Modeling and numerical implementation of managed pressure drilling systems for the assessment of pressure control systems

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Abstract

Automated Managed Pressure Drilling (MPD) is a method to enhance downhole pressure control performance and safety during drilling operations. It is becoming more common to use model-based simulation for the evaluation of pressure control systems designed for MPD automation before using those in the field. This demands a representative hydraulics simulation model which captures the relevant aspects of a drilling system. This paper presents such a model and, additionally, an approach to numerically implement that model for simulation studies. The complexity of this simulation model should be limited to, firstly, support effective numerical implementation and, secondly and most importantly, to allow for the analysis of the behaviour and performance of the automated pressure control systems during the controller design phase. To this end, aspects of a drilling system that can considerably affect the performance of the automated MPD system are captured in the model. This hydraulics model incorporates both the distributed and multi-phase flow nature of a drilling system. Moreover, it captures nonlinear boundary conditions at the inlet of the drillstring, at the drill bit and choke manifold, and also the variations in the cross-sectional area of the flow path. Model validations against field data from real-life MPD operations and simulations of industry-relevant scenarios indicate that these aspects are effectively captured in the model and preserved during the numerical implementation.

Introduction

Conventionally, the task of pressure control is accomplished by changing the mud density during drilling operations. However, this method of controlling the pressure is slow and inaccurate, and it lacks a means of compensating and responding to transient pressure fluctuations (e.g., occurring during pipe connection operations or drilling into high-pressure formations). Besides, this method cannot be used for the drilling of deep wells with narrow drilling windows because of its low accuracy. To overcome such drawbacks of conventional pressure control methods, managed pressure drilling (MPD) has been introduced. A main objective of MPD is to provide a means of fast, accurate and efficient control of the bottom-hole pressure (BHP), as opposed to conventional methods. As we illustrate in Fig. 1, in MPD, the annulus is sealed off at the top with a rotating control device to direct the mud flow from the annulus to a choke valve with a variable opening (see, e.g., Stamnes et al. (2008); Godhavn (2010)). This equipment, which is often accompanied by a back-pressure pump, pressurizes the fluid inside the wellbore by providing an active back pressure. The back pressure, and thus the BHP, can be controlled by manipulating the choke opening. In automated MPD systems, the task of manipulating the choke opening is primarily performed by an automatic pressure control system. This enhances safety and performance, and reduces drilling time and cost, see (Godhavn et al. 2011). In particular, if equipped with advanced control systems, automated MPD can make it possible to handle many well-control events automatically without operator intervention and using conventional well-control methods (Berg et al. 2019a).

A control system designed for automated MPD should pass some virtual and representative test scenarios on a simulation level before it can be used in the field. This is done because any failure in the drilling system, especially in the pressure control system, can have catastrophic consequences. Training new operators for drilling operations and well control incidents in a controlled environment and also well monitoring are other important reasons for performing model-based simulation studies in drilling. (Cayeux et al. 2012; Vishnumolakala et al. 2015). However, simulations performed for training purposes often need to be well-supported by graphical interfaces, which is not the case when it comes to controller design. These simulations rely on a mathematical model of the drilling system dynamics, the complexity of which varies depending on the required purpose. In particular, the complexity of an MPD model developed for testing pressure controllers should be limited to facilitate the performance analysis and design of the control system by neglecting less important system aspects. Such a model, called the simulation model in this context, should only contain: 1) aspects of a drilling system considered in the controller designed and, 2) aspects that are not considered in the controller design but can have detrimental effects on the performance of the controller. Models used in the controller design are called the design models. A design

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model is often much simpler than a simulation model, as it usually contains only the mass transport dynamics, neglecting the distributed nature of drilling systems; see, e.g., (Nygaard and Nævdal 2006; Kaasa et al. 2012; Pedersen et al. 2018; Di Meglio et al. 2009; Aarsnes et al. 2016b). Below are listed a number of the drilling aspects that can be detrimental to an automated MPD system.

- **Pressure wave propagation:** pressure controllers are usually developed based on simplified models in which pressure dynamics (i.e., wave propagation effects) are ignored partially (Landet et al. 2013) or totally (Stamnes et al. 2008; Kaasa et al. 2012; Nikoofard et al. 2017; Aarsnes et al. 2016b; Ambruš et al. 2016; Pedersen et al. 2018; Di Meglio et al. 2009). The essential time scale associated to these dynamics can be in the range of tens of seconds, or even minutes in the case of gas influx into the wellbore. Not only can such dynamics deteriorate the control performance, but they can also cause instability (Pedersen et al. 2015), if not accounted for during the controller design stage;

- **Dynamics of the flow in the drillstring:** in many cases, e.g., in (Aarsnes et al. 2016a), the dynamics of the flow in the drillstring (as opposed to the flow through the annulus), whether fast or slow, are ignored throughout the controller design stage. This part can, however, have a significant contribution to the system behaviour, e.g., by changing the location of the major resonance frequencies of the system or generating additional resonance frequencies. Thus, the closed-loop performance in practice can be worse than expected from simulation studies if system aspects imposed by the drillstring are ignored in the design model;

- **Nonlinear behaviour of the drill bit:** once the drillstring flow path is ignored in the design model, the drill bit is replaced by an independent source of flow. The flow through the bit is, however, nonlinearly dependent on the pressure drop over the bit. In particular, in the case of standpipe pressure control during gas influxes, this nonlinearity can be detrimental.

- **The variable structure (i.e., switching nature) of the model:** this variable structure is induced mainly by the presence of a non-return valve in the bottom-hole assembly. During operations such as pipe connection, the non-return valve usually remains closed, changing the system properties and behaviour;

- **Lastly, variations in the cross-sectional area of the flow path:** these variations, especially those in the annulus, may have significant contribution to frequency responses of a drilling system. Therefore, if not included in the design model, these aspects may compromise the control performance.

Other dynamical effects, such as temperature transients, evolve so slowly, see, e.g., Fig. 14 in (Cayeux et al. 2012), that these can be neglected in the simulation model. These effects may be modelled in terms of uncertainties in the system parameters. Effects such as well expansion can also be approximately lumped into the system parameters, see, e.g., (Berg et al. 2019b).

In this paper, we rely on physical simplifications to derive a simulation model for MPD which is consistent with the modelling choices above. The simulation of the resulting model is based on a numerical discretization method. As these numerical methods are incapable of exactly preserving all the characteristics of the model, particular care should be taken in choosing, developing and using these numerical methods. Thus, we also provide a dedicated numerical approach for the simulation of this model.

The majority of existing hydraulics simulation models for drilling are single-phase models, often based on the (isothermal) Euler model, (Landet et al. 2012; Naderi Lordejani et al. 2018), and two-phase models, which are often based on either the two-fluid model or the Drift-Flux Model (DFM), see, e.g., (Strecker et al. 2017; Lage et al. 2003; Nygaard and Nævdal 2006; Shekari et al. 2013; Udegbunam et al. 2015; Aarsnes et al. 2016d; de Carvalho et al. 2019). In MPD modelling, it is key to specify the boundary conditions of the system carefully because exactly these boundary conditions differentiate MPD from conventional drilling methods and are means of applying control inputs. Moreover, one should consider the fact that the flow path in a drilling system, from the rig pump to drill bit all the way up to the choke, experiences discontinuities in its cross-sectional area. These discontinuities have a considerable contribution to the transient and steady-state behaviour of the flow and pressure along the flow path. These issues have not been addressed adequately in the literature. Lage et al. (2003) showed, by means of simulations and experiments on test wells, the predictive capabilities of the DFM for drilling operations. Though providing an advanced numerical setting for the simulation of their model, they performed their studies in a conventional drilling setting, not MPD. They did also not consider variations in the cross-sectional area of the flow. The work by Wei et al. (2018) is similar to (Lage et al. 2003), but the drilling model is solved with a different numerical scheme to improve the solution accuracy. Aarsnes et al. (2016b) used the DFM as a basis for simulating the annulus, testing controllers and model complexity reduction for MPD. The issue of variations in the cross-sectional area was, however, not addressed. A similar hydraulics model was studied in
Udegbunam et al. (2015), where only the annulus was considered in the model and an extrapolation method was used for solving the considered boundary conditions for simulations. Shekari et al. (2013) used the two-fluid model for the simulation of the flow in the annulus. Cayeux et al. (2012) provided a survey on advances in drilling simulators, but no technical details were given on the underlying mathematical models. In this paper, we propose a model for two-phase MPD drilling operations based on the DFM. This model takes all the relevant aspects, mentioned above, into account.

**Contributions.** The main contribution of this paper is a simulation platform suitable for evaluating controller performance for MPD operations, which includes both the physical model and the tool to implement it. In particular, a comprehensive formulation of a hydraulics model for MPD-relevant two-phase flow drilling scenarios is presented first. In this model, interactions between the different parts of a drilling system are formulated in terms of boundary conditions. The complexity of the model is limited to contain control relevant hydraulics aspects of a drilling system dynamics which can in some way be detrimental to the closed-loop performance of an automated MPD. The developed model allows for the simulation of many drilling scenarios ranging from making pipe connections, choke plugging and choke swapping, and bit nozzle plugging to liquid and gas influx scenarios. Then, we provide a numerical approach to support simulation tooling for fast scenario testing. In particular, we adapt a characteristics-based method to solve the nonlinear and boundary conditions, and also propose a dynamical model for the drill bit to circumvent numerical issues which appear at low pump flow rates. Since the effects of variations in the cross-sectional area of the flow path can be significant both on transient and steady-state response of the system, we explicitly address these variations during the numerical implementations of the model. Finally, we validate the hydraulics model for relevant MPD scenarios of single-phase flow by comparing it with a set of field data obtained from commissioning tests of an automated MPD system.

**Outline.** The next section is devoted to providing a short introduction to the DFM. Next, the mathematical modelling of the system is discussed. The steady-state solution of the model is discussed afterwards. After completing the modelling part, we present a numerical approach for the implementation of the model, which is later illustrated by means of a simulation study.

**Drift-flux model**

This section provides a short introduction to the DFM (Gavrilyuk and Fabre 1996) as it is the cornerstone of the MPD model to be developed. Flow behaviour in a transmission line can, to some extent, be described by the DFM. Because of its relative simplicity yet favourable capabilities in capturing the pressure and mass transport dynamics of two-phase flows, the DFM is probably the most widely used model in literature on control and simulation of two-phase drilling scenarios (Aarsnes et al. 2016d). Consisting of two mass conservation equations and one combined momentum conservation equation, the DFM reads as

\[ \frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = S, \]  

(1)

with

\[ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} := \begin{bmatrix} \rho_l \alpha_l \\ \rho_g \alpha_g \\ \rho_l \alpha_l v_l + \rho_g \alpha_g v_g \end{bmatrix}, \quad f := \begin{bmatrix} \rho_l \alpha_l v_l \\ \rho_g \alpha_g v_g \\ \rho_l \alpha_l v_l^2 + \rho_g \alpha_g v_g^2 + p \end{bmatrix}, \quad S := \begin{bmatrix} \Gamma \\ -\Gamma \\ s(u(t,x)) \end{bmatrix}, \]  

(2)

Here, \( x \in (0, L) \) and \( t > 0 \) are the spatial and time variables, respectively, with \( L \) the length of the computational domain and the well in this case. The volume fraction, density, velocity and pressure are denoted by \( \alpha = \alpha(t,x), \rho = \rho(t,x), v = v(t,x) \) and \( p = p(t,x) \), respectively, where the subscript \( l \) denotes the liquid phase and \( g \) refers to the gas phase. The vector of primitive variables (individual variables which have a clear physical meaning) is indicated by \( u = [\alpha_l, \rho_l, v_l, \alpha_g, \rho_g, v_g, p]^T \), while \( q \) represents the vector of conservative variables, and \( f(\cdot) \) is the flux function. The source term is represented by \( s(u(t,x)) \), and \( \Gamma_l \) and \( \Gamma_g \) model mass exchange between the phases, which are often assumed to be zero. The source term consisting of a gravitational and frictional term is given by

\[ s(u(t,x)) = \rho_m v_m |v_m|, \]  

(3)

where \( g \) and \( \theta(x) \) are the gravitational acceleration and the inclination of the transmission line with reference to the horizontal direction and \( D \) is hydraulic diameter. Moreover, \( \rho_m = \alpha_l \rho_l + \alpha_g \rho_g \) and \( v_m = \alpha_l v_l + \alpha_g v_g \) are the mixture density and velocity, respectively. Here, \( \nu \) is the Fanning friction factor which is in general a challenging parameter to determine. It is well-known that drilling muds in general...
exhibit non-Newtonian behaviours (Reed and Pilehvari 1993). There are a number of models describing these types of flows such as the Herschel-Bulkley, Bingham plastic and the Power Law model. Of these three, the three-parameter model of Herschel-Bulkley is the most accurate, as it includes the other two models as special cases. However, the respective equations are highly nonlinear and challenging to solve and, moreover, complex models with too many parameters are less useful from a control and estimation perspective. Here, we adapt the two-parameter Power Law model to trade off between complexity and accuracy. In this model, we define the Generalized Reynolds number as \( \text{Re} = \rho_m v_m D_{eff} / \mu_m \cdot \mu_{app} \), where \( D_{eff} = 4n_m D / (3n_m + 1) \) is the effective diameter and

\[
\mu_m = \mu_m^0 \left( \frac{3n_m + 1}{4n_m} \right)^{n_m - 1}
\]

is the apparent mixture viscosity (Reed and Pilehvari 1993). Here, \( n_m = \alpha_l n_l + \alpha_g \), with \( n_l \) the liquid behaviour index, is the mixture behaviour index, and \( \mu_m = \mu_l m_l + \mu_g m_g \), with \( \mu_l \) and \( \mu_g \) the liquid and gas viscosity, respectively, is the mixture viscosity. The Fanning friction factor for laminar flow, when \( \text{Re} < 3250 - 1150n_m \), is given by

\[
v = \frac{16}{\text{Re}^{0.5}}
\]

while for the turbulent flow, when \( \text{Re} > 4250 - 1150n_m \), \( v \) is the solution to

\[
\frac{1}{\sqrt{v}} + 4 \log \left( \frac{D_{eff}}{\nu} \left( \frac{D_{eff}}{\nu} \frac{\rho_m}{\mu_m} \right)^{0.75} \right) = 0,
\]

where \( \nu \) is the pipe roughness (Reed and Pilehvari 1993). As can be seen, this equation is highly nonlinear and no exact explicit solution is currently available to it, thus we use an approximate solution, see Appendix A. For transition flow, when \( \text{Re} \in [3250, 4250] - 1150n_m \), we compute \( v \) by a linear interpolation from (5) and (6). We stress again that one may use a different frictional model depending on the application and required accuracy, see (Livescu 2012) for a review of these models.

**Remark 1.** The Power Law model is in general considered as complex when it comes to the design of controllers and estimators for MPD automation, given that it is far less common compared to simpler Newtonian models used for these purposes.

**Remark 2.** Using mixture parameters and variables \( \mu_m, \rho_m \) and \( v_m \) is a common approach to extend liquid frictional models to two-phase flows. Likewise, we have used a mixture behaviour index \( n_m \) to be able to use the Power Law model for two-phase flow scenarios.

The DFM is completed by four other equations to be, potentially, solvable uniquely. These, often known as the closure laws, can be expressed as follows:

\[
\begin{align*}
p - p_0 - c_l^2 (p_l - p_0) &= 0, \quad (7a) \\
p - c_g^2 p_g &= 0, \quad (7b) \\
\alpha_l + \alpha_g - 1 &= 0, \quad (7c) \\
v_g - v_l - \Phi(u) &= 0, \quad (7d)
\end{align*}
\]

where \( p_0, p_l, c_l \) and \( c_g \) are the reference pressure, liquid reference density, sound velocity in the liquid and sound velocity in the gas, respectively. The first two equations are, respectively, known as the equations of state for the liquid and gas. An equation of state describes the state of matter in terms of physical variables such as temperature, density and pressure. Equations of state can be rather complex in general, but we here use linear variants approximating only the relation between pressure and density, as in (7a) and (7b). We note that these equations still capture the liquid and gas compressibility. Moreover, the volume balance between the phases is imposed by (7c), and the slip law (7d) describes the relative velocity between the two phases depending on the function \( \Phi(\cdot) \). Here, the slip law is given by (Ishii and Hibiki 2011)

\[
\Phi(u) = C_0(u) v_m + V_d(u) - v_l
\]

where \( C_0(\cdot), V_d(\cdot) \) are the distribution parameter and drift velocity, respectively. Several descriptions, which are mostly obtained based on experiments and function fitting, for these parameters can be found in the literature. Depending on well conditions, mud
properties and expected drilling scenarios, a particular description of these parameters can be selected to be used in the hydraulics model, assuming only bubbly and slug flows. For a review of a variety of descriptions for these parameters and their validity conditions, see (Bhagwat and Ghajar 2014).

It can be shown that the DFM admits three distinct eigenvalues in a wide region of the variable space, see (Evje and Fjelde 2002). These eigenvalues are $\lambda_1 = v_g$, $\lambda_2 = v_g - c_m$, $\lambda_3 = v_g + c_m$, with $c_m(u)$ the sound velocity in the mixture. Currently, no exact analytical expression is available for $c_m(u)$. Thus, we use an approximation of $c_m(u)$ based on the local definition of the bulk moduli, as follows:

$$c_m(u) = \sqrt{\frac{\beta_m(u)}{\rho_m(u)}}, \quad \text{................................................................. (9)}$$

where $\beta_m$ is mixed bulk modulus, defined as $\beta_m = \frac{\beta_f \rho_f}{\rho_g^2 + \alpha_g \rho_f^2}$, with $\beta_g = p$ and $\beta_l = c_l^2 \rho_l$ the bulk moduli of gas and liquid, respectively (Aarsnes et al. 2016a; Kaasa et al. 2012).

**MPD Description and Hydraulics Modelling**

Consulting Fig. 1, an MPD system may be simply regarded as two equivalent hydraulic transmission lines (or simply pipes) which are connected through a drill bit in the middle and one of which ends up with a controllable choke valve. The exposed zone of the annulus (so-called open-hole section) is susceptible to gas and liquid influx from the surrounding formations that may potentially contain hydrocarbons. Therefore, to have a good description of the flow and pressure transients along the flow path, it is necessary to use a multi-phase flow model for the annulus. However, except in some specific drilling operations, such as operations performed in underbalanced drilling, it is quite reasonable to use a single-phase flow model to describe the flow in the drillstring.

**Hydraulics modelling.** The DFM, as in (1), can be used only for the description of the flow lines with constant cross-sectional area, while in practice there are variations in the cross-sectional area of the flow path, due to changes in the diameter of pipes and open hole, that impact the flow behaviour. This urges the use of a modified version of the DFM that accounts for the variations in the cross-sectional area. The modified DFM for the annulus reads as follows (Rommelset and Vefring 1991):

$$\frac{\partial (A_u q_u)}{\partial t} + \frac{\partial (A_u q_u)}{\partial x} = A_u S_a + \frac{\partial A_u}{\partial x} P_a, \quad \text{................................................................. (10)}$$

where $A_u(x)$ is the cross-sectional area of the annulus and we have denoted $P_a := \{0, p_a\}^T$. In this context, a sub/superscript $a$ refers to the annulus and a $d$ to the drillstring. Compared to the model (1), changes in the cross-sectional area lead especially to the term $P_a \partial A_u / \partial x$. We should note that $t$, $x$, $q_u$, $f_u(\cdot)$ and $S_a$ are the same as in (1) computed from the variables in the annulus. In the annulus, as illustrated in Fig. 1, $x = 0$ marks the well bottom and $x = L$ a point in the annulus that is in the same level as the choke. Moreover, $D_a = D_m - d_u$, with $D_m$ the diameter of the annulus and $d_u$ the outer diameter of the drillstring, and $\theta_g(x) = -\theta(L-x)$.

As will be explained in later sections, we need to switch between the primitive variables $u_a$ and the conservative ones $q_a$ to numerically solve the model under development. Based on the closure laws in (7) and also the relation between the vectors $q_a$ and $u_a$, as in (2), we derive the following relation for the pressure in terms of conservative variables:

$$p_a = \frac{-b + \sqrt{b^2 - 4d}}{2}, \quad \text{................................................................. (11)}$$

where $b = \rho_o c_l^2 - \rho_o - q_{u,1} - c_g^2$ and $d = -(\rho_o c_l^2 - \rho_o) c_g^2 q_{u,2}$. Next, given the pressure from (11), (7a) and (7b) can be used to compute $\rho_l$ and $\rho_g$, respectively. Next, the definition of $q_a$ leads to expressions for the volume fractions:

$$\alpha_l = \frac{q_{u,1}}{\rho_l}, \quad \alpha_g = \frac{q_{u,2}}{\rho_g} \quad \text{................................................................. (12)}$$

Next, we can compute $v_l$ using

$$v_l = \frac{q_{u,3} (1 - C_0 \alpha_g) - q_{u,2} V_d}{q_{u,1} + C_0 \alpha_g q_{u,2}} \quad \text{................................................................. (13)}$$

Then from the slip law (7d), with (8), we obtain

$$v_g = \frac{C_0 \alpha_g v_l + V_d}{1 - C_0 \alpha_g} \quad \text{................................................................. (14)}$$
Remark 3. Depending on the choice of $C_0(u)$ and $V_d(u)$, if these parameters are dependent on $v_l$ and $v_g$, then (13) and (14) can become nonlinear equations with respect to $v_l$ and $v_g$ which need to be solved simultaneously using nonlinear solvers. However, these are often independent of $v_l$ and $v_g$.

As already mentioned, the main reason for using a two-phase model for the annulus is to enable modeling of a gas influx from the formation into the annulus. On the contrary, it is reasonable to use a single-phase model for describing the flow behavior inside the drillstring. It is worth mentioning that there are certain drilling operations, such as operations in underbalanced drilling, where some rate of gas is intentionally injected into the drillstring. This gas injection in turn gives rise to a two-phase medium in the drillstring. The following isothermal Euler equation accounting for the variations in the cross-sectional area describes the flow behavior in the drillstring:

$$\frac{d(A_d q_d)}{dt} + \frac{d f_d(A_d q_d)}{dx} = A_d S_d + \frac{d A_d}{dx} p_d,$$

where $f_d(A_d q_d) = [A_d \rho v, A_d \rho v^2 + A_d p_d]^T$, $S_d(u_d, t, x) = [0, s_d(u_d, t, x)]^T$, $P_d := [0, p_d]^T$. Moreover, $\rho = \rho(t, x), v = v(t, x), p_d = p_d(t, x)$ are the mud density, velocity and pressure profiles along the drillstring. The vectors of primitive and conservative variables are indicated by $u_d = [\rho, v, p_d]^T$ and $q_d = [\rho, p_d]^T$, respectively. Moreover, $A_d(x)$ is the cross-sectional area of the drillstring. For the drillstring, $x = 0$ marks a point in the drillstring which is at the same level as the pump whereas $x = L$ marks its outlet at the bit. To avoid unnecessary notational complexities, we do not use a subscript to refer to the primitive variables in the drillstring, except for the pressure. As before, $s_d$ is the source term acting on the flow in the drillstring, and the same model as in (3) is used to determine it, with the mixture variables and parameter reducing to liquid variables. Moreover, we have $D_d = d_{in}$, with $d_{in}$ the inner diameter of the drillstring, and $\theta_d(x) = \theta(x)$.

The equation of state considered for the liquid in the drillstring is the same as (7a), the one used in the annulus.

Remark 4. It should be noted that (15) may be obtained from (10) by setting $\alpha_g = 0$.

Boundary conditions To potentially be able to solve (10) and (15) uniquely, one needs to specify a set of boundary conditions. In this regard, a careful observation of Fig. 1 reveals that the hydraulics behavior of an MPD system is largely dictated by three main physical boundary conditions, which are the boundaries at the drillstring inlet, the bit together with the behavior of the formations around the open-hole, and the choke valve, as the annulus outlet.

The boundary condition at the drillstring inlet is expressed in a general form as follows:

$$f_{ibc}(\rho(t, 0), v(t, 0), t) = 0,$$

where $f_{ibc}(\cdot, \cdot)$ is the boundary condition at the drillstring inlet, and it is determined depending on the ongoing drilling operation. For example, during normal operations, when the drillstring is connected to the mud pump, we can define

$$f_{ibc} := A_d(0) \rho(t, 0) v(t, 0) - J_p(t),$$

where $J_p(t)$ denotes the mass flow rate of the mud pumped into the drillstring. It should be noted that in practice we are often provided the pump strokes per minutes $n_p(t)$ rather than the mass flow rate. In that case, the mass flow rate can be computed using $J_p(\rho(t, 0), t) = V_p n_p(t) \rho(t, 0)$, where $V_p$ is the volume that the pump sweeps per stroke. As another example, during a bleed-off operation, an operation to slowly release the trapped pressure within the drillstring before detaching it from the top-derive, a valve equation should be used to model this boundary condition. Next, at the bottom of the well, one can write three boundary equations, consisting of the bit equation that describes the liquid mass flow rate through the bit in terms of the pressure drop over the bit, the liquid mass balance equation between both sides of the bit and the gas balance between the formations and the annulus, respectively:

$$A_d(L) \rho(t, L) v(t, L) - c_d A_n \sqrt{2 \rho(t, L) r(p_d(t, L) - p_d(t, 0))} = 0,$$

$$A_d(L) \rho(t, L) v(t, L) + J_p(t, 0), p_r) - A_d(0) \alpha_l(t, 0) \rho(t, 0) v(t, 0) = 0,$$

$$J_g^e(p(t, 0), p_r) - A_d(0) \alpha_g(t, 0) \rho(t, 0) v(t, 0) = 0,$$

where the function $r(e) = \begin{cases} e, & \text{if } e > 0 \\ 0, & \text{if } e \leq 0 \end{cases}$ is used to model the non-return valve installed above the bit inside the Bottom-Hole Assembly (BHA), $A_n$ is the effective area of the bit nozzles and $c_d$ is the discharge coefficient. Also, $J_p^l(\cdot, \cdot)$ and $J_g^e(\cdot, \cdot)$ respectively represent the
mass flow rates of the liquid and gas exchanged between the well-bore and the formations with a pressure \( p_r \), known as the reservoir pressure. Here, we approximate these variables using a linear static reservoir model as follows (see, e.g., (Aarsnes et al. 2016c)):

\[
J^i_f = \kappa_i r(p_r - p_a(t,0)), \quad i \in \{g,l\},
\]

where \( \kappa_i \) is the production index for the phase \( i \). Coupling with a more intricate reservoir model is also possible and it can be substituted into (21). However, near well-bore reservoir modelling is out of the scope of this paper. At the top side of the annulus, the boundary condition is determined by the choke equation describing the mass flow rate of the mixture through the choke as a function of the pressure drop over the choke (see, e.g., (Di Meglio et al. 2011))

\[
J_c(t, u_a(t, L)) - J_{bpp}(t) - J_{w}(u_a(t, L)) = 0, \quad J_c(t, u_a(t, L)) = \sum_{i=1}^{n_c} k_{c,i} G_i(z_{c,i}(t)) \sqrt{2 p_m(t,L) r (p_a(t, L) - p_0)},
\]

where \( k_{c,i}, z_{c,i} \) and \( G_i(t) \) are the choke flow factor, the choke opening and the choke characteristic of the choke valve \( i \), respectively. Here, \( n_c \) is the number of choke valves in the MPD set up and \( J_{bpp}(t) \) is the mass flow rate from the back-pressure pump. Moreover, \( J_c^g = A_g(L) \alpha_g(t, L) \rho_g(t, L) v_g(t, L) + A_g(L) \alpha_g(t, L) \rho_g(t, L) v_g(t, L) \) is the mass flow rate upstream the choke whereas \( J_c \) is that downstream the choke. Again, more accurate models of multi-phase flow through valve can be derived to replace (22).

The combination of Eqs. (7)-(22) constructs our MPD simulation model. Specifically, in this model, we have accounted for variations in the cross-sectional area of the flow path and also the nonlinear boundary conditions of an MPD system. Now the MPD model has been specified, and next the steady-state solution of the system can be found based on this model.

**Steady-state solution of the model**

Clearly, to be able to solve the MPD model derived in the previous section, an initial condition is required. As most of the drilling time is occupied by normal drilling operation, it is reasonable to start a simulation study from a drilling ahead condition. In this case, the system shows a steady-state behaviour; therefore, all the derivatives with respect to the time variable \( t \) may be discarded from the Equations (10) and (15), resulting in the steady-state differential equations

\[
\frac{df_a(A_a q_a)}{dx} = A_a \ddot{s}_a + \frac{\partial A_a}{\partial x} \ddot{p}_a, \\
\frac{df_d(A_d q_d)}{dx} = A_d \ddot{s}_d + \frac{\partial A_d}{\partial x} \ddot{p}_d.
\]

Now, using the closure laws (7) together with (23), we obtain

\[
\frac{d\ddot{y}_a(x)}{dx} = M_a^{-1}(\ddot{y}_a) H_a(\ddot{y}_a, x), \\
\frac{d\ddot{y}_d(x)}{dx} = M_d^{-1}(\ddot{y}_d) H_d(\ddot{y}_d, x),
\]

where \( \ddot{y}_a(x) = [\ddot{v}, \ddot{p}]^T, \ddot{y}_d(x) = [\ddot{v}, \ddot{p}, \ddot{\alpha}, \ddot{\bar{p}}]^T \), with the a bar (i.e., \( \bar{\cdot} \)) indicating the variables and vectors in the steady-state. Moreover,

\[
M_a = \begin{bmatrix}
1 - \frac{\partial \Phi}{\partial \rho_g} & -1 & - \frac{\partial \Phi}{\partial v_l} & - \frac{\partial \Phi}{\partial p_l} \\
(1 - \alpha_g) \bar{p}_l & 0 & - \bar{p}_l \bar{v}_l & - \frac{\partial \phi}{\partial \bar{p}_l}
\end{bmatrix}
\]

\[
\frac{d\ddot{y}_a}{dx} = \ddot{S}_a - \frac{d A_a}{dx} \begin{bmatrix}
0 \\
\frac{(1 - \alpha_g) \bar{p}_l \bar{v}_l}{\alpha_g \bar{p}_l \bar{v}_l}
\end{bmatrix}
\]

\[
H_a = \ddot{S}_a - \frac{d A_a}{dx} \begin{bmatrix}
\ddot{\bar{p}} \\
0
\end{bmatrix}
\]

\[
M_d = \begin{bmatrix}
\ddot{\bar{p}} & \ddot{\bar{v}} & 0
\end{bmatrix}
\]

\[
\frac{d\ddot{y}_d}{dx} = \ddot{S}_d - \frac{d A_d}{dx} \begin{bmatrix}
\ddot{\bar{p}} \\
0
\end{bmatrix}
\]
with $\Phi(\mathbf{u})$ as in (8). The boundary conditions of the ordinary differential equation (24) are given by the physical boundary conditions (17), (18) and (22) in the steady-state conditions:

$$
(A_u \bar{p} \mathbf{v})|_{x=0} - J_p = 0,
$$
$$
(A_u \bar{p} \mathbf{v})|_{x=L} - (A_u (1 - \alpha_g) \bar{p} \mathbf{v})|_{x=0} + J_{L}^f (\bar{p}_u(0), \bar{p}_r) = 0,
$$
$$
J_{L}^p (\bar{p}_u(0), \bar{p}_r) - (A_u \bar{g} \bar{p}_g \bar{v}_g)|_{x=0} = 0,
$$
$$
(A_u \bar{p} \mathbf{v})|_{x=L} - A_n c_d \sqrt{2\bar{p}(L) r (\bar{\rho}_d(L) - \bar{\rho}_u(0))} = 0,
$$
$$
(A_u \bar{c} \bar{p}_l \mathbf{v}_l + A_u \bar{g} \bar{p}_g \bar{v}_g)|_{x=L} + J_{ppp} - \sum_{i=1}^{n_c} k_{c,i} G_i (z_{c,i}) \sqrt{2\bar{p}_m(L) r (\bar{\rho}_d(L) - \bar{\rho}_u(0))} = 0.
$$

Eq. (24) together with (26) construct a two-point boundary value problem in terms of $x$ as the independent variable.

Note that the presence of $dA/dx$, which contains impulses due to area discontinuities in $A(x)$, in the steady-state equations (24) causes discontinuities in the steady-state solution. After we have initialized our problem by solving (24), we start with the numerical implementation of the model in the sequel.

The developed model in this paper includes the system dynamics and aspects that are essential to the control performance. Considering only the control-relevant aspects of the system in the model keeps its complexity relatively low, such that it permits, for example, a semi-analytical assessment of its dynamical properties (an assessment that relies partially on theoretical analyses and partially on numerical analyses). For instance, semi-analytical analyses are used in (Di Meglio 2011, Chapter 1). This type of assessment can be computationally expensive, if not impossible in the case of high-complexity models. Moreover, simulation studies performed during the controller design might need to be performed several times. Thus, it is important for these simulations to run fast. Moreover, the relatively low complexity of the model will allow designers and engineers to more easily identify the reason for or the source of problems in the case of poor simulation results.

### Numerical implementation

The MPD model derived in the modelling section cannot be solved analytically, due to its complexity (e.g., infinite-dimensional nature and nonlinearities). To solve and then use this model for simulation purposes, we employ a numerical scheme based on a finite-volume method (FVM) discretization. As illustrated in Fig. 2, in FVM (LeVeque 2002), the spatial domain of a hyperbolic partial-differential equation (PDE) is divided into a finite number of control volumes or cells. All the variables are assumed to have a predefined distribution in each control volume. As illustrated in the figure, the spatial domain is discretized into $N$ cells denoted by $G_i = (x_{i-1/2}, x_{i+1/2}), i = \{1, 2, ..., N\}$, of length $\Delta x$, with $x_{i+1/2} = i\Delta x$ called the $i$th cell interface and $x_i = (i - 1/2)\Delta x$ marking the middle point of that cell. The variable $U_i^n$ ($Q_i^n$) is an approximation of the spatial average of the vector $u(n\Delta t, x)$ ($q(n\Delta t, x)$) over $G_i$ and the approximate variables at right and left sides of each interface are indicated by $U^-$ and $U^+$, respectively. Here, $\Delta t$ is the time discretization step length. A finite volume Godunov-type method has the following general form (LeVeque 2002):

$$
Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left( F \left( U_{i+1/2}^{n+1}, U_{i-1/2}^{n+1} \right) - F \left( U_{i-1/2}^{n+1}, U_{i-1/2}^{n} \right) \right) + \Delta t S_i^n, \quad \text{where } F(\cdot, \cdot), \text{ a conventional numerical flux function assuming a fixed cross-sectional area, is determined by the numerical scheme, and } S_i^n = S(U_i^n, n\Delta t, x_i) \text{ is the discretized source term. A starred variable } U^\ast, \text{ yet to be computed, is an update of the variable } U \text{ which accounts for variations in the cross-sectional area of the flow. Note that, since the same formula as in (27) is used for solving both of (10) and (15), the sub/superscripts } a \text{ and } d \text{ are omitted for readability. The time step size } \Delta t \text{ is determined based on the Courant-Friedrichs-Lewy (CFL) condition (Courant et al. 1967). In particular, the numerical implementation of the DFM (1) has been extensively studied in the last few decades (see, e.g., (Fjelde and Karlsen 2002)). These studies mostly aim at developing accurate but computationally low-cost numerical schemes for computing the numerical flux function.}
$$

To treat the variations in cross-sectional area of the flow path of the MPD model, the method proposed by Kröner and Thanh (2005) for (15) and an extension of that method proposed by Abbasi et al. (2019) for the modified DFM (10) are exploited here to compute the starred variables in (27). In this method, we use a coordinate transformation that gives the equivalent values of the variables in a cell if the geometry of that cell changes. In particular, when updating the variables in the cell $G_i$, we assume that the cells $G_{i-1}$ and $G_{i+1}$ have the same geometries as $G_i$ and use this transformation to compute the equivalent of the variables in those cells, considering this
change of geometry. In this way, we are still able to use a Godunov setting with numerical flux functions as in (27) to numerically solve our MPD simulation model. Following this approach, \( U_{i+1}^{+} = [p_{i+}, v_{i+}]^T \) of the interface \( i + 1/2 \) in the drillstring is obtained through a nonlinear coordinate transformation that is given in terms of the following system of equations:

\[
\begin{align*}
\rho^{+} v^{+} A_{d} - \rho^{+} v^{+} A_{d}^+ &= 0, \\
(v^{+})^2 - (v^{+})^2 + c_l^2 \ln \left( \frac{\rho^{+}}{\rho^{+}} \right)^2 &= 0,
\end{align*}
\]  

where \( A^- \) and \( A^+ \) denote, as illustrated in Fig. 2, the cross-sectional area \( A \) immediately at the left and right side of an interface, respectively. Moreover, \( U_{i+}^{+} \) of the interface \( i - 1/2 \) is obtained by replacing the superscripts + and − by − and +, respectively, in (28) and then solving the resulting equation. Next, \( U_{i+}^{+} \) for the DFM of the annulus is obtained by a coordinate transformation given by the following equations together with closure relations (7a)-(7d) and (8):

\[
\begin{align*}
\alpha_l^{+} \rho_l^{+} v_l^{+} &= \alpha_l^+ \rho^+ v^+ \left( \frac{v^{+}}{2} \right)^2 + \alpha_l^+ \left( \frac{v^{+}}{2} \right)^2 - \alpha_l^+ \rho^+ v^+ \left( \frac{v^{+}}{2} \right)^2 - \alpha_l^+ \left( \frac{v^{+}}{2} \right)^2 + c_l^2 \ln \left( \frac{\rho^{+}}{\rho^{+}} \right)^2 = 0, \\
\alpha_g^{+} \rho_g^{+} v_g^{+} &= \alpha_g^+ \rho^+ v^+ \left( \frac{v^{+}}{2} \right)^2 + \alpha_g^+ \left( \frac{v^{+}}{2} \right)^2 - \alpha_g^+ \rho^+ v^+ \left( \frac{v^{+}}{2} \right)^2 - \alpha_g^+ \left( \frac{v^{+}}{2} \right)^2 + c_g^2 \ln \left( \frac{\rho^{+}}{\rho^{+}} \right)^2 = 0.
\end{align*}
\]

As before, \( U_{i-}^{+} \) of the interface \( i - 1/2 \) is obtained by replacing + and − in (29) with − and +, respectively.

**Remark 5.** A close observation of (28) reveals that variations in the pressure profile \( p_d(i, x) \) in the drillstring due to changes in the cross-sectional area at the location of these changes are not significant. This is mainly because the mud velocity \( v \) in the drillstring is far smaller than the sound velocity \( c_l \). On the contrary, when there is gas inside the annulus, variations in the cross-sectional area can cause considerable variations in the pressure. Given this explanation, without losing much accuracy, we may assume that \( p^{+} = \rho^{+} \) for the drillstring and solve only the mass-balance equation of (28), which is linear, for computing \( v^{+} \).

After computing \( Q_i^{n+1} \) from (27), the vector of primitive variables are computed using (11)-(14). Then, if a first-order scheme is used, a uniform distribution is considered for the variables \( u(n \Delta t, x) \) over a cell \( G_i \), thus

\[
\hat{u}(n \Delta t, x) = U_j^n, \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).
\]

When a second-order scheme is used, this approximation is obtained by a linear interpolation as follows:

\[
\hat{u}(n \Delta t, x) = U_j^n + (u_e)_j^n (x - x_j), \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),
\]

where \( (u_e)_j^n \) is an approximation of the exact derivatives \( \partial u(n \Delta t, x) / \partial x \) at \( x = x_j \), computed using a flux limiter, see, e.g., (Sweby 1984).

**Remark 6.** We reasonably assume that \( A(x) \) is piecewise continuous, with a discontinuity occurring only at a cell interfaces such that \( A(x) \) is constant in each cell \( G_i \) (i.e., for \( x \in (x_{i-1/2}, x_{i+1/2}) \), \( i \in \{1, 2, \ldots, N\} \)).

In the next part of this section, we explain how to combine the implicit boundary conditions of the problem with the numerical scheme used for updating the internal domains of the model.

**Boundary conditions treatment.** Expanding the scheme (27) for \( i = 1 \) and \( i = N \) reveals explicit dependencies on \( U_0 \) and \( U_{N+1} \), and also implicit dependencies on \( U_{-1}, U_{N+2} \) for the case of second-order schemes. These variables are required to incorporate the boundary conditions of the boundary value problem in the described scheme in the previous section. Extrapolation is a common method for determining the boundary variables. It is effective and can provide accurate results if there are no source terms in the model (Prebeg et al. 2015). However, it lacks a sound theoretical support and can cause large spikes in the solution in the presence of source terms. Contrary to extrapolation, a method known as the characteristics-based method (Cook et al. 2018) offers more accurate and reliable solutions and it also has a more reliable theoretical foundation. This method involves breaking a two-point boundary value problem into two initial value problems and solving those separately at their respective boundaries. Now, by the use of a nonlinear coordinate transformation and approximations, the DFM with the closure laws (7a)-(7d) can be decomposed into its characteristic equations. In this
form, two of the PDEs describe the propagation of the pressure waves, also called fast dynamics of the DFM, inside the domain and one PDE, called the slow dynamics, describes the migration of the gas phase. For the DFM (characterizing flow the annulus), these relations come in the following form, see (Fjelde and Karlsen 2002):

\[ \alpha_g (1 - C_0 \alpha_g) \frac{d_1 p_a}{d_1 t} + p_a \frac{d_1}{d_1 t} \alpha_g = 0, \quad \text{............................................................................................................................................... (32)} \]

\[ \frac{d_2 p_a}{d_2 t} - \frac{\rho_i c_m (v_g - v_l)}{d_2 t} \frac{d_2 g}{d_2 t} - \rho_i \alpha_i (v_g - v_l + c_m) \frac{d_2 v_l}{d_2 t} = (v_g - v_l + c_m) s_a, \quad \text{............................................................................................................................................... (33)} \]

\[ \frac{d_3 p_a}{d_3 t} + \frac{\rho_i c_m (v_g - v_l)}{d_3 t} \frac{d_3 g}{d_3 t} - \rho_i \alpha_i (v_g - v_l - c_m) \frac{d_3 v_l}{d_3 t} = (v_g - v_l - c_m) s_a, \quad \text{............................................................................................................................................... (34)} \]

where in this case we have defined \( \frac{d_j}{d_j t} := \frac{\partial}{\partial t} + \lambda_{a,i} \frac{\partial}{\partial x}, i = 1, 2, 3 \), which is a directional derivative along the vector \( V = [1, \lambda_{a,i}]^T \), with \( \lambda_{a,i} \) being an eigenvalue of the DFM of the annulus. Eqs (32)-(34) correspond to the gas volume wave travelling at a speed \( \lambda_{a,i} = v_g \) from downstream the annulus, the pressure waves propagating at \( \lambda_{a,2} = v_l - c_m \) upstream the annulus and the pressure wave travelling at a speed of \( \lambda_{a,3} = v_l + c_m \) towards the choke, respectively. The characteristic relations of the isothermal Euler equation describing the single-phase flow in the drillstring are given by

\[ \frac{\partial p_j}{\partial t} - \lambda_{d,1} \frac{\partial p_j}{\partial x} - c_i \rho \frac{\partial v_j}{\partial t} - c_i \rho \lambda_{d,1} \frac{\partial v_j}{\partial x} = c_i s_d, \quad \text{............................................................................................................................................... (35)} \]

\[ \frac{\partial p_j}{\partial t} + \lambda_{d,2} \frac{\partial p_j}{\partial x} + c_i \rho \frac{\partial v_j}{\partial t} + c_i \rho \lambda_{d,2} \frac{\partial v_j}{\partial x} = -c_i s_d, \quad \text{............................................................................................................................................... (36)} \]

where (35) corresponds to the pressure wave travelling upstream the flow with a velocity of \( \lambda_{d,1} = v - c_l \), while (36) corresponds to the pressure wave travelling the opposite direction at a velocity of \( \lambda_{d,2} = c_l + v \).

**Remark 7.** Note that the characteristic relations (32)-(34) are obtained under several simplifying assumption such as an incompressible liquid phase, \( \partial C_0(\alpha) / \partial u = 0 \) and \( \partial V_0(\alpha) / \partial u = 0 \), and that \( \alpha_g \rho_g \ll \alpha_i \rho_i \) holds, because otherwise the derivation of such relations is highly challenging, if not impossible. Naturally, these assumptions lead to some degree of inaccuracy in the computation of the boundary variables. Nonetheless, for small gas volume fractions \( \alpha_g \) (less than 0.25) at the boundaries, these assumption are rather realistic, especially when managed pressure drilling operations are supported by high-performance kick detectors and pressure control systems that prevent large gas kicks. It is mentioned that the relations (35) and (36) are exact.

In the remainder of this section, we propose a method for computing the boundary variables based on the characteristic relations (32)-(36) together with the physical boundary conditions introduced in the previous section.

We solve the drillstring inlet boundary condition in (16) together with the characteristic relation (35), forming an initial value problem as

\[ f_{\text{ibc}}(p(t,0), v(t,0), t) = 0, \]

\[ \frac{\partial p_j}{\partial t} - \lambda_{d,1} \frac{\partial p_j}{\partial x} - c_i \rho \frac{\partial v_j}{\partial t} - c_i \rho \lambda_{d,1} \frac{\partial v_j}{\partial x} = c_i s_d, \quad \text{............................................................................................................................................... (37)} \]

This partial differential algebraic equation, if solved at \( x = 0 \), gives \( u_d(t,0) = [v(t,0), p(t,0)]^T \). Finding the analytical solution of this nonlinear partial differential algebraic equation is however challenging. Therefore, we solve (37) numerically by performing a first-order Euler discretization (both spatial and temporal) on (37), yielding

\[ f_{\text{ibc}}(p_{0,j}^j, v_{0,j}^j, n \Delta t) = 0, \]

\[ \frac{p_{d,0}^j - p_{d,0}^{j-1}}{\Delta t} - c_i \rho_{0,j}^{n-1} \frac{v_{0,j}^n - v_{0,j}^{n-1}}{\Delta t} = - (\lambda_{d,1})^n_1 \frac{p_{d,0}^{n-1} - p_{d,1}^{n-1}}{\Delta x} - c_i \rho (\lambda_{d,1})_1^{n-1} \frac{v_{0,j}^n - v_{0,j}^{n-1}}{\Delta x} + c_i s_d \left( U_{d,0}^{n-1}, (n-1) \Delta t, 0 \right), \quad \text{............................................................................................................................................... (38)} \]

where the variables \( p_{0,j}^j \) and \( v_{0,j}^j \) are the solutions of (38), are approximations of \( p(n \Delta t, 0) \) and \( v(n \Delta t, 0) \), respectively, and the notation \( (\cdot)^n_j \) stands for the term inside the parentheses evaluated at the time instance \( i \) and the point \( x_j \) in the special domain. The resulting nonlinear algebraic equation (38), together with the equation of state (7a), can be solved with a proper zero-finder algorithm, such as a Newton solver. After this equation is solved, we can compute \( U_{d,0}^{n} = [v_{0,j}^n]^T \) and \( Q_{d,0}^{n} = [p_{0,j}^n, v_{0,j}^n]^T \), as required in (27) for the drillstring. At the
where \( \Delta P_{db}(t) = \frac{2A_d(L)A_u(0)}{AI(A_d(L)+A_u(0))} \left( \Delta p_{db} - \frac{z^2}{2p(t,L-L/2)(A_dL)^2} + \frac{s_d}{A_d(0)} + \frac{s_d}{A_d(L)} \right) \). (40)

The operator \( \max(\cdot, \cdot) \) in (39) is used to account for the non-return valve installed in the drillstring to prevent a back-flow from the annulus into the drillstring and we take \( \Delta t \) as a parameter that determines the inertia of the dynamics of \( z(t) \). Now, using the other characteristic relation in the drillstring and performing an Euler discretization over space and time, we can approximately compute the drillstring boundary variables at the bit by solving

The boundary vectors \( Q_{d,N+1}^m \) and \( Q_{a,0}^m \), respectively, and \( z^n \), an approximation of \( z(n\Delta t) \), is obtained from the time discretization of (39) using an Euler method, i.e.,

where \( \chi^j \) is an auxiliary variable such that \( \chi^0 = z^{-1} \) and \( \Delta t_z \) is the corresponding discretization time step length. Note that, to avoid numerical oscillations when the time step \( \Delta t \) is large, we can design \( \Delta t_z \) to be smaller than \( \Delta t \). To this end, we set \( \Delta t_z = \Delta t/m, m \in \mathbb{N} \), where \( m \) is chosen to be large enough.

Analogously, we can compute \( U_{d,n}^m \), as an approximation of \( u_d(n\Delta t,0) \), by solving the following system of nonlinear algebraic equations as a result of spatial and temporal discretization of (33) using the Euler method with (19)-(20):

Note that the above equations need to be solved together with the closure laws to return (generally) a unique solution. After solving (41) and (43), we can compute \( Q_{d,N+1}^m \) and \( Q_{a,0}^m \). At the choke boundary, the initial value problem consists of the choke equation (22), and all
closure laws of the DFM (7a)-(7d) together with the two characteristic equations (32) and (34). This problem is approximated in terms of a nonlinear algebraic equation, similar to (38), using a first-order Euler discretization over space and time domains. The solution of the resulting algebraic equation is then used to compute \( U'^{n}_{a,N+1} \) and \( Q'^{n}_{a,N+1} \).

**Remark 8.** If a second-order scheme is used, in addition to the boundary vectors \( U_{a,0} \) and \( U_{a,N+1} \), the vectors \( U_{a,-1} \) and \( U_{a,N+2} \) also need to be determined. Although these variables are less crucial than the boundary variables for the accuracy of the MPD model, the way we compute these can have a significant impact on the solution. A common approach in this regard is to assume that \( U_{a,-1} = U_{a,0} \) and \( U_{a,N+2} = U_{a,N+1} \). However, one can use more advanced approaches such as the one proposed by Prebeg et al. (2017), which comes at a higher computational expense.

### Field data comparisons and simulation studies of industry-relevant scenarios

To evaluate the predictive capabilities of the simulation model and the performance of the proposed numerical implementation of the model, simulations and model validations against experimental data are performed in this section.

**Comparisons with field data** We have performed comparisons for single-phase flow scenarios between the hydraulics model presented in this paper and field data obtained during commissioning tests of an MPD operation on a real drilling well. These tests were performed after running casing and before resuming drilling ahead at the length of 1647 m to adjust the MPD control system. The geometries of the drillstring and wellbore are reported in Fig. 4. The other parameters used in the model are listed in Table B-1. The considered measurements correspond to a time period when the drillstring was stationary. In this experiment, the mass flow of the mud pumped into the drillstring varies between low, medium and high values at different rates.

In this paper, instead of identifying \( G_i(z_{ci}) \) as a function of \( z_{ci} \), we approximate it as a function of time, i.e., we compute the implicit choke characteristic \( G(t) \) from the measurements and use it in our simulations. In particular, we use the following relation to approximately compute \( G(t) \):

\[
G(t) = \sum_{i=1}^{3} G_i(z_{ci}) \simeq \frac{J_c(t)}{k_{c,1} \sqrt{2 \rho_c(t) r (p_c(t) - p_0)}}
\]

where \( p_c(t) = p_a(t, L) \) and \( \rho_c(t) = \rho_a(t, L) \) are the measured pump pressure and flow density upstream the choke, and it is assumed that \( k_{c,2} = k_{c,1} \). The choke flow \( J_c(t) \) is also a measured variable in this equation. We note that this relation is directly obtained from the choke equation in (22). The pump flow rate together with the choke opening signals and implicit choke characteristic \( G(t) \) are reported in Fig. 5.

In Fig. 6, the measured and simulated mass flow rates \( J_p \) are plotted in comparison to the measured pump flow rate \( J_p \). We can observe a good match between these two signals. Since the entire length of the wellbore was cased throughout this scenario, we set \( J'_p = 0 \). Next, we compare the measured and simulated pressure signals at the choke and pump. In Fig. 7, the left side, the modelled and measured choke pressures \( p_c \) are compared, where a good match is observed between the measurements and the model. The quality of this match is also an indication of the good accuracy of the implicit choke characteristic \( G(t) \). Moreover, by comparing this figure to Fig. 6, reporting the flow rates, during periods when the pump flow is steady, we can observe that transients in the choke pressure \( p_c \) correspond to transients in the choke flow \( J_c \). This is due the compressibility of the mud, which is well captured by the hydraulics model. The model-based and measured pump pressures \( p_p = p_a(t, L) \) are plotted in Fig. 7, on the right side. We can clearly see a good match between these two signals. However, there are some discrepancies between the two signals as well. These discrepancies are primarily due to imperfections in the Power Law model, used for computing the friction factor. We could expect a higher accuracy by using more advanced frictional models, such as the Herschel-Bulkley model, but at the expense of additional computational complexity and one additional parameter to identify.

A careful observation of the pump and choke pressure measurements reveals that there is a delay of about 4 seconds between transients in the pump pressure and those in the choke pressure, which is exactly due to the fact that pressure waves propagate at the limited speed of sound velocity. These delays are well captured by the model, which is another indication of the high predictive capability of the hydraulics model in terms of capturing fast transients and wave propagation effects. Moreover, in this figure, the high accuracy of the surface pressure control system of MPD can be well observed, when comparing the measured choke pressure to its reference.

### Simulation studies

In this part, we present simulation studies. The geometries of the drillstring and the annulus considered in the simulations are reported in Fig. 8, and the parameters are listed in Table B-2. In this section we try, among other things, to convey the
importance of having a simulation model the complexity of which is and should be kept relatively low by including pressure control-relevant aspects of an MPD system. This is achieved by demonstrating through simulations that theoretical analyses based on simple design models and simulation studies are not always sufficient for obtaining a comprehensive and reliable assessment of an MPD pressure control system, and further types of assessments (semi-analytic assessments for instance) based on a simulation model might be needed. This type of assessment may itself become impossible if the simulation model includes irrelevant or less relevant aspects that can cause excessive complexity. However, these type of assessments are beyond the scope of this paper.

To compute the numerical functions \( F_d(\cdot) \) in (27) for the DFM of the annulus, a second-order flux-vector-splitting (FVS) scheme is used (Evje and Fjelde 2002). For the drillstring, a first-order upwind scheme (LeVeque 2002, Chapter 4) is used to compute \( F_d(\cdot) \) in (15). After this linearization, which is obtained by considering \( c_f \gg v(x) \), for all \( x \in (0, L) \), the flux function in (15) reduces to \( f_d^T(q_d) = [\rho v, c_f^2 \rho] \), which is now linear in terms of \( q_d \). The maximum value that the time step \( \Delta t \) can take is determined by the Courant-Friedrichs-Lewy (CFL) condition, which is a necessary condition for the convergence of a numerical solution as in (27). Here, we use the following relation to compute the time step:

\[
\Delta t = CFL \frac{\Delta x}{\lambda}, \quad \tilde{\lambda} = \max\{|\lambda_{a,1}|,|\lambda_{a,2}|,|\lambda_{d,1}|,|\lambda_{d,2}|\},
\]

where \( CFL \) is called the CFL number that should be less than one (Courant et al. 1967). Note that while one can choose different time steps for the drillstring and the annulus in computing (27), here we choose both to be equal and assume that \( \tilde{\lambda} = c_f \). For the case when there is gas in the annulus, the time step (45) can be highly conservative (in the sense that it is chosen much smaller than the maximum value it can take) for computing (27) for the annulus, causing a diffusive solution for the annulus. This is because even for small values of \( \alpha_g \) in a cell, the sound velocity in that cell can drop substantially.

We perform the simulations for three common and representative drilling scenarios as described below:

1. As the first case study, we run the implemented model for a **choke plugging** scenario, that is, a contingency where the choke effective area drops due to, for example, partial or complete blockage of the orifice by drilling cuttings. Here, we replicate such a scenario by a sudden decrease in the choke flow factor \( k_{c1} \) in (22) during drilling ahead.

2. Making a **pipe connection** is a common normal drilling operation that takes place around every two to ten hours, depending on the rate of penetration. A pipe connection operation entails halting drilling by slowly ramping down the pump flow to zero and, then, bleeding off the trapped pressure inside the drillstring by opening a bleed-off valve. Afterwards, the top drive is detached from the drillstring, and a new stand of drillpipe is screwed onto the drillstring. After that, the rig pump flow is ramped up again, resuming the drilling.

3. Gas influx, or gas kick, and subsequent **gas migration** in the annulus is the third scenario that will be studied. A kick usually happens when the reservoir pressure exceeds the well pressure, which can occur for reasons, such as drilling into a high-pressure zone, a pressure drop during a pipe connection or swab and surge effects. We simulate such a scenario by increasing the reservoir pressure to replicate running into an unexpected high-pressure zone.

**Control system:** To maintain the downhole pressure during these scenarios, a simple pressure control system is used. We design this control system based on a simple lumped-parameters model in (Kaasa et al. 2012). This model consists of three ordinary differential equations and in this model wave propagations effect is compromised in exchange for simplicity. Although this model is derived based on a single-phase flow assumption, it partially accounts for two-phase scenarios through the parameters related to the bulk moduli. A comparison between this design model and the simulation model is provided in Table B-3.

**Remark 9.** The focus of this paper is not on controller design and the used controller does not necessarily provide a satisfactory pressure control performance. The focus is rather on assessing how certain model/system aspects, taken into account in the proposed model, affect closed-loop system performance.

Following the work by Kaasa et al. (2012), this control system is made up of two parts: 1) a proportional-integral (PI) controller that regulates the surface pressure \( p_s \) through the choke opening \( z_c \) as the control input and 2) an estimator that generates a reference for the controller from the surface and downhole measurements and also the reference given for the downhole pressure \( p^*_{dh} \). The reference
generator consists of a parametrized model, approximating the surface pressure based on a given reference for the downhole pressure, and an estimator that generates an estimate for the parameters of this model. This estimator is designed with a recursive least square (RLS) method with a forgetting factor, see (Nelles 2001). Assuming laminar flow, the parametrized model is given by

\[ p_{ch}^*(t) = p_{dh}(t) - (J_c(t) - J_{pp}(t))(1 - \Theta_f(t)) F - (1 - \Theta_p(t)) G, \]

where \( F = \int_0^t \frac{32 \rho_l}{A^2} \theta dx + G = g \int_0^t p_0 \sin(\theta_p(x)) dx \), \( \Theta_f(t) \) and \( \Theta_p(t) \) are the to-be-estimated parameters, the estimates of which are indicated by \( \hat{\Theta}_f(t) \) and \( \hat{\Theta}_p(t) \), respectively. Here, we assume that the surface measurements are available at a high sampling rate while the downhole measurements are performed at a low rate, which is often the case in realistic drilling scenarios. Here, we take the downhole sampling period during normal operations to be \( \Delta t = 20 \) s. In practice, and especially in the case of long wells, there is also some delay in transmitting the downhole measurements to the surface because of using mud pulse telemetry. However, we here assume that the downhole data are immediately available after measurement. Moreover, we assume that the only choke, described by (22), has a linear characteristic in its operating range (i.e., \( G_1(z_c) = z_c \) for \( 0 \leq z_c \leq 1 \), and \( G_1(z_c) = 0 \) and \( G_1(z_c) = 1 \) for \( z_c < 0 \) and \( z_c > 1 \), respectively). However, one can consider more complex characteristics for the choke through \( G(\cdot) \).

**Results for a choke plugging scenario.** Here, the results for a choke plugging scenario are shown. In this scenario, the choke flow factor \( k_c \) drops by 50% from its nominal value at \( t = 400 \) s. Because a laminar flow has weaker damping effects on propagating pressure waves and an objective of this section is to illustrate distributed aspects of the model, we consider a laminar flow with \( n_l = 1 \) along the entire flow path in this scenario. The corresponding results are reported in Fig. 9; on the left side of which are located the pressure signals, for both the design model and the simulation model, and on the right side snapshots of the liquid velocity along the flow path are shown. As seen from the left figure, the overall closed-loop responses in both models are similar; the design and simulation models show close dynamical behaviours in this scenario. The difference between the two models in terms of the steady-state values of the pressures \( p_c \) and \( p_p \) is because the compressibility of the liquid is not captured in the design model.

The developed simulation model in this paper captures the wave propagation effects to a large extent and these effects can be clearly observed in the left figure (for the simulation model) in the time-delays that exist between the time when the pressure wave is generated at the surface and the times when it affects the downhole and pump pressures. In addition, the jagged behaviour observed in the pressure signals in this figure is due to the wave propagation effects. A more insightful illustration of this effect is given in the right figure by the snapshots of the liquid velocity along the flow path at a variety of time instances. In particular, this figure illustrates the propagation and reflection of pressure waves when striking obstacles (such as the bit) and geometrical changes in the flow path. As can be seen, right before the choke gets plugged, the system is experiencing a (almost) steady-state condition. When the choke is plugged at \( t = 400 \) s, it causes some fluctuations in the velocity (and also pressure) profiles of the system that keep propagating along the system afterwards, until those are mitigated on a longer time scale, due to 1) frictional effects and control suppression and 2) numerical dissipations.

Summarising, these results show the value of the proposed model in the scope of the performance evaluation of MPD control systems.

**Results for a pipe connection scenario.** Next, we present the simulation results in a pipe connection scenario. The results are presented for two sets of control parameters to further illustrate how neglecting the fast transients (such as wave propagation effects resulting from the distributed nature of the system) in the design model can deteriorate the closed-loop performance.

In this scenario, the mass flow rates of the mud pump and the back-pressure pump change as in Fig. 10. When the pump flow rate reaches a level less than half its nominal value, we stop sampling the downhole variables and updating the estimator, to replicate a realistic connection scenario during which the downhole measurements are not available because of the lack of mud circulation.

We first implement this scenario by considering the set of control parameters, referred to as the parameter set 1, which are designed to lead to slow and gentle control signals and a rather slow closed-loop system in terms of recovering from disturbances such as changes in the pump flow. We report the results of these simulations in Figs. 11 and 12. We apply the controller to both the design model and simulation models. It is observed from Fig. 11, that the simulation model exhibits a transient behaviour that is similar to that of the design model. This observation indicates that the simulation model reduces to the design model when the operations are performed slowly, such as in pipe connections. We have also shown the flow and pressure drop of the bit in Fig. 12. As expected, when the pressure drop is negative (i.e. \( p_d(t, I) - p_d(t, 0) < 0 \)) the flow through the bit becomes zero because of the non-return valve.

Results of these simulations can, however, be misleading as it might lead one to conclude that the distributed nature of a drilling
system with MPD need not be taken into account while designing a pressure controller. To show that this can be a wrong conclusion and to further highlight the considerable effects of the distributed nature the system on the closed-loop performance, we repeat the same simulation scenario, but with the second set of control parameters, referred to as the parameter set 2, which should lead to faster control signals and better closed-loop performance in terms of recovering from disturbances. The results are plotted in Fig. 13. As expected, the closed-loop performance with the design model as the plant has improved. However, the response quality when this controller is applied to the simulation model has degraded unexpectedly. This has three important implications: 1) the simulation model is more realistic than the design model and it provides a more accurate prediction of the flow and pressure behaviour in a drilling system. 2) One should not rely only on the theoretical results based on the design model and the simulations on a high-fidelity simulation model, as those might show perfect performance in some scenarios and poor performance in yet other scenarios. This can be because this controller with the parameter set 2 results in closed-loop dynamics which are too fast that the fast dynamical aspects of the drilling system (such as the pressure wave propagation effects) are no more negligible. 3) By having the time scale of the fast dynamics of the system, we can already predict intuitively which PI control parameters can result in a poor performance without a need to perform time-consuming simulations on the simulation model. However, doing so might be challenging or even impossible when the control system is more complicated. In such cases, one approach to determine the performance of the closed-loop system can be carrying out semi-analytical system analyses together with simulation studies on the simulation hydraulics model, which includes only the pressure control-aspects of MPD to allow for fast analyses and also simulations.

Results for a gas influx scenario. This part illustrates the ability of the hydraulics model in capturing gas influx and migration scenarios in the annulus in a closed-loop setting with MPD. Throughout this scenario the pump flow rate is kept constant at $J_p = 60$ kg/s. A rapid 4% increase in the reservoir pressure $p_r$ is applied at $t = 400$ s, resembling the scenario of encountering a high pressure zone while drilling. Before this time, the flow is single-phase all along the flow path. Here, we assume that we can detect the resulting gas kick and also identify the new reservoir pressure some time after it begins. Afterwards, a new reference, larger than the reservoir pressure, is set for the downhole pressure to prevent further gas influxes into the wellbore. To prevent a potential control failure, this operation, changing $p_{dh}^*$, is performed slowly. We should mention that a common practice during a kick is to control the pump pressure rather than the downhole pressure.

The simulation results are depicted in Figs. 14 and 15. The choke, downhole and pump pressures are shown on the left side in Figs. 14. On the right side of this figure, we illustrate the choke opening together with the parameters of the estimator. We can clearly observe that when the gas reaches the surface, it leads to a rapid change in the choke opening. Moreover, the gas expansion in the annulus causes $\hat{p}_{dh}$ to be increasing, as expected, for some time, given the fact that gas expansion phenomenon lowers the hydrostatic pressure in the annulus. Fig. 15 gives an illustration of the gas migration and its effects on the flow rate through the choke. We can see from snapshots of the gas void fraction $\alpha_g$, on the right side of Fig. 15, that as the kick moves closer to the surface it expands more, due to a lower pressure. The gas expansion is also illustrated in Fig. 16. The gas expansion also increases the mass flow rate of the choke, as can be clearly observed from $J_c(t)$. On the contrary, when the kick reaches the surface and starts leaving the annulus through the choke, the mass flow rate of the choke rapidly drops, starting at around $t = 33.5$ min. The steady-state difference between the accumulative mass of the influx and the accumulative mass of the gas through the choke shows that the used scheme does not exactly preserve the mass flow rate. In order to obtain more accuracy in this respect, one may adapt well-balanced schemes that are more capable in preserving the steady-state response, see, e.g., (Chertock et al. 2015). In general, this type of scheme is, however, highly expensive computationally. It is observed that the control system successfully, in terms of remaining stable, circulated out the kick in this scenario. However, it is not unreasonable to expect the failure of the designed control system in events such as a choke plugging when there is a two-phase flow in the annulus. The reason for so is large changes in the system behaviour that can occur in such cases. In particular, even for small amounts of gas void fraction $\alpha_g$, the sound velocity of the flow in the annulus drops drastically, causing the fast dynamics of the system not to be regarded as fast any more. This observation, again, implies the importance of performing a semi-analytical assessment of the performance of a pressure control system based on simulation models, in addition to simulation studies.

Simulations starting from a transient state. In general, drilling systems with automated MPD are close to some steady-state condition for most of the drilling time. Therefore, it is reasonable to start the simulation of many drilling scenarios from a steady state of the hydraulics model. Nonetheless, a reliable MPD control system should also show robustness to situations where the system is already in a transient state when the control system takes over the control task. Therefore, in this part, we present simulation results which have
been started from a transient initial condition. To design a transient initial condition for these simulations, we consider a steady state of the model without any gas in the wellbore and, then add a pocket of gas to perturb it. The gas migration corresponding to this scenario is reported on the right side of Fig. 17. In this figure, the snapshot caught at the zero time shows the considered initial condition for \( \alpha_g \). The pump, downhole and choke pressure signals are reported on the left side of Fig. 17. These simulations show that this hydraulics model and the numerical tool can also be used to simulate drilling scenarios starting from a transient state.

Conclusions
A two-phase hydraulics model in the form of a two coupled systems of partial-differential equations has been derived for managed pressure drilling (MPD). The model complexity is limited by incorporating only the mass transport, pressure dynamics and other aspects of an MPD system that can impact the performance of a pressure control system in real world drilling scenarios. Therefore, it provides a basis for evaluating the performance of pressure control systems in virtual test scenarios. Moreover, an approach has been presented for numerical implementation of the model. Variations in the cross-sectional area of the flow path as well as the nonlinear boundary conditions are often not considered in the control design but do exist in reality and can significantly jeopardize the performance and stability of a pressure control system. These aspects have been captured in the model and are accounted for during numerical implementations. The predictive capability of the model and the performance of the numerical implementations have been demonstrated through illustrative case studies representing a choke plugging, connection and gas influx scenario. Through these studies, we have also demonstrated the importance of keeping the complexity of an MPD simulation low. However, the developed model is not suitable for handling scenarios related to vertical motions of the drillstring, such as washing-stand and tripping scenarios. Adding this aspect to the model is an ongoing research. Moreover, we have illustrated the high accuracy of the model by comparing it with field data from a real-life drilling well for single-phase scenarios. A direction of our current work is the further validation of the developed MPD model in this paper by comparing simulation results and drilling field data for two-phase flow drilling scenarios.

Acknowledgement
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References


### Nomenclature

- \( A_a \) = annulus area, \( m^2 \)
- \( J_{bpp} \) = back pressure pump flow rate, \( kg/s \)
- \( c_d \) = bit discharge coefficient, -
- \( A_n \) = bit nozzles area, \( m^2 \)
- \( CFL \) = CFL number, -
- \( G \) = choke characteristic, -
- \( p_c \) = choke pressure, \( Pa \)
- \( p_{dh} \) = downhole pressure, \( Pa \)
- \( V_d \) = drift velocity, \( m/s \)
- \( A_d \) = drillstring area, \( m^2 \)
- \( D_{eff} \) = effective diameter, \( m \)
- \( k_c \) = flow factor of choke, \( m^2 \)
- \( V_g \) = gas velocity, \( m/s \)
- \( Re \) = generalized Reynolds number, \( Pa \)
- \( D \) = hydraulic diameter, \( m \)
- \( d_{in} \) = inner diameter of drillstring, \( m \)
- \( n_l \) = liquid behaviour index, -
- \( V_l \) = liquid velocity in annulus, \( m/s \)
- \( v \) = liquid velocity in drillstring, \( m/s \)
- \( n_m \) = mixture behaviour index, -
- \( V_m \) = mixture velocity, \( m/s \)
- \( n_c \) = number of choke valves, -
- \( N \) = number of cells in discretization, -
- \( z_c \) = opening of choke, -
- \( d_o \) = outer diameter of drillstring, \( m \)
- \( p_d \) = pressure in drillstring, \( Pa \)
- \( p_a \) = pressure in annulus, \( Pa \)
- \( C_0 \) = profile parameter, -
- \( p_P \) = pump pressure, \( Pa \)
- \( p_0 \) = reference pressure, \( Pa \)
- \( p_r \) = reservoir pressure, \( Pa \)
- \( J_p \) = rig pump flow rate, \( kg/s \)
- \( c_g \) = sound speed in gas, \( m/s \)
- \( c_l \) = sound speed in liquid, \( m/s \)
- \( n \) = temporal discretization index, -
- \( t \) = temporal variable, \( s \)
- \( D_w \) = wellbore diameter, \( m \)
- \( L \) = well length, \( m \)
\[\hat{\theta}\] = average well inclination, rad  
\[\Delta l\] = bit control volume length, m  
\[\rho_c\] = choke density, kg/m\(^3\)  
\[\nu\] = fanning friction factor, -  
\[\theta_f\] = frictional correction parameter, -  
\[\beta_g\] = gas bulk modulus, Pa  
\[\rho_g\] = gas density, kg/m\(^3\)  
\[\mu_g\] = gas dynamic viscosity, Pa.s  
\[\kappa_g\] = gas production index, kg/(Pa.s)  
\[\alpha_g\] = gas void fraction, -  
\[\theta_g\] = gravitational correction parameter, -  
\[\beta_l\] = liquid bulk modulus, Pa  
\[\rho_0\] = liquid density at \(p_0\), kg/m\(^3\)  
\[\rho_l\] = liquid density in annulus, kg/m\(^3\)  
\[\mu_l\] = liquid dynamic viscosity, Pa.s  
\[\alpha_l\] = liquid void fraction, -  
\[\rho_m\] = mixture density, kg/m\(^3\)  
\[\mu_m\] = mixture viscosity, Pa.s  
\[\varepsilon\] = pipe roughness, m  
\[\Delta x\] = spatial discretization step size, m  
\[\Delta t\] = time discretization step size, s  
\[\theta\] = well inclination, rad
Appendix A- Approximate solution for turbulent flows

In this section, we present an approximate solution to (6) based on a first-order Taylor series expansion. From (6), we define

$$H(v) := \frac{1}{\sqrt{v}} + 4\log \left( \frac{0.27\nu}{D_{eff}} + \frac{1.26^{m-1/2}}{(v^{1-\frac{m}{2}})Re} n_m^{0.35} \right), \quad \text{subject to} \quad H(v) = 0.$$ \hspace{1cm} (47)

such that $H(v) = 0$. We take

$$v = v_0 - \Delta v, \quad \text{subject to} \quad v = v_0 - \Delta v.$$ \hspace{1cm} (48)

where

$$v_0 = \frac{1}{16\log \left( \frac{0.27\nu}{D_{eff}} + \frac{5.74}{Re^{0.9}} \right)^2}, \quad \text{subject to} \quad v = v_0.$$ \hspace{1cm} (49)

is a well-known approximate solution to Colebrook equation, which is recovered from (47) for $n_m = 1$, see (Swanee and Jain 1976). Moreover, $\Delta v$ is a parameter that is to be approximated. Now, using a Taylor expansion of $H(v)$ around $v_0$, we obtain

$$H(v) \approx H(v_0) - H'(v_0)\Delta v, \quad \text{subject to} \quad H(\Delta v) = 0.$$ \hspace{1cm} (50)

where $H'(v_0) = \frac{dH}{dv}(v_0)$ is available analytically. Following (50), designing $\Delta v$ as

$$\Delta v = \frac{H(v_0)}{H'(v_0)}, \quad \text{subject to} \quad \Delta v = \frac{H(v_0)}{H'(v_0)}.$$ \hspace{1cm} (51)

leads to $H(v_0 - \Delta v) \approx 0$. Substituting this design of $\Delta v$ into (48) further leads the following explicit approximation for the friction factor:

$$v \approx v_0 - \frac{H(v_0)}{H'(v_0)}, \quad \text{subject to} \quad v_0 - \Delta v.$$ \hspace{1cm} (52)

One may expand this equation to obtain a closed-form description of it to reduce the computational burden during simulations. Our numerical evaluations verified the high accuracy of this approximate solution over a wide range of Reynolds’s numbers and $0.6 < n_m < 1.4$.

Appendix B-Derivation of the dynamical bit equation

The bit equation is obtained by considering a control volume filled with only liquid over the bit, as illustrated in Fig. 3, and averaging the momentum conservation equation for liquid over it. For the first and second half of this control volume, each of the length $\Delta l/2$, located in the drilling and annulus, we respectively obtain

$$\frac{\Delta l}{2} \frac{dz(t)}{dt} = A_d(L) \left( \frac{z^2(t)}{A_d(L)\rho(t, L - \Delta l/2)} - \frac{z^2(t)}{A_d(L)\rho(t, L)} + P_d(t, L - \Delta l/2) - P_d(t, L) \right) + \tilde{s}_d(t), \quad \text{subject to} \quad \frac{\Delta l}{2} \frac{dz(t)}{dt} = A_d(L) \left( \frac{z^2(t)}{A_d(L)} - \frac{z^2(t)}{A_d(L)} + P_d(t, \Delta l/2) - P_d(t, 0) \right) + \tilde{s}_d(t),$$ \hspace{1cm} (53) \hspace{1cm} (54)

where $z(t)$ is an approximation of the average mass flow rate in this control volume, which also gives an approximation of the flow through the bit. Note that, we have also assumed a single-phase flow in the half of this control volume that is located in the annulus. Moreover, $\tilde{s}_d = A_d(L) \int_{-\Delta l/2}^{\Delta l/2} s_d(u^t, t, x) \, dx$ and $\tilde{s}_a = A_d(0) \int_{-\Delta l/2}^{\Delta l/2} s_a(u^t, t, x) \, dx$. These terms can be approximated as follows:

$$\tilde{s}_d(t) \simeq A_d(L) \frac{\Delta l}{2} \tilde{s}_d \left( U_{d,N}, t, L - \frac{\Delta l}{2} \right), \quad \tilde{s}_a(t) \simeq A_d(0) \frac{\Delta l}{2} \tilde{s}_a \left( U_{a,1}, t, \frac{\Delta l}{2} \right).$$ \hspace{1cm} (55)

Next, considering that $\rho(t, L - \Delta l/2) \simeq \rho(t, L)$ and $\rho_a(t, \Delta l/2) \simeq \rho_a(t, 0)$, subtracting (53) from (54) results in

$$A_d(0)P_a(t, 0) + A_d(L)P_d(t, L) = A_d(0)P_a(t, \Delta l/2) + A_d(L)P_d(t, L - \Delta l/2) - \tilde{s}_d(t) + \tilde{s}_a(t). \quad \text{subject to} \quad A_d(0)P_a(t, 0) + A_d(L)P_d(t, L) = A_d(0)P_a(t, \Delta l/2) + A_d(L)P_d(t, L - \Delta l/2) - \tilde{s}_d(t) + \tilde{s}_a(t).$$ \hspace{1cm} (56)
Now, given the bit equation in (18), we have

\[ P_d(t,L) - P_a(t,0) = \frac{1}{2\rho(t,L)} \left( \frac{z(t)}{c_d A_n} \right)^2, \quad z > 0. \]  

(57)

If we solve (56) and (57) for \( P_d(t,L) \) and \( P_a(t,0) \) and substitute the solution into the summation of (53) and (54), we obtain the bit equation (39).

**Appendix B-Figures**

![Diagram of a drilling system with MPD equipment](image-url)

**Fig. 1**—A simplified schematic diagram of a drilling system with MPD equipment.

![Illustration of a spatial discretization performed in FVM](image-url)

**Fig. 2**—An illustration of a spatial discretization performed in FVM.
Fig. 3—A schematic of the control volume assumed over the bit to facilitate solving the boundary equations at the bit.

Fig. 4—The diameter of the annulus and drillstring of the drilling well used in the model validation studies.
Fig. 5—Field data: (top) the pump mass flow rate, (bottom) the choke opening signals $z_{c,1}$ and $z_{c,2}$ and the implicit choke characteristic $G(t)$.

Fig. 6—A comparison between the measured and modelled choke mass flow rates, together with the measured pump flow.
Fig. 7—A comparison between (left) the measured and modelled choke pressures, together with the reference for the choke pressure as the reference for the MPD pressure control system, and (right) the measured and modelled pump pressures.

Fig. 8—The diameter and cross-sectional area of the annulus and of the drillstring considered for the simulation case studies.
Fig. 9—Simulation results for a choke plugging event: (left) the choke, downhole and pump pressure signals while comparing the design model to the simulation model, (right) Snapshots of the spatial velocity profile for the simulation model before and after the event.

Fig. 10—Mass flow rates of the mud pump and the back-pressure pump in the connection scenario.
Fig. 11—Simulation results for a pipe connection scenario for the parameter set 1: the choke, downhole and pump pressure signals from (left) the design model and (right) the simulation model.

Fig. 12—Simulation results for a pipe connection scenario for the parameter set 1: (left) the pressure drop over the bit and the flow through it, (right) the adaptive parameters of the estimator.
Fig. 13—Simulation results for a pipe connection scenario for the parameter set 2: the choke, downhole and pump pressure signals from (left) the design model and (right) the simulation model.

Fig. 14—Simulation results for the gas influx scenario: (left) the choke, downhole and pump pressures, (right) the choke opening with the adaptive parameters of the estimator.
Fig. 15—Simulation results for the gas influx scenario: (left) mixed mass flow rate downstream the choke $J_c$, mass flow rate of the gas influx $J_g$ and that of the gas through the choke $J_{gc}$, and accumulative mass of the gas influx and that of the gas through the choke, (right) snapshots of the gas void fraction $\alpha_g$ along the annulus.

Fig. 16—Simulation results for the gas influx scenario: the total volume of the gas in the wellbore over time.
Fig. 17—Simulation results for a transient initial condition: (left) the pump, downhole and choke pressure signals, (right) snapshots of the gas void fraction $\alpha_g$ along the annulus.
### Appendix B-Tables

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
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<td>Average well inclination</td>
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<td>-</td>
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<tr>
<td>Choke flow factor</td>
<td>$k_{c,1}$</td>
<td>0.0026</td>
<td>m²</td>
</tr>
<tr>
<td>Choke flow factor</td>
<td>$k_{c,2}$</td>
<td>0.0026</td>
<td>m²</td>
</tr>
<tr>
<td>Liquid viscosity</td>
<td>$\mu_l$</td>
<td>0.177</td>
<td>Pa.s</td>
</tr>
<tr>
<td>Liquid behaviour index</td>
<td>$n_l$</td>
<td>0.93</td>
<td>-</td>
</tr>
<tr>
<td>Bit nozzles area</td>
<td>$A_n$</td>
<td>$5.69 \times 10^{-4}$</td>
<td>m²</td>
</tr>
<tr>
<td>Bit discharge coefficient</td>
<td>$c_d$</td>
<td>0.8</td>
<td>-</td>
</tr>
</tbody>
</table>

Table B-1—Parameters used in the hydraulics model for model validation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of the well</td>
<td>$l$</td>
<td>4000</td>
<td>m</td>
</tr>
<tr>
<td>Well inclination</td>
<td>$\theta(x)$</td>
<td>$\pi/2$</td>
<td>rad</td>
</tr>
<tr>
<td>Liquid bulk modulus</td>
<td>$\beta_l$</td>
<td>$1.1 \times 10^9$</td>
<td>Pa</td>
</tr>
<tr>
<td>Sound speed in gas</td>
<td>$c_g$</td>
<td>316</td>
<td>m/s</td>
</tr>
<tr>
<td>Reference pressure</td>
<td>$p_0$</td>
<td>$10^5$</td>
<td>Pa</td>
</tr>
<tr>
<td>Liquid density at $p_0$</td>
<td>$\rho_0$</td>
<td>1500</td>
<td>kg/m³</td>
</tr>
<tr>
<td>Number of chokes</td>
<td>$n_c$</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>Choke flow factor</td>
<td>$k_{c,1}$</td>
<td>0.0025</td>
<td>m²</td>
</tr>
<tr>
<td>Average velocity</td>
<td>$V_d$</td>
<td>0.5</td>
<td>m/s</td>
</tr>
<tr>
<td>Liquid viscosity</td>
<td>$\mu_l$</td>
<td>0.04</td>
<td>Pa.s</td>
</tr>
<tr>
<td>Gas viscosity</td>
<td>$\mu_g$</td>
<td>$5 \times 10^{-6}$</td>
<td>Pa.s</td>
</tr>
<tr>
<td>Liquid behaviour index</td>
<td>$n_l$</td>
<td>0.95</td>
<td>-</td>
</tr>
<tr>
<td>Bit nozzles area</td>
<td>$A_n$</td>
<td>$5.77 \times 10^{-4}$</td>
<td>m²</td>
</tr>
<tr>
<td>Space discretization step length</td>
<td>$\Delta x$</td>
<td>12.5</td>
<td>m</td>
</tr>
<tr>
<td>Bit control volume length</td>
<td>$\Delta l$</td>
<td>$1.5 \Delta x$</td>
<td>m</td>
</tr>
<tr>
<td>Gas production index</td>
<td>$\kappa_g$</td>
<td>$8 \times 10^{-7}$</td>
<td>kg/(Pa.s)</td>
</tr>
<tr>
<td>Bit discharge coefficient</td>
<td>$c_d$</td>
<td>0.8</td>
<td>-</td>
</tr>
<tr>
<td>Profile parameter</td>
<td>$C_0$</td>
<td>1.1</td>
<td>-</td>
</tr>
<tr>
<td>Number of discretization cells</td>
<td>$N$</td>
<td>320</td>
<td>-</td>
</tr>
<tr>
<td>CFL number</td>
<td>$CFL$</td>
<td>0.9</td>
<td>-</td>
</tr>
<tr>
<td>Discretization parameter</td>
<td>$m$</td>
<td>20</td>
<td>-</td>
</tr>
</tbody>
</table>

Table B-2—The simulation parameter values.
<table>
<thead>
<tr>
<th>Model condition</th>
<th>Simulation model</th>
<th>Design model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complexity</td>
<td>5 PDEs+5 closure laws</td>
<td>3 ODEs</td>
</tr>
<tr>
<td>Number of dimensions</td>
<td>1D</td>
<td>1D</td>
</tr>
<tr>
<td>Number of phases</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Captures liquid-liquid flows</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures gas-liquid flows</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures flow compressibility</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures wave propagation</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures gas migration</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures liquid influx</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Captures gas influx</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures flow pattern transitions</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Captures variation in cross sectional area</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Isothermal condition assumption</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Radially homogeneous flow assumption</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Axial flow assumption</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table B-3—Conditions of the simulation model versus the design model.