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Theory of nonlinear quantum tunneling resistance in one-dimensional disordered systems

D. Lenstra and R. T. M. Smokers*

Department of Physics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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A novel generalized Landauer formula is derived and used to study the voltage-dependent resistance in a one-dimensional (1D) disordered system. A finite voltage difference introduces energy integration and gives the system self-averaging behavior to a certain extent. The quantum resistance of a 1D system generally shows a rich structure in its dependence on applied voltage and length. Resistance fluctuations are shown to decrease with increasing voltage. In spite of the self-averaging, the mean resistance at large voltage turns out to scale superlinearly with length.

I. INTRODUCTION

The quantum resistance of a one-dimensional (1D) system with static disorder reflects the coherent wave-propagation nature of electrons at low temperatures.¹⁻³ A simple formula which expresses this resistance in terms of the coherent transmission and reflection properties of the structure is Landauer's formula.⁴ This formula is valid only for sufficiently small voltages, such that the transmission probability $T(E)$ can be treated as a constant in the corresponding energy window of width ΔE in which the net conduction takes place. Let L be the system and $N(E_F)$ the density of states per unit length at the Fermi energy, $N(E_F) \simeq (\pi\hbar v_F)^{-1}$, with v_F the Fermi velocity. The typical energy scale on which $T(E)$ will develop large variations is given by $[LN(E_F)]^{-1}$. Therefore, the Landauer formula is only applicable so long as $\Delta E \ll [LN(E_F)]^{-1}$.

The scaling approach to localization in 1980 by Anderson *et al.*⁵—already anticipated by Landauer⁴ in 1969—is based on the careful investigation of the statistical properties of the electrical resistance as a function of the number of elastic scattering segments placed behind each other and leads to the famous exponential scaling law for the resistance, i.e.,

$$R = (\pi\hbar/e^2)[\exp(L/\ell) - 1],$$

with L the system length and ℓ the localization length. This result being derived with the use of Landauer's formula is valid in the limit $V \rightarrow 0$, where V is the voltage over the system.

In this article we present the derivation of a new Landauer-type formula for the resistance, which generalizes a formula given by Büttiker *et al.*⁶ in the sense that we will not linearize in V . Our formula is well suited to deal with energy integration associated with substantial differences in chemical potential of the two reservoirs (which emit and absorb the electrons at both sides of the system) as well as with temperature broadening in the reservoirs. Using this extended resistance formula, we study both the dependence on voltage for a system of given length and the dependence on length (i.e., the scal-

ing behavior) at given voltage.

The central quantity of interest turns out to be a properly energy-averaged transmission probability \bar{T} . In the limit of vanishingly small voltage over the system we recover the exponential scaling regime,⁵ characterized by $\langle \ln T \rangle = -L/\ell$, where $\langle \dots \rangle$ means ensemble averaging. However, with increasing voltage the scaling properties of \bar{T} may become such as to induce exponential scaling with a scaling length generally *larger* than the Anderson localization length ℓ . Although the resistance values at zero voltage are heavily dependent on the sample (universal conductance fluctuations⁷⁻⁹), on increasing the voltage difference they were found to converge to a sample-independent value which is significantly smaller than the zero-voltage scale resistance. For a given fixed system with length L , the resistance turns out to be a very complicated function of voltage with typical fluctuations which for small voltages are in agreement with universal conductance fluctuations, but for larger values of V tend to decrease. These effects are due to self-averaging behavior as a consequence of the conduction taking place in an energy window of *finite* rather than infinitesimally small width.

Several authors have recognized earlier that the presence of an electric field, such as associated with a finite voltage difference, may be relevant for localization behavior¹⁰⁻¹² as well as for transport properties.^{2,13-20} References 15, 16, 18, and 19 treat the quantum dynamics of electrons under the influence of an electric field. The work of Refs. 10-12 and 17 is more in the spirit of the present stationary approach, but the energy integration associated with the finite voltage difference over the system was disregarded by these authors.

In Sec. II we will set up the model and derive our central formula. Systems of variable length are numerically analyzed in Sec. III using a Kronig-Penney-type random potential simulating the presence of scattering centers. The scaling predictions based on the resistance formula are derived in Sec. IV leading to a novel exponential scaling law for the mean resistance, in which the scale length is directly related to the fluctuations (on energy averaging) of the transmittivity. In Sec. V the main results are summarized and discussed in relation to existing theory.

II. DERIVATION OF THE RESISTANCE FORMULA

The configuration to be considered here is sketched in Fig. 1. It is the same configuration studied in Ref. 6, but our derivation will be different in that the potential difference over the system will not be assumed to be infinitesimally small. The two reservoirs in Fig. 1 are held at fixed chemical potentials μ_l and μ_r . The crucial point first elucidated by Landauer⁴ is that the voltage difference over the system is smaller than $|\mu_l - \mu_r|/e$ by an amount to be determined from self-consistency requirements.

Let the respective equilibrium distributions in the left- and right-hand reservoir be given by the Fermi-Dirac distribution functions

$$f_j(E) = \frac{1}{1 + \exp[(E - \mu_j)/k_B T_j]} \quad (j=r,l), \quad (1)$$

where it is noted that the temperature of the reservoirs, T_r and T_l , may be different and k_B is Boltzmann's constant. It is assumed that the electrons propagate fully coherently both in the *ideal* conductors (see Fig. 1) and in the wire between them. The role of temperature is restricted to dictating the precise shape of the two distribution functions. All scattering processes present in the wire are assumed to be coherent processes, i.e., the electrons are scattered by a time-independent rigid potential (no phonon creation or annihilation; no electron-electron interactions).

In order to obtain the net current through the wire we will need the transmission and reflection probabilities for an electron at given energy E . These quantities are completely determined by the full scattering structure of the system which in turn is self-consistently related to the charge distribution in the wire and the leads. In an exact procedure one would have to use an iterative method for integrating Schrödinger's equation subject to the given boundary conditions associated with the prescribed incoming fluxes at both reservoir-contact boundaries. Fortunately, a first-order solution to this complicated self-consistency problem can be obtained by relatively simple

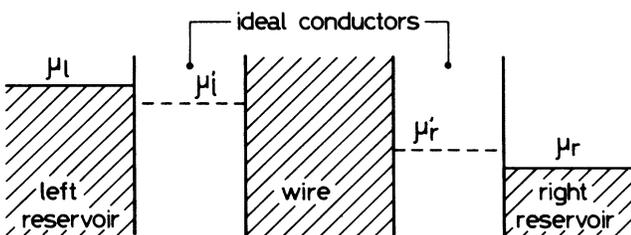


FIG. 1. One-dimensional model for the determination of the quantum resistance of a scattering segment ("wire") connected through ideally conducting "contacts" to two reservoirs which are at different chemical potentials μ_l and μ_r . Due to redistribution of charge, the actual voltage difference over the wire is related to the local chemical potentials at both ends of the wire, μ'_l and μ'_r , which have to be determined in a self-consistent way.

means.⁴⁻⁶ In this method, one introduces the local chemical potentials μ'_l and μ'_r and identifies the voltage difference over the wire as $V = |\mu'_l - \mu'_r|/e$.

For convenience, we will assume in Fig. 1 the symmetric situation with $\mu_l - \mu'_l = \mu'_r - \mu_r$. It will be shown below that this is a very reasonable assumption indeed, fully consistent with the approximation made. The potential felt by an electron is sketched in Fig. 2. Drops in potential occur when an electron moves from inside a reservoir to the point of entering the wire ($0 < x < L$). In the left reservoir ($x < x_l$) the potential is $\Delta\mu$. In the ideal conductor ($x_l < x < 0$) the potential has dropped by an amount $\frac{1}{2}(\Delta\mu - eV)$, where V is the voltage over the wire. In the wire we put the total potential equal to $U(x) - (eVx/L) + \frac{1}{2}(\Delta\mu + eV)$, where $U(x)$ is the static-disorder potential. In the other ideal conductor ($L < x < x_r$) the potential is again at a constant value, given by $\frac{1}{2}(\Delta\mu - eV)$, while in the right reservoir ($x > x_r$) the potential is zero. All ingredients necessary for solving the transmission and reflection problem have now been given.

The probability of emitting an electron at energy E by the left-hand reservoir is $f_l(E)$. When this electron is incident on the wire, its wave number equals

$$k'_l = [(2m/\hbar^2)(E - \frac{1}{2}\Delta\mu - \frac{1}{2}eV)]^{1/2}.$$

After transmission the electron will leave the wire with wave number

$$k'_r = [(2m/\hbar^2)(E - \frac{1}{2}\Delta\mu + \frac{1}{2}eV)]^{1/2}.$$

The current contribution due to electrons emitted from the left is

$$I_l = \frac{e}{\pi} \int_0^\infty dk f_l(E) v(k) T(E), \quad (2)$$

where k is the wave number in the left reservoir,

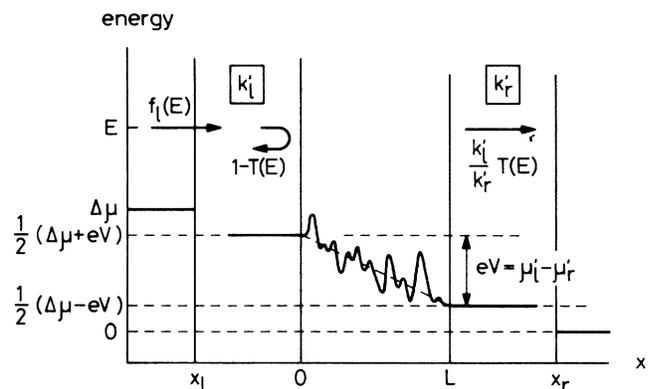


FIG. 2. Potential felt by an electron in the configuration of Fig. 1. An electron emitted at energy E by the left reservoir [probability $f_l(E)$ given by (1)] has wave number $k'_l = [(2m/\hbar^2)(E - \frac{1}{2}\Delta\mu - \frac{1}{2}eV)]^{1/2}$ when incident on the wire [reflectivity $1 - T(E)$] and has wave number $k'_r = [(2m/\hbar^2)(E - \frac{1}{2}\Delta\mu + \frac{1}{2}eV)]^{1/2}$ when transmitted [transmittivity $k'_l T(E)/k'_r$], as indicated in the upper part of the figure.

$$k = [(2m/\hbar^2)(E - \Delta\mu)]^{1/2},$$

$v(k) = \hbar k/m$, and $T(E)$ is the transmission probability for an electron with energy E . As a matter of course $T(E)$ will strongly depend on V . A precise definition and treatment of the quantity $T(E)$ will be postponed to Sec. IV. We mentioned already that one has to be careful in introducing T because of the wave numbers being different at both ends of the scattering system.

The right-hand side of (2) is converted into an integral over energy, using the density of states (with positive k) per unit length, $[\pi\hbar v(k)]^{-1}$,

$$I_l = \frac{e}{\pi\hbar} \int dE f_l(E) T(E). \quad (3)$$

A similar expression can be written down for the current contribution due to electrons emitted from the right-hand reservoir. In this expression, the transmission probability is equal to the one in (3), as will be shown in Sec. IV. Subtracting this current from (3), we arrive at the net current through the wire

$$I = \frac{e}{\pi\hbar} \int dE [f_l(E) - f_r(E)] T(E). \quad (4)$$

Next, we must determine the voltage V that stands over the wire by relating it to the local chemical potentials μ'_l and μ'_r in the ideally conducting contacts at both sides of the wire. First, we notice from (4) that the net current is carried only by electrons with energies in a narrow interval (width $\approx \Delta\mu$) around the Fermi energy $E_F = \frac{1}{2}(\mu_l + \mu_r)$. Directly associated with the current there is a difference between the densities of electrons in the reservoir and in its perfectly conducting connector to the wire. To a good approximation the density difference is given by

$$\Delta\rho_l = \frac{1}{\pi\hbar v_F} \int dE [f_l(E) - f_r(E)] T(E) = I/(ev_F), \quad (5)$$

where the velocity v_F is assumed to be a constant on the relevant energy interval around E_F . It was assumed, in deriving (5), that the density in the ideal conductor is given by the *incoherent* sum of densities due to electrons incident from the left and right reservoir, respectively. A justification for this might be given in terms of the randomizing dynamical nature of the reservoirs leading to temporal averaging of phase relations, but strictly speaking we are faced here with a basic difficulty which confronts us with an inconsistency of the stationary-state approach.

The chemical potential difference corresponding to $\Delta\rho_l$ equals

$$\mu_l - \mu'_l = \Delta\rho_l \frac{\partial E}{\partial \rho} \approx \Delta\rho_l \frac{1}{2} \pi\hbar v_F = \frac{\pi\hbar}{2e} I. \quad (6)$$

Similarly, we find in this approximation, at the right-hand side

$$\mu'_r - \mu_r = \frac{\pi\hbar}{2e} I. \quad (7)$$

From (6) and (7) we obtain

$$V = \Delta\mu/e = \frac{\pi\hbar}{e^2} I, \quad (8)$$

which is the same relation as found for infinitesimally small voltages.

The resistance can be written, using (4) and (8),

$$R = V/I = \frac{\pi\hbar}{e^2} \left[\frac{1}{\bar{T}} - 1 \right], \quad (9)$$

where \bar{T} is the energy-averaged transmission probability,

$$\bar{T} = \int dE W(E) T(E), \quad (10)$$

with the energy weight function

$$W(E) = \frac{f_l(E) - f_r(E)}{\Delta\mu}. \quad (11)$$

Note that $W(E)$ is to a good approximation normalized to unity, but not necessarily a non-negative function. At zero temperature, $W(E)$ is equal to the uniform distribution on the interval $[\mu_r, \mu_l]$. If both reservoirs are at equal finite temperature, then $W(E)$ will be a non-negative distribution. $W(E)$ will assume negative values as well when the reservoirs are at different temperatures.

Our formula (9) is fully compatible with the result derived by Hu²⁰ [see Eqs. (8) and (9) in this reference and let the chemical potential difference be much smaller than the Fermi energy]. Hu's result is derived with application to larger potential differences in mind. In view of our scope with $\Delta\mu \ll E_F$, we prefer to deal with the more suggestive formula (9). Sinkkonen¹³ and Eränen and Sinkkonen¹⁴ use linearized Boltzmann expressions for the distribution functions in the connectors at both sides of the wire. Our approach is different in that these connectors are ideally conducting, i.e., scatterer-free regions separating the wire from the reservoirs. Both our Eq. (9) and Hu's result²⁰ are not directly derivable from the results in Refs. 13 and 14, presumably because the linearized Boltzmann approximation to the distribution function in the connectors is inconsistent with nonlinear current response.

Equation (9) is a surprisingly simple result which takes into account self-averaging effects on the transmission probability through the energy integration in (10). Note that (9) is not an exact result, but nevertheless an extension of existing Landauer formulas. The latter are valid for sufficiently small potentials only, that is, as long as $|eV| \ll [LN(E_F)]^{-1}$, where the right-hand side is just the energy scale on which $T(E)$ will show large fluctuations. Our extension (9) is valid as long as $|eV| \ll E_F$. It is easily seen that for $\mu_l - \mu_r \rightarrow 0$ and $T_l, T_r \rightarrow 0$ the expression (9) coincides with the usual Landauer formula, since in that case we have $\bar{T} \rightarrow T(E_F)$. It will be shown in the next sections that both the statistical and the scaling properties of the resistance given by (9) will drastically change with increasing voltage.

III. NUMERICAL RESULTS FOR A KRONIG-PENNEY-TYPE POTENTIAL

The model system that we will numerically explore in this section, and to which we shall refer as 1D wire, is de-

picted in Fig. 3. The wire has length L and consists of a chain of N δ functions with equal weights H placed at irregular positions x_j along the wire. A constant electric field $F = V/L$ is assumed to be present inside the wire, where V is the voltage difference over the wire. In a fully self-consistent treatment charges would pile up in between δ -function scatterers in the presence of a dc current, leading to a much more complicated background potential than the one corresponding to the constant electric field. However, our model must be considered as a first attack of the self-consistency problem in terms of a first-order self-consistent approximation to the potential. Throughout this paper, the emphasis is put on physical principles rather than accurate numbers.

The stationary Schrödinger equation for an electron in the wire ($0 \leq x \leq L$) is

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \sum_{j=1}^N H\delta(x-x_j) + E_0 + eF(L-x) \right] \psi(x) = E\psi(x), \quad (12)$$

where $E_0 = \frac{1}{2}(\Delta\mu - eV)$. Solutions to this equation will be obtained by using the transmission-reflection formalism. In this formalism there corresponds to each given energy E a 2×2 complex scattering matrix \mathcal{M} , which relates the plane-wave amplitudes on one side of the wire to those on the other side (Fig. 4)

$$\begin{pmatrix} A_r \\ B_r \end{pmatrix} = \mathcal{M} \begin{pmatrix} A_l \\ B_l \end{pmatrix}, \quad (13)$$

where it is noted that the wave numbers k_r and k_l are different according to the potential difference eV .

Quite generally, \mathcal{M} can be obtained as an ordered product of matrices \mathcal{M}_j of the individual δ -function scatterers and matrices \mathcal{T}_j which account for variations in amplitude and phase in the areas of constant electric field between the scatterers,

$$\mathcal{M} = \mathcal{M}_N \mathcal{T}_{N-1} \mathcal{M}_{N-1} \cdots \mathcal{M}_2 \mathcal{T}_1 \mathcal{M}_1. \quad (14)$$

The scattering matrix for a δ function of weight H_j and local wave number

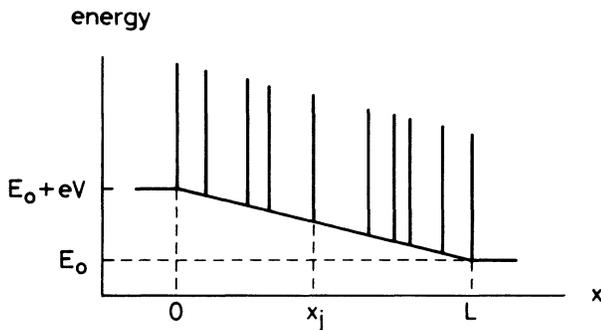


FIG. 3. Potential energy of an electron in the model wire used in the numerical calculations. The vertical bars denote δ -function potentials $H\delta(x-x_j)$, where the positions x_j are random. The energy E_0 equals $\frac{1}{2}(\Delta\mu - eV)$ and is chosen such that the potential in the right-hand reservoir equals zero.

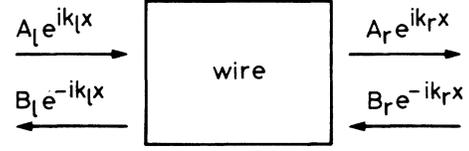


FIG. 4. Illustrating the model wire of Fig. 3 in the reflection-transmission formalism. The wave vectors are given by $k_l = [(2m/\hbar^2)(E - E_0 - eV)]^{1/2}$ and $k_r = [(2m/\hbar^2)(E - E_0)]^{1/2}$, where E_0 is defined in Fig. 3.

$$k_j = [(2m/\hbar^2)(E - \frac{1}{2}\Delta\mu - \frac{1}{2}eV + eFx_j)]^{1/2},$$

where x_j is the position of the scatterer ($x_1=0; x_N=L$), can be written as

$$\mathcal{M}_j = \begin{pmatrix} 1 - i\beta_j & -i\beta_j \\ i\beta_j & 1 + i\beta_j \end{pmatrix} \quad (j=1, \dots, N) \quad (15)$$

with

$$\beta_j = mH_j / (\hbar^2 k_j). \quad (16)$$

The matrix for the translation from x_j to x_{j+1} in a constant electric field F has been derived in a Wentzel-Kramers-Brillouin (WKB) approximation,²¹

$$\mathcal{T}_j = \begin{pmatrix} \exp(i\phi_{j,j+1}) & 0 \\ 0 & \exp(-i\phi_{j,j+1}) \end{pmatrix}, \quad (17)$$

where

$$\phi_{j,j+1} = \frac{2mE_1}{\hbar^2} \left[L_j + \frac{eF}{4E_1} L_j^2 \right] \quad (L_j = x_{j+1} - x_j), \quad (18)$$

and $E_1 = E - \frac{1}{2}\Delta\mu - \frac{1}{2}eV$. For the present purpose of giving a qualitative analysis with emphasis on principles, the Wentzel-Kramers-Brillouin approximation (18) is satisfactory since it adequately accounts for the change in effective path length due to the electric field. Once the matrix \mathcal{M} has been obtained, the transmission probability T can easily be determined by [see also Sec. IV, Eqs. (22) and (24)]

$$T^{-1} = \frac{k_r}{k_l} |\mathcal{M}_{11}|^2. \quad (19)$$

The calculational procedure is now as follows. For a wire of given length and given positions of δ functions, we calculate T using the zeroth-order approximation to the self-consistent voltage $V = V_0 = \Delta\mu/e$. Then we calculate the first-order approximation to the voltage difference by using (8), and repeat the scheme until satisfactory convergence for V is reached. In our calculations it was never necessary to do more than two iterations. With the self-consistent V we can now calculate the resistance by performing the energy integration in (10) and substitution of the result in (9).

Figure 5 shows for a given wire with 10 scattering centers the transmission probability T versus wave number $k = (2mE/\hbar^2)^{1/2}$ where the energy E varies in a small

region. In Fig. 5(a) no voltage is applied across the wire, whereas in Fig. 5(b) the voltage corresponds to 2.5% of the Fermi energy ($E_F = 2.7$ eV). Both characteristics are for the same geometry of scatterers; the field present in Fig. 5(b) has changed all effective lengths, thus leading to a similar, but markedly different T versus E characteristics.

The calculated effect of voltage on the resistance at zero temperature is depicted in Fig. 6 for three different wires of equal length, equal scatterer density, and equal scatterers but different random positions. Note that the horizontal (logarithmic) scale gives the applied chemical potential difference $\Delta\mu$ between the reservoirs. The actual voltage difference over the wire is simply related to $\Delta\mu$ by $eV = \Delta\mu / (1 + \pi\hbar/e^2R)$, as follows from (8) after substitution of $I = V/R$. All numerical results on resistance to be presented pertain to the situation with both reservoirs having zero temperature. Hence, the energy weight function $W(E)$ defined in (11) is just the normalized uniform distribution on $[\mu_r, \mu_l]$.

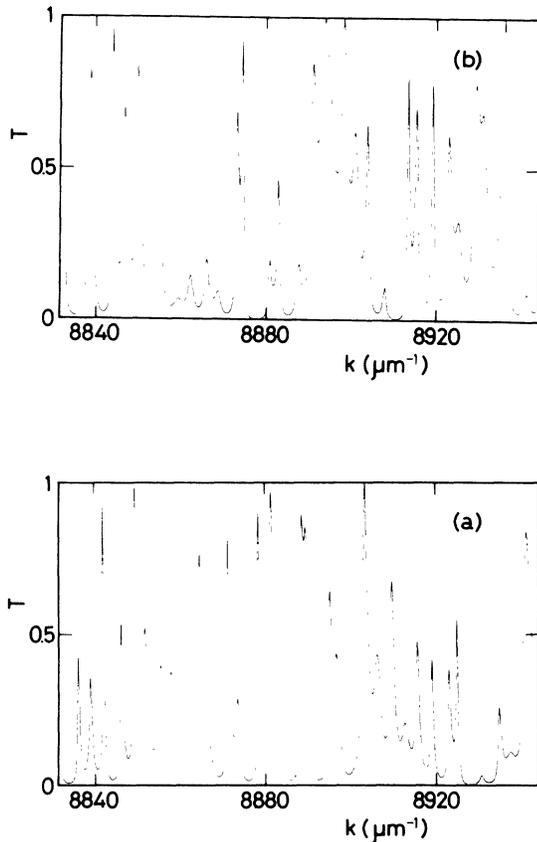


FIG. 5. Transmission probability T vs wave number $k = (2mE/\hbar^2)^{1/2}$, where E is the energy. The wire has length $L = 1 \mu\text{m}$ and contains 10 δ -functions at random positions and equal weight $H = 2.7$ eV \AA . (a) No voltage difference over the wire. (b) The voltage difference equals 67.5 mV. Due to the electric field present in (b), all effective lengths have changed, resulting in a different T - E characteristic.

At small voltage values the three resistances differ by as much as an order of magnitude. This phenomenon, i.e., the strong dependence of the resistance at $V = 0$ on the actual configuration of scattering centers, is a manifestation of the universal conductance fluctuation phenomena.⁷⁻⁹ For values of $\Delta\mu$ well below 1 meV, the resistance is independent of voltage. This is in agreement with the expectation that self-averaging effects should develop when $\Delta\mu$ becomes of the order of $[LN(E_F)]^{-1}$, where the latter is equal to 1.6 meV. For sufficiently large voltage the resistance values tend to approach each other and become less dependent on voltage. In fact, the voltage-induced averaging over energy dramatically improves the statistical distribution of resistance values over ensemble members, that is, the width of the distribution decreases with increasing voltage.

Also indicated in Fig. 6 is the scale resistance (dashed line) obtained at zero voltage from averaging $\ln(1/T)$ over 2000 ensemble members, i.e., the resistance introduced by Anderson *et al.*⁵ The corresponding value of 4.95 (in units $\pi\hbar/e^2$) is significantly larger than the mean resistance value (2.7 in these units) at large voltage. We conclude that the proper inclusion of transmission channel

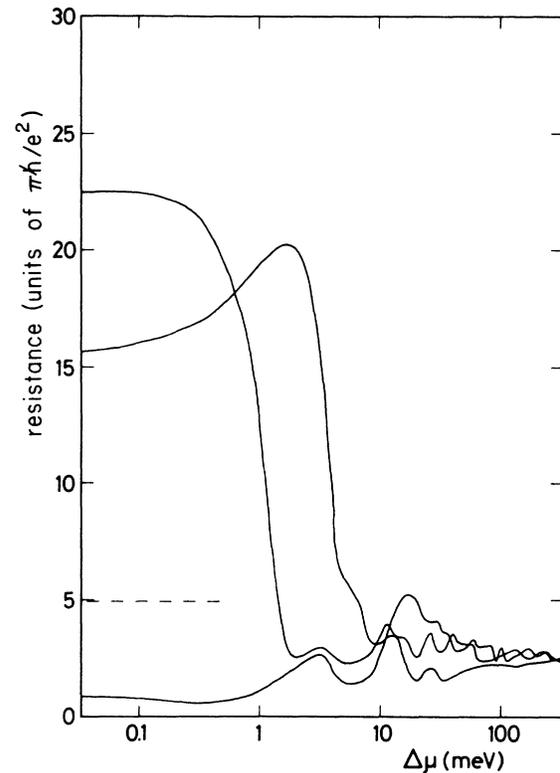


FIG. 6. Resistance R vs applied chemical potential difference for three wires of equal length ($1 \mu\text{m}$), equal scatterer density (10^7 m^{-1}), equal scatterers ($H = 2.7$ eV \AA), equal Fermi energy ($E_F = 2.7$ eV), but different random scatterer positions. The dashed line indicates the scale-resistance value equal to 4.95 (in units $\pi\hbar/e^2$) obtained from the ensemble average of $\ln(1/T)$ at zero voltage.

broadening due to finite voltage difference leads to a substantial decrease of the quantum resistance in a disordered system.

IV. SCALING BEHAVIOR OF THE RESISTANCE

The scaling analysis of the resistance by Anderson *et al.*⁵ is based on the Landauer formula for infinitesimal voltage, i.e., Eq. (9) with \bar{T} equal to $T(E_F)$. The resistance values obtained with this formula are known to be highly sensitive to the precise position of the Fermi energy E_F . This is related to the statistical distribution of R values over an ensemble of similar wires not being a regular, but rather a singular one. In fact, it was shown by the authors of Ref. 5 that the ensemble distribution of $\ln(1+e^2R/\pi\hbar) = -\ln T(E_F)$ behaves much more regularly with its mean scaling linearly with length and its variance scaling no worse than linearly.

As far as we know, there exists as yet no quantum-resistance scaling theory which includes the effects of a finite potential difference. The idea, however, of a finite voltage influencing the scaling and other statistical properties of the resistance is most interesting, as it may extend our understanding or alter our view on the relevance of localization in disordered systems.

In the first instance, one might expect that due to self-averaging at larger voltages the resistance would exhibit linear scaling behavior, in which case we would have a simple Drude formula

$$R = (\pi\hbar/e^2)L/l_e, \quad (20)$$

with l_e the elastic mean free path, not to be identified with the localization length ℓ . In fact, it is noted in Ref. 5 [see Eq. (19) of this reference] that the result of averaging $T(E_F)$ may lead to perfect additivity, i.e., linear scaling, of the resistance. However, we will show that linear scaling will not occur when the electrons propagate coherently. The coherence is not destroyed by the field-induced averaging process, nor by having the reservoirs at finite temperature. The coherence can only be destroyed by introducing inelastic scattering events inside the wire.

In order to establish the scaling theory that is valid for sufficiently large potential differences, we first consider a wire segment of length L as illustrated in Fig. 7. The presence of a voltage difference over the system implies different wave numbers k_1 and k_2 at both sides. Therefore, we represent the wave function in region j ($j=1,2$) as

$$A_j \exp(ik_j x) + B_j \exp(-ik_j x).$$

The relation between (A_2, B_2) and (A_1, B_1) can be expressed as

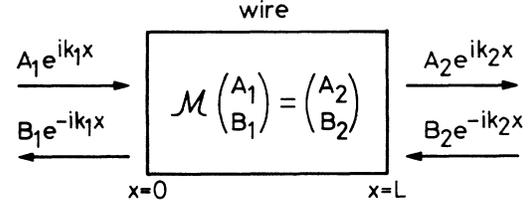


FIG. 7. Illustrating the transmission matrix \mathcal{M} .

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad (21)$$

where \mathcal{M} denotes a 2×2 matrix which summarizes all scattering properties at a given energy. The most general representation of \mathcal{M} (assuming time-reversal symmetry, i.e., no magnetic scattering centers) is

$$\mathcal{M} = \begin{pmatrix} 1/t & r/t \\ r^*/t^* & 1/t^* \end{pmatrix}, \quad (22)$$

where t and r are complex-valued quantities which satisfy the current-conservation condition

$$k_1 |t|^2 = k_2 (1 - |r|^2). \quad (23)$$

In terms of r and t the transmission and reflection probabilities are given by

$$\begin{aligned} T_{21} &= |t|^2 = (k_2/k_1)T \quad (\text{transmission from 2 to 1}), \\ R &= R_1 = R_2 = |r|^2 = 1 - T \quad (\text{reflection}), \end{aligned} \quad (24)$$

$$T_{12} = (k_1/k_2)^2 T_{21} = (k_1/k_2)T \quad (\text{transmission from 1 to 2}).$$

Note that the relationships are a little more complicated than usual due to the occurrence of different wave numbers k_1 and k_2 . In the absence of a potential difference we have $k_1 = k_2$, and this implies the usual transmission symmetry $T = T_{21} = T_{12}$.

Let us now add to the right-hand side of our wire a segment of length ΔL , which is sufficiently short for the corresponding transmission probability $T(\Delta L)$ to be independent of energy in the integration interval in (10) and such that

$$T(\Delta L) = 1 - \Delta R, \quad (25)$$

with $\Delta R \ll 1$. The transmission probability $T(L + \Delta L)$ for the total segment with length $L + \Delta L$ can be written as (see also Ref. 5)

$$T(L + \Delta L) = \frac{T(L)(1 - \Delta R)}{1 + [1 - T(L)]\Delta R + 2\{[1 - T(L)]\Delta R\}^{1/2} \cos\psi}, \quad (26)$$

where the angle ψ expresses phase information collected in the segment of length L .

Let us consider the E dependence of the right-hand side of (26). As a consequence of the random positions of scatterers, $\cos\psi$ will be a heavily fluctuating, more or less random, function of the energy E in the integration interval, at least when $\Delta\mu \gg [LN(E)]^{-1}$. Moreover, if we assume that the distribution of $\cos\psi$ values over energy can be considered as uncorrelated to the distribution of $T(L)$ values over energy, then we can perform two independent averaging procedures in order to arrive at an expression for $\overline{T(L + \Delta L)}$ in terms of $\overline{T(L)}$ and ΔR . Assuming that any ψ -value on the interval $[0, 2\pi]$ is equally probable, the ψ -averaging yields

$$\overline{T(L + \Delta L)}^\psi = \frac{T(L)[1 - \Delta R]}{1 - [1 - T(L)]\Delta R}. \quad (27)$$

Performing the energy averaging, i.e., the E integration of (27), we can express the result in lowest order ΔR as

$$\overline{T(L + \Delta L)} = \overline{T(L)} - \overline{T(L)^2} \Delta R,$$

from which we find that $1/\overline{T} - 1$ obeys the scaling relation

$$\frac{1}{\overline{T(L + \Delta L)}} - 1 = \frac{1}{\overline{T(L)}} - 1 + \Delta R + \frac{\overline{T(L)^2} - \overline{T(L)}^2}{\overline{T(L)}^2} \Delta R. \quad (28)$$

This result is most convenient as it expresses the amount by which the scaling is superlinear in terms of the variance of the $T(L)$ distribution. Hence, the only possibility for arriving at linear scaling, i.e., classical additivity, is when the variance vanishes. This would imply $T(L)$ to be fully independent of energy, which is impossible in the context of the model studied here.

We have not yet been able to find *general* properties for the variance of $T(L)$, i.e.,

$$\text{var} T(L) = \overline{T(L)^2} - \overline{T(L)}^2,$$

where we recall that the averaging is in fact energy integration according to (10). However, we have deduced properties which the variance should have in order to lead to simple scaling behavior. To be precise, if the quantity f defined by

$$f = \frac{\overline{T(L)^2} - \overline{T(L)}^2}{\overline{T(L)}[1 - \overline{T(L)}]} \quad (29)$$

would (for some reason not yet understood) be independent of length L , then (28) can be written as

$$\rho(L + \Delta L) - \rho(L) = [1 + f\rho(L)]\Delta R(L), \quad (30)$$

where $\rho(L) = 1/\overline{T(L)} - 1$. Transforming (30) into a differential equation, we obtain

$$\frac{d\rho}{dL} = [1 + f\rho(L)]\alpha(L), \quad (31)$$

with

$$\alpha(x) = \lim_{\Delta L \rightarrow 0} \frac{\Delta R(x)}{\Delta L}. \quad (32)$$

The solution to (31) is given by

$$\rho(L) = \frac{1}{f} (e^{\int_0^L dx \alpha(x)} - 1). \quad (33)$$

The interesting thing about this result is that it can be related to the Anderson localization length ℓ by writing

$$\begin{aligned} \int_0^L dx \alpha(x) &\approx L \langle \alpha \rangle \\ &= \lim_{x \rightarrow 0^+} \frac{L}{x} \langle \Delta R \rangle \\ &= - \lim_{x \rightarrow 0^+} \frac{L}{x} \langle \ln T(x) \rangle = L/\ell. \end{aligned}$$

Hence, we may write (33) as

$$\rho(L) = \frac{1}{f} (e^{fL/\ell} - 1). \quad (34)$$

Equation (34) is a very interesting scaling relation, since it unifies three different scaling laws into one form. The scaling law due to Landauer^{4,5} is obtained by putting $f=2$ in (34), while the scale resistance derived by Anderson *et al.*⁵ corresponds to $f=1$. In the case here studied we always have $0 < f < 1$, as can easily be shown from (29), by using $0 \leq T(L) \leq 1$. Of course, the unification aspect of (34) is only of formal significance.

Let us recall that (34) is based on the assumption that f defined by (29) is independent of length. We were not able to find a general proof for this property, but numeri-

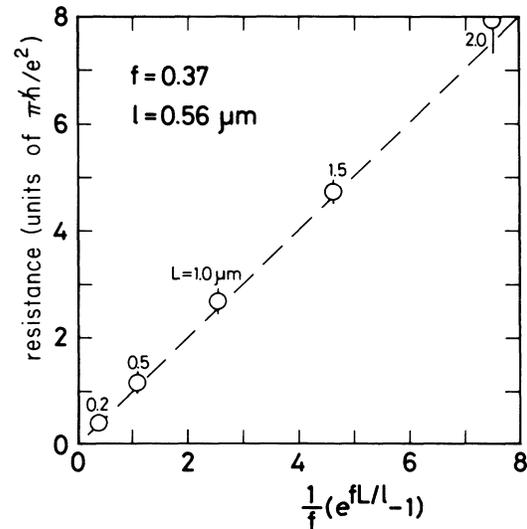


FIG. 8. Comparison between calculated resistance values (vertical axis) and the predictions based on the high-voltage scaling law (34) (horizontal axis), for systems of several different lengths ranging from 0.2 to 2 μm as indicated. The scatterer density is 10^7 m^{-1} in all cases. The scaling indicator f was determined numerically (see text) at $f=0.37$.

cal calculations on model systems of variable length with randomly placed δ -function potentials in the presence of a finite voltage difference indeed indicate that f is a constant, independent of L . Moreover, consistent with the above-given theory, the resistance scaling law (34) was numerically confirmed for these model systems.

In Fig. 8 the average resistance and standard deviations (six members each) are plotted versus the quantity $f^{-1}[\exp(fL/\ell)-1]$, for the L values indicated. Here, $\ell=0.56 \mu\text{m}$ while f was determined directly from transmission data for each length separately using (29), whereafter all f values thus obtained were averaged again, yielding $f=0.37$. The data depicted in Fig. 8 reveal the remarkably good agreement between the numerical results obtained straightforwardly using (9) and the theoretical scaling prediction (34). It should be mentioned, however, that we have not found the reason why the scaling indicator f , defined by (29), is a characteristic quantity not depending on length.

V. SUMMARY AND CONCLUSIONS

With the purpose of studying the voltage-dependent resistance of a one-dimensional disordered system, we have derived a straightforward extension of the Landauer resistance formula. In a model system with δ -function scattering centers and a constant electric field we have found significant dependences of the resistance on increasing voltage, ranging from decreasing resistance fluctuations to an appreciable decrease of the scale resistance. A finite voltage difference introduces self-averaging over energy and, therefore, may lead to a sharply peaked and well-behaving resistance distribution over ensemble members. Exponential scaling of the mean resistance with system length at large voltage difference has been obtained while the corresponding scale length is markedly larger than the zero-field scale length, i.e., the localization length. A scaling theory has been presented which is valid in the large voltage regime where sufficient self-averaging occurs, i.e., $eVN(E_F)L \gg 1$. A simple universal scaling formula is derived which combines the existing exponential scaling laws and the one here obtained in one formal expression.

Let us discuss how our findings compare with reported results on electric-field-induced effects in the stationary state.^{10-12,17} In none of these works the aspect of integration over a finite energy interval was considered; all references discuss the influence of an electric field on the localization of wave functions. They agree that an increasing electric field induces a crossover from exponential localization at zero field to power-law localization at large fields. References 11 and 17 have also dealt with the field-dependent resistance which they evaluated, how-

ever, using the zero-field Landauer formula. It is not surprising then that the scaling of resistance thus obtained strongly resembles the above localization crossover from exponential scaling at zero field to power-law scaling at large fields.

The reported field-induced effects in the transmittivity typically are eV/E_F effects, that is, one needs to consider rather large fields in order to see anything happen. In our view, the discussion of the influence of these effects on the resistance suffers from incompleteness so long as the intrinsic energy averaging associated with a finite voltage is not included. That is what we have done here and we have shown that the averaging leads to substantial effects on resistance which typically go with $eVN(E_F)L$, where $N(E_F)$ is the density of levels per unit length at the Fermi energy. Hence, since $N(E_F)L \gg E_F^{-1}$, the voltage-averaging effects and the resistance saturation predicted by us will have developed long before a power-law scaling will become manifest. The approach followed by us is closely related both to the work by Sinkkonen¹³ and Eränen and Sinkkonen¹⁴ and to the work by Hu.²⁰ However, these authors seem to have had other applications in mind as they do not mention the self-averaging in a disordered system due to a finite voltage difference.

Finally, let us attempt to extrapolate our findings to real, i.e., 3D systems. One has to be careful in doing so, because 1D systems are very special in view of their discrete spectral properties. As compared to 1D systems, 2D and 3D systems already have intrinsic self-averaging behavior, which is due to the naturally available energy degeneracy, even for small voltage. This is the main reason for strong localization in a random potential in 2D or 3D being hard if not impossible to realize. One aspect of electronic coherence which is nevertheless present in real samples is the occurrence of universal conductance fluctuations, i.e., the conductances in real, macroscopically identical, samples show large variations from one sample to the other in a universal manner. We may expect from our findings for 1D systems that these sample-to-sample variations will diminish and gradually disappear due to the energy averaging on increasing the voltage difference. This is not to say that the conduction will become diffusive, since there will still be a quantum-interference contribution to the resistance which cannot be accounted for in a classical Drude-like approach in which subsequent scatterings are uncorrelated events.

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*Present address: Research Institute for Materials, University of Nijmegen, Toernooiveld, NL-6525 ED Nijmegen, The Netherlands.

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