_\infty$. In analogy, the unstructured block $\Delta_p \in \mathcal{RH}_\infty$. Together with $\Delta_u$ it makes up the augmented block $\Delta := \text{diag}(\Delta_u, \Delta_p)$.

The proposed IO selection method is aimed at achieving RP, *i.e.*, performance must be guaranteed for a class of uncertainties. The required performance is quantified by the $\mathcal{H}_\infty$ norm of the uncertain closed-loop $\tilde{M}$ between $w^*$ and $z^*$. The following provides a necessary and sufficient condition for an RP level $\gamma$ [11, Chapter 11]:

**Robust Performance:** Assume $M$ is stable and let $\gamma > 0$. For all $\Delta_u \in \boldsymbol{\Delta_u}$ with $\|\Delta_u\|_\infty \leq 1/\gamma$, the uncertain closed-loop $\tilde{M}$ is stable and $\|\tilde{M}\|_\infty < \gamma$ if and only if:

$$\|M\|_\mu := \sup_\omega \mu_{\boldsymbol{\Delta}}(M(j\omega)) < \gamma. \quad (7)$$

So, for a specified frequency grid, $\mu$ can be used for control system analysis. In analogy to the constant matrix case, the frequency-dependent $\mu$ is replaced by its upper bound, *i.e.*, inequality (7) is replaced by:

$$\|M\|_{\bar{\mu}} := \sup_\omega \inf_{D \in \boldsymbol{D}} \bar{\sigma}(D_z(\omega) M(j\omega) D_w^{-1}(\omega)) < \gamma \quad (8)$$

and is used to test the RP property of a given closed-loop $M$. Inequality (8) serves as the starting point for the development of the IO selection method in [4]. This is summarized in the next section.

## 3 Input Output Selection Theory

This section provides the conditions for IO selection documented in [4]. The derivation is not given, only the main steps are mentioned and the resulting tools commented. $\Delta$ is not restricted to be made up of square blocks, as in [4]. The formulas for this more general case are straightforwardly obtained by using the same derivation as in [4].

The basic idea for the IO selection is to test if $\min_{K \in \mathcal{K}_S} \|M(G, K)\|_\mu < \gamma$, with $\mathcal{K}_S$ the set of all proper, real-rational, and stabilizing controllers (actually, $\gamma = 1$ in [4]). As announced before, this test is replaced by:

$$\min_{K \in \mathcal{K}_S} \sup_\omega \inf_{D \in \boldsymbol{D}} \bar{\sigma}(D_z M(G, K) D_w^{-1}) < \gamma, \quad (9)$$

yielding a sufficient (and generally almost necessary) condition for existence of a robustly performing controller. The Youla controller parameterization (see, *e.g.*, [3]) is invoked to create an expression for $M$ which is affine in the "new controller" $Q$, whereas (6) is not affine in $K$. Inequality (9) is then replaced by its *equivalent:*

$$\min_{Q\in\mathcal{RH}_\infty} \sup_{\omega} \inf_{D\in\boldsymbol{D}} \bar{\sigma}(D_z M(N,Q) D_w^{-1}) < \gamma, \qquad (10)$$

with $N$ a modified version of $G$. Note, that the requirement of $K$ being stabilizing ($K$ itself need not be stable) is replaced by the requirement of $Q$ itself being stable.

To arrive at the controller-independent IO selection conditions in [4], $Q \in \mathcal{RH}_\infty$ in (10) is replaced by $Q \in \mathcal{RM}$. Here, $\mathcal{RM}$ denotes all proper, real-rational TFMs, which need *not* be stable ($\mathcal{RH}_\infty \subset \mathcal{RM}$). Hence, the requirement that the controller must be stabilizing is dropped, which is a major shortcoming of the IO selection method. It is claimed in [4], that dropping the stability requirement on $Q$ is equivalent to dropping the causality requirement on $Q$; see also [11, Section 4.3], where a TFM which is analytic and bounded in the open left-half-plane (*i.e.*, a TFM in $\mathcal{H}_\infty^-$) is defined as anti-stable *or* anti-causal. Without going into further detail here, the following condition for IO selection results:

*A necessary condition for existence of a $K \in \mathcal{K}_S$ achieving $\|M\|_{\bar{\mu}} < \gamma$ is the existence of an $X_z(j\omega) \in \boldsymbol{D_z}$ and $X_w(j\omega) \in \boldsymbol{D_w}$ such that for all $\omega$:*

$$\hat{G}_{21\perp}\{G_{11}^* X_z G_{11} - \gamma^2 X_w\}\hat{G}_{21\perp}^* < 0 \qquad (11)$$

("output selection"),

*and*

$$\hat{G}_{12\perp}^*\{G_{11} X_w^{-1} G_{11}^* - \gamma^2 X_z^{-1}\}\hat{G}_{12\perp} < 0 \qquad (12)$$

("input selection").

Here, $\hat{G}_{12} = G_{12}(G_{12}^* G_{12})^{-1/2}$, $\hat{G}_{21} = (G_{21}G_{21}^*)^{-1/2}G_{21}$ and $\{\cdot\}_\perp$ denotes the orthogonal complement. Like the $D$-scales in (4), $X_z$ and $X_w$ share the same variables and only differ in size (in [4], $X_z = X_w = X$, due to the restriction to square blocks in $\Delta$). $\hat{G}_{12}$ and $\hat{G}_{21}$ exist if and only if $G_{12}$ and $G_{21}$ have full column rank and full row rank respectively for all $\omega$. This is assumed in the sequel, implying $n_z \geq n_u$ and $n_w \geq n_y$.

With respect to the IO selection conditions, a few comments are made. First, (11) and (12) are each convex feasibility problems in the form of Linear Matrix Inequalities (LMIs). However, they must be checked jointly, which is a *non-convex* feasibility problem. According to [4], this problem is currently only solved straightforwardly if $\Delta$ consists of two full blocks, *e.g.*, for an RP problem with $\Delta_p$ and one full block $\Delta_u$. In that case, $X_z$ and $X_w$ consist of two real diagonal blocks (see (4)), the second of which is identity. As a result, each LMI could be solved for one scalar: (11) for $s$ associated with $X_z$ and $X_w$; (12) for $t$ corresponding to $X_z^{-1}$ and $X_w^{-1}$. Conditions (11) and (12) are jointly met if the solution intervals for $s$ and $1/t$ intersect. To check this, the solution interval for one LMI is determined by a minimization and a maximization subject to feasibility of the LMI and next, this solution interval is used as an additional constraint when checking the feasibility of the other LMI. The MATLAB package LMITOOL [2] is used for these computations.

Second, it is remarked, that (11) only depends on $G_{11}$ and $G_{21}$. This implies that (11) *only* depends on the outputs $y$, provided there are no uncertainties in $\Delta_u$, which are directly linked to the inputs $u$, *e.g.*, multiplicative or additive input uncertainties. Therefore, (11) is referred to as the "output selection" condition. In analogy, the "input selection" condition (12) only depends on the inputs $u$ if there are no uncertainties directly linked to $y$. Under these restrictions to the uncertainty model, one possible approach to IO selection is to check (11) for all candidate output sets and (12) for all candidate input sets. This approach is not restricted by the number of blocks in $\Delta$, but it may eliminate less nonviable IO sets than the joint test.

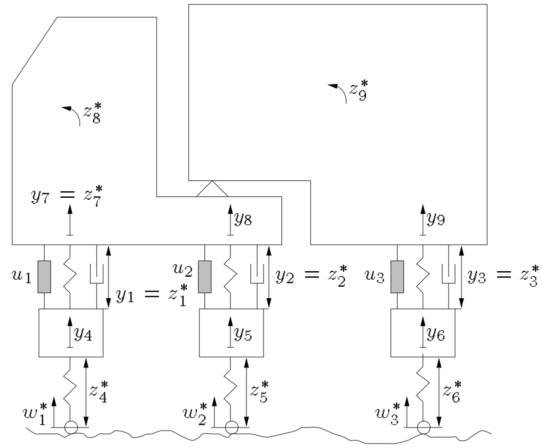


Figure 2: The 6 DOF tractor-semitrailer combination

Third, the input selection condition (12) drops out if $n_z = n_u$ ($\hat{G}_{12}$ is nonsingular and square and hence $\hat{G}_{12\perp}$ is empty). In this case, the input set is seen as perfect by the IO selection method. In analogy, the output selection condition (11) drops out if $n_w = n_y$ ($\hat{G}_{21}$ is nonsingular and square, so $\hat{G}_{21\perp}$ is empty) and the output set is seen as perfect. Finally, *all* IO sets with $n_z = n_u$ and $n_w = n_y$ will pass the necessary conditions for IO selection, but (9) is not guaranteed to be met. Usually (and for the application studied here), actuator weights and sensor noise are included in $z$ and $w$ respectively. Then $n_z > n_u$ and $n_w > n_y$ and neither (12), nor (11) drops out.

# 4 Control Problem Formulation

Conditions (11) and (12) are used to select sensors and actuators for the active suspension applied in the 6 Degree-Of-Freedom (DOF) model of the tractor-semitrailer combination in Fig. 2. This model is detailed in [7] and a MATLAB file generating the model can be requested. The three candidate actuators ($u_1$, $u_2$, $u_3$) are placed between the axles and the tractor and semitrailer chassis. The nine candidate sensors measure the suspension deflections ($y_1$, $y_2$, $y_3$), the axle accelerations ($y_4$, $y_5$, $y_6$), and the vertical tractor and semitrailer accelerations ($y_7$, $y_8$, $y_9$). Combining these inputs and outputs yields 3,577 candidate IO sets. The control objectives are basically the same as in [8].

## 4.1 Performance Specifications

The exogenous input $w^*$ contains the road surface height ($w_1^*$, $w_2^*$, $w_3^*$) and measurement noise ("$y$-noise," $w_4^*, \ldots, w_{12}^*$). Assuming the road surface height to be lowpass-filtered white noise, the corresponding shaping filters in $V$ are chosen as:

$$V_{1,2,3} = \frac{v_0}{s/\omega_0 + 1}. \quad (13)$$

For a fair motorway and a vehicle speed of 25 [m/s], $\omega_0 = 2\pi \cdot 0.25$ [rad/s] and $v_0 = 8.0\,10^{-3}$ [-] are representative.

The measurements are assumed to be disturbed with zero-mean noises. Manufacturer data of displacement and acceleration sensors is consulted, as well as experimental data from vehicle tests. The suspension deflection measurements $y_1, \ldots, y_3$ have an accuracy of $10^{-3}$ [m]. The acceleration measurements $y_4, \ldots, y_9$ are disturbed with noise with an RMS level of $2.5\,10^{-2}$ [m/s$^2$]. Also, these sensors are sensitive to transverse accelerations: 1.3% of the transverse accelerations is passed through to $y_4, \ldots, y_9$ as if they were vertical. An RMS level of $2.5\,10^{-2}$ [m/s$^2$] is used here as a worst-case situation. The RMS levels for these two acceleration noise sources are simply added to give $V_7, \ldots, V_{12}$. The following constant entries in $V$ now result:

$$\begin{aligned} V_i &= 1.0\,10^{-3} \quad i = 4,5,6 \qquad (y_1, \ldots, y_3), \\ V_i &= 5.0\,10^{-2} \quad i = 7, \ldots, 12 \quad (y_4, \ldots, y_9). \end{aligned} \quad (14)$$

Three main design goals are distinguished in $z^*$. The first and second one are limiting the suspension deflections (due to space limitations) and tire deflections (for good handling and minimum road surface damage) respectively. To represent these objectives in the $\mathcal{H}_\infty$ norm setting, constant weights are chosen as an approximation:

$$\begin{aligned} W_1 &= \rho_{1_f} \quad W_2 = \rho_{1_r} \quad W_3 = \rho_{1_t} \quad (z_1^*, \ldots, z_3^*), \\ W_4 &= \rho_{2_f} \quad W_5 = \rho_{2_r} \quad W_6 = \rho_{2_t} \quad (z_4^*, \ldots, z_6^*). \end{aligned} \quad (15)$$

The third design goal is to limit the tractor's vertical and rotational accelerations $z_7^*$, $z_8^*$ (to guarantee good driver comfort) and to limit the semitrailer's rotational acceleration $z_9^*$ (to avoid cargo damage). The weighting filters for $z_7^*$ and $z_8^*$ are chosen to represent the human sensitivity to these accelerations respectively. Therefore,

$$W_7 = \rho_3\, \omega_1^2 \frac{s/\omega_2 + w_{1_0}}{(s + \omega_1)^2}, \quad (16)$$

with $\rho_3 = 1$ [-], $w_{1_0} = 0.4$ [-], $\omega_1 = 2\pi \cdot 10$ [rad/s], and $\omega_2 = 2\pi \cdot 5$ [rad/s]. The weight for $z_8^*$ is given by:

$$W_8 = \rho_4 \frac{w_{2_0}}{s/\omega_3 + 1}, \quad (17)$$

with $w_{2_0} = 1$ [-] and $\omega_3 = 2\pi \cdot 2$ [rad/s]. The choice of $W_9$ for the semitrailer's acceleration strongly depends on the cargo. Here, it is taken as a constant: $W_9 = \rho_5$.

The final control objective is to limit $u$ ("$u$-weights," $z_{10,11,12}^* = u_{1,2,3}$). Since the actuator bandwidths are limited, high-frequency inputs cannot be realized. This is accounted for by the bi-proper weighting filter:

$$W_{10,11,12} = \rho_6 \frac{s/\omega_4 + 1}{s/\omega_5 + 1}, \quad (18)$$

where $\rho_6 = 5\,10^{-6}$ [-], $\omega_4 = 2\pi \cdot 5$ [rad/s], and $\omega_5 = 100 \cdot \omega_4$ [rad/s].

Some $\rho$-parameters in $W$ still have to be chosen. They are determined by simulations for the full IO set's nominal closed-loop after $\mathcal{H}_\infty$ optimization. Rounded pulses [8] as a class of deterministic road surfaces were used for this purpose. Controllers are designed iteratively until the closed-loop responses are acceptable. This process yields the following settings:

$$\begin{aligned} &\rho_{1_f} = 104.88 \quad \rho_{1_r} = 29.18 \quad \rho_{1_t} = 6.62, \\ &\rho_{2_f} = 0.61 \quad \rho_{2_r} = 38.39 \quad \rho_{2_t} = 6.94\,10^{-2}, \\ &\rho_4 = 0.66 \quad \rho_5 = 0.40. \end{aligned}$$

### 4.2 Uncertainty Model

The vehicle model has six natural frequencies $\omega_{n_i}$ and damping factors $\beta_i$. Uncertainty in $\omega_{n_i}$ could be used to account for uncertain spring and mass parameters, the latter of which are especially important due to large cargo mass variations. However, because of the self-leveling air suspension, the natural frequencies are assumed to remain approximately the same in practice. Therefore, damping factor variations are used to account for uncertainty.

The system matrix of the plant's state-space representation can be chosen so it incorporates $\omega_{n_i}$ and $\beta_{n_i}$ explicitly. This involves a similarity transformation (for details, see [7]), yielding a new block-diagonal system matrix, made up of the following blocks:

$$J_i = \begin{bmatrix} 0 & 1 \\ -\omega_{n_i}^2 & -2\beta_i\omega_{n_i} \end{bmatrix}, \; i = 1, \ldots, 6. \quad (19)$$

These blocks are employed to model the uncertain damping factors. With $\beta_i$ the nominal value of the uncertain parameter $\tilde{\beta}_i$, the relation $\tilde{\beta}_i = \beta_i(1 + b_i\delta_i)$, $\|\delta_i\|_\infty \leq 1$ is used for uncertainty modeling. The six $\delta_i$'s together make up the diagonal block $\Delta_u$, while the $b_i$'s express the relative amount of uncertainty. The corresponding entries in $V$ and $W$ are chosen $\sqrt{b_i}$.

Recall, that checking the IO selection conditions jointly requires $\Delta_u$ to be unstructured. Therefore, it is decided to consider only that particular $\beta_i$ uncertain, which varies mostly with the semitrailer mass. The six $\beta_i$'s are computed for semitrailer masses ranging from 15% of the maximum mass to the maximum mass (fully loaded semitrailer). The mean value of the strongest varying damping factor is $\beta = 1.34$ (with $b = 0.72$), found for 24% of the maximum semitrailer mass. This mass with the corresponding spring stiffnesses yield the nominal model. An optimal $\mu$-synthesis for the full IO set now gives $\|M\|_{\bar{\mu}} = 0.90$ and an $\mathcal{H}_\infty$ optimization without uncertainty gives $\|M\|_\infty = 0.12$.

## 5 Input Output Selection Application

Two major practical aspects of the IO selection method are investigated: effectiveness and efficiency. For ten typical IO sets, Section 5.1 compares the results of optimal $\mu$-synthesis and the IO selection conditions under decreasing test values $\gamma$. IO selection is performed in Section 5.2.

### 5.1 Results for Typical IO Sets

Due to dropping the stabilizing property of the controller, there may be a difference between the lowest achievable $\|M\|_{\bar{\mu}}$ with a stabilizing controller and the $\gamma$ for which the IO set is first eliminated by the test conditions ($\|M\|_{\bar{\mu}} \geq \gamma$). To investigate this source of ineffectiveness and to gain

Table 1: Results of optimal $\mu$-synthesis and IO selection with decreasing $\gamma$ for ten typical IO sets

| | IO set | | $\|M\|_{\bar{\mu}}$ | $\gamma^*_{opt}$ | $\gamma_{opt}$ | Failing condition |
|---|---|---|---|---|---|---|
| 1 | $y_1,\dots,y_9$ | $u_1\,u_2\,u_3$ | 0.90 | 0.89 | 0.23 | input |
| 2 | $y_1,\dots,y_9$ | $u_1$ | 2.60 | 2.60 | 2.60 | input |
| 3 | $y_1,\dots,y_9$ | $u_2$ | 2.57 | 2.57 | 2.56 | input |
| 4 | $y_1,\dots,y_9$ | $u_3$ | 2.59 | 2.59 | 0.23 | input |
| 5 | $y_1\,y_2\,y_3$ | $u_1\,u_2\,u_3$ | 0.90 | 0.89 | 0.23 | input |
| 6 | $y_4\,y_5\,y_6$ | $u_1\,u_2\,u_3$ | 2.61 | 2.61 | 2.61 | output |
| 7 | $y_7\,y_8\,y_9$ | $u_1\,u_2\,u_3$ | 2.59 | 2.22 | 2.21 | output |
| 8 | $y_1\,y_4\,y_7$ | $u_1\,u_2\,u_3$ | 1.57 | 1.52 | 1.52 | output |
| 9 | $y_2\,y_5\,y_8$ | $u_1\,u_2\,u_3$ | 0.90 | 0.89 | 0.23 | input |
| 10 | $y_3\,y_6\,y_9$ | $u_1\,u_2\,u_3$ | 0.90 | 0.89 | 0.23 | input |

insight into the preferable number, place, and type of actuators and sensors, Table 1 is composed.

First, for ten typical IO sets, optimal $\mu$-synthesis is performed, yielding $\|M\|_{\bar{\mu}}$. For the involved $D$-$K$ iteration (see, *e.g.*, [11, Chapter 11]), a frequency grid of 51 logarithmically spaced points between $10^{-1}$ and $10^3$ [rad/s] is used. All computations are done with the MATLAB $\mu$-Analysis and Synthesis Toolbox, Version 3.0 [1].

Second, for the same ten IO sets, the output and input selection conditions (11) and (12) are checked for distinct $\gamma$ values. Starting with $\gamma = 3$, a bisection algorithm finds the smallest $\gamma$ for which the selection condition passes. Like in the $\mathcal{H}_\infty$ optimization part of $D$-$K$ iteration, `tol`$= 10^{-2}$ is used as the bisection termination criterion. For the *joint* input and output selection conditions, $\gamma^*_{opt}$ in Table 1 is the smallest $\gamma$ for which the IO set is accepted. In analogy, for the *separate* conditions, $\gamma_{opt}$ is the smallest $\gamma$ for which the IO set is accepted and the condition which fails first is explicitly indicated. The IO selection is based on the same frequency grid as $\mu$-synthesis, starting with $10^{-1}$ [rad/s] and working upwards until a "frequency fails" or $\omega = 10^3$ [rad/s] is reached. For each frequency, the feasibility of the LMIs (11) and (12) is checked. In case of the joint test, the solution interval for $t$ associated with the input selection LMI (12) is determined and imposed as an additional constraint on $s$ when checking the feasibility of the output selection LMI (11), see Section 3.

The preference of actuators and sensors is studied based on $\|M\|_{\bar{\mu}}$ in Table 1. None of the IO sets 2, 3, and 4 is viable for the RP level $\gamma = 1$ and there is no clear preference for a particular actuator. Obviously, all other IO sets based on a single actuator are also nonviable, since eliminating sensors will never improve the best achievable control. $\|M\|_{\bar{\mu}}$ for IO sets 5, 6, and 7 shows, that IO sets 6 and 7 (using only acceleration sensors) are nonviable for $\gamma = 1$, while IO set 5 is viable and equally good as the full IO set 1. This suggests, that at least one suspension deflection measurement should be incorporated. A comparison between IO sets 8, 9, and 10 shows, that sensors at the tractor's rear and at the semitrailer are preferred and that IO sets 9 and 10 are equally good as the full IO set.

Table 1 also shows, that for all IO sets except 7 and 8, the differences between $\|M\|_{\bar{\mu}}$ and $\gamma^*_{opt}$ are within the bisection tolerance 0.01. Apparently, dropping the stabilizing property of the controller is not an important source of ineffectiveness for the considered (generalized) plant and most of the typical IO sets.

For IO sets 1, 4, 5, 9, and 10 there is a difference between $\gamma^*_{opt}$ (joint check) and $\gamma_{opt}$ (separate checks). Recall from Section 3, that output selection based on (11) implicitly assumes that the input set is perfect. In analogy, input selection based on (12) assumes the output set to be perfect. As a result, $\gamma_{opt} \le \gamma^*_{opt}$ and the joint test may eliminate additional IO sets. Apparently, this source of ineffectiveness only plays a role for the above-mentioned IO sets. For the separate tests, it is indicated whether the input or output selection condition fails. For the full IO set, the input selection condition (12) fails first. So, if inputs are eliminated, (12) should fail first again, which is supported by the results for IO sets 2, 3, and 4. IO sets $5,\dots,10$ use fewer sensors and hence the output selection condition may now break down first. This is true for IO sets 6, 7, and 8, but for IO sets 5, 9, and 10 the input selection condition still fails first.

## 5.2 IO Selection Results

The results of IO selection for $\gamma = 1$ are discussed. For the nine candidate sensors and three candidate actuators, there are 511 candidate output sets, seven candidate input sets, and 3,577 candidate IO sets. The following three-step strategy is implemented. The *first* step subjects the candidate output sets to the output selection condition (11) for the first frequency. Starting with the full output set, subsets are tested, but nonviable output sets and their subsets are directly eliminated. For the output sets passing the first frequency, this procedure is repeated for the next frequency. The *second* step uses a similar procedure for input selection based on (12). The *third* step performs the joint test for all remaining output sets and input sets. Determining the solution interval for $s$ or $t$ takes more time than a feasibility check and therefore the solution interval is computed for the LMI related to the fewest remaining sets. So, if the number of remaining input sets is smallest, the solution interval is determined for (12). Next, starting from the largest remaining IO set, the feasibility of the other LMI is checked, subject to the corresponding solution interval constraint. Again, subsets of nonviable IO sets are directly eliminated. The joint test takes much more time than the separate tests and so efficiency may be improved if the number of joint tests is reduced beforehand, like in this three-step procedure.

Checking the output selection LMI (first step), 384 candidate output sets are termed viable. At least one of the suspension deflection sensors $y_2$ (tractor's rear) or $y_3$ (semitrailer) must be used. This is in line with Table 1, where 1) suspension deflection sensors are preferred to acceleration sensors and 2) sensors mounted at the tractor's rear or at the semitrailer are preferred to sensors mounted at the tractor's front. Checking the input selection LMI (second step), four out of seven candidate input sets are viable. At least the semitrailer actuator $u_3$ must be used. Though the acceptance of input set $u_3$ is in line with the $\gamma_{opt}$ values in Table 1, the single $u_3$ cannot yield a viable IO set, since $\|M\|_{\bar{\mu}} = 2.59$ for IO set 4. For the accepted input and output sets, 1,536 IO sets are formed and subjected to the joint test (third step). The 768 IO sets with $u_2u_3$ or $u_1u_2u_3$ as input set are accepted, so accounting for the coupling of outputs and inputs rejects additional IO sets. The smallest viable IO sets are $y_2/u_2u_3$ and $y_3/u_2u_3$.

Table 2 compares CPU times (Silicon Graphics, Indy, 200MHz, R4400SC) for the LMI-based IO selection with

Table 2: CPU times [s] for the two IO selection approaches

| IO selection: | LMI-based | $\mu$-based |
|---|---|---|
| Separate conditions | $4.01\,10^3$ | $6.80\,10^4$ |
| Joint conditions | $9.05\,10^3$ | $1.51\,10^5$ |
| Total CPU | $1.31\,10^4$ | $2.19\,10^5$ |

CPU times for IO selection based on suboptimal $\mu$-synthesis. In the latter case, the existence of a *stabilizing* controller achieving RP is checked. The $D$-$K$ iteration employs the same frequency grid and third order $D$-scale approximations. The iteration is stopped in two cases: if $\|M\|_{\bar{\mu}} < 1$ (set is viable), or if the reduction in $\|M\|_{\bar{\mu}}$ is less than 0.01 and $\|M\|_{\bar{\mu}} > 1$ (set is nonviable). It appeared, that one of these criteria is always met within five steps. For this $\mu$-based IO selection, a similar strategy is used as before. First, the candidate output sets are investigated, using the full input set; second, the candidate input sets are investigated, using the full output set; third, the possibly viable IO sets generated from the accepted candidate output and input sets are checked. In each step, all subsets of nonviable candidates are directly eliminated.

In the first step of the $\mu$-based IO selection, the same 384 output sets are accepted as for the LMI-based approach. In the second step, the $\mu$-based approach accepts two of the four input sets termed viable by the LMI-based approach ($u_2u_3$ and $u_1u_2u_3$). As a result, 768 candidate IO sets remain for the third step. These are all accepted and are the same as those accepted by the LMI-based approach. Optimal $\mu$-synthesis for the two smallest viable IO sets yields $\|M\|_{\bar{\mu}} = 0.96$ for $y_2/u_2u_3$ and $\|M\|_{\bar{\mu}} = 0.96$ for $y_3/u_2u_3$.

In the current implementation, the CPU time for the $\mu$-based IO selection is 17 times larger than for the LMI-based IO selection. It is emphasized, that the differences in the CPU times for the two approaches strongly depend on implementation aspects and on the particular application. The implementation affects the CPU time, *e.g.*, by the $D$-scale order and the tolerance in $\mu$-synthesis, by the frequency grid definition, and by the order of checking the frequencies and the choice of the initial estimates for $X_z$ and $X_w$ in the LMI-based IO selection. The application affects the CPU time, *e.g.*, by the convergence in the $\mu$-synthesis and the first encountered frequency for which nonviable candidates drop out. Clearly, the CPU times listed in Table 2 must be taken with a grain of salt.

## 6 Conclusions and Recommendations

The IO selection method proposed in [4] was used for an active suspension control problem. The method relies on necessary conditions, since the stabilizing requirement of the controller is dropped. Two other possible sources of ineffectiveness are the following. First, control problems with one repeated or multiple (repeated or full) blocks in $\Delta_u$ involve solving a non-convex optimization, for which currently no efficient solution is known. To partially resolve this, all possible combinations of $\Delta_p$ and two full blocks in $\Delta_u$ could be studied, but an IO set accepted by this strategy may not pass for the original $\Delta_u$. Second, critical frequencies may be overlooked in the specified grid.

For the application, the IO selection method eliminated the same nonviable IO sets as $\mu$-synthesis: the method was effective. However, previous researches made clear (see, *e.g.*, [7,8]), that the effectiveness is strongly affected by the choice of the design filters, possibly leading to poor results: in one situation, twice too many IO sets were accepted, while the CPU time was only three times smaller than for $\mu$-synthesis. At present, it is unclear for which type of problems the LMI-based IO selection method can be applied effectively. For the application, the IO selection method was considerably more efficient than $\mu$-synthesis. In general, the required computation time is hard to predict, since this is affected by many aspects. The IO selection method may especially be useful if only a small number of important frequencies is considered, *e.g.*, $\omega = 0$ as in [4]. A more efficient IO selection method is discussed in [10], but due to sufficiency candidates may incorrectly be rejected.

Future research could focus on the following idea. For the full information case, [6] gives a necessary and sufficient condition for the existence of a stabilizing $K$ and a constant $D$-scale such that $\|D_z M(G,K) D_w^{-1}\|_\infty < 1$. This is sufficient for existence of a stabilizing $K$ achieving $\|M(G,K)\|_\mu < 1$. The condition is an LMI feasibility problem, involving the matrices in the state-space formulation of $G$. It is expected, that an LMI condition can also be derived for the full control case, but that the *combination* of the LMIs is again a non-convex problem. If so, an IO selection method analogous to the one discussed here could be used. On the one hand, this method would be based on sufficiency and viable IO sets may be eliminated. On the other hand, the stabilizing property would not be dropped and the efficiency may be improved, since the tests for multiple frequencies are replaced by a single test.

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