Application of the Wigner distribution function in optics

Application of the Wigner distribution function in optics

Martin J. Bastiaans

Technische Universiteit Eindhoven, Faculteit Elektrotechniek,
Postbus 513, 5600 MB Eindhoven, Netherlands

Abstract

This contribution presents a review of the Wigner distribution function and of some of its applications to optical problems. The Wigner distribution function describes a signal in space and (spatial) frequency simultaneously and can be considered as the local frequency spectrum of the signal. Although derived in terms of Fourier optics, the description of a signal by means of its Wigner distribution function closely resembles the ray concept in geometrical optics. It thus presents a link between Fourier optics and geometrical optics.

The concept of the Wigner distribution function is not restricted to deterministic signals; it can be applied to stochastic signals, as well, thus presenting a link between partial coherence and radiometry. Some interesting properties of partially coherent light can thus be derived easily by means of the Wigner distribution function.

Properties of the Wigner distribution function, for deterministic as well as for stochastic signals (i.e., for completely coherent as well as for partially coherent light, respectively), and its propagation through linear systems are considered; the corresponding description of signals and systems can directly be interpreted in geometric-optical terms. Some examples are included to show how the Wigner distribution function can be applied to problems that arise in the field of optics.

1. INTRODUCTION

In 1932 Wigner introduced a distribution function in mechanics that permitted a description of mechanical phenomena in a phase space. Such a Wigner distribution function was introduced in optics by Walther in 1968, to relate partial coherence with radiometry. A few years later, the Wigner distribution function was introduced in optics again (especially in the area of Fourier optics), and, since then, a great number of applications of the Wigner distribution function have been reported. It is the aim of this contribution to review the Wigner distribution function and some of its applications to problems that arise in the field of optics.

The Wigner distribution function is introduced in Section 2, and some of its most elementary properties are discussed there. In Section 3 we consider some examples of Wigner distribution functions, mainly chosen from the field of optics. Properties of the Wigner distribution function are studied in Section 4. In Sections 5 and 6 we study the propagation of the Wigner distribution function through linear systems by formulating input-output relations and transport equations. While in Sections 2 through 6 we confine ourselves to completely coherent light, the use of the Wigner distribution function to describe partially coherent light is considered in Section 7. Finally, in Section 8, we present some applications of the Wigner distribution function to optical problems.

We conclude this introduction with some remarks about the signals that we are dealing with. We will consider scalar optical signals, which will be described by, say, $\tilde{\varphi}(x, y, z, t)$, where x, y, and z denote space variables and where t represents the time variable. The temporal Fourier transform $\varphi(x, y, z, \omega)$ of such a signal will be defined as $\varphi(x, y, z, \omega) = \int \tilde{\varphi}(x, y, z, t) \exp[i\omega t] dt$. (Unless otherwise stated, all integrations and summations in this contribution extend from $-\infty$ to $+\infty$.) For the sake of convenience, we shall omit the temporal-frequency variable ω from the formulas, since in the present discussion the temporal-frequency description is of no importance. In other words, we can restrict ourselves to a *time-harmonic* signal $\varphi(x, y, z) \exp[i\omega t]$, which is completely described by its *complex amplitude* $\varphi(x, y, z)$. Very often we shall consider signals in a plane $z = constant$, in which case we can omit the space variable z from the formulas. Furthermore, we shall restrict ourselves to the one-dimensional case, where the signals are functions of the space variable x, only; the extension to two dimensions is straightforward. The signals that we are dealing with are thus described by a function $\varphi(x)$.

2. WIGNER DISTRIBUTION FUNCTION AS A LOCAL FREQUENCY SPECTRUM

It is sometimes convenient to describe a space signal $\varphi(x)$, say, not in the space domain, but in the frequency domain by means of its *frequency spectrum*, i.e., the *Fourier transform* $\bar{\varphi}(u)$ of the function $\varphi(x)$, which is defined by

$$\bar{\varphi}(u) = \int \varphi(x) \exp[-iux] dx; \tag{2.0-1}$$

a bar on top of a symbol will mean throughout that we are dealing with a function in the frequency domain. The frequency spectrum shows us the *global* distribution of the energy of the signal as a function of frequency. However, one is often more interested in the *local* distribution of the energy as a function of frequency. Geometrical optics, for instance, is usually treated in terms of rays, and the signal is described by giving the directions (i.e., frequencies) of the rays that should be present at a certain point. Hence, we look for a description of the signal that might be called the *local frequency spectrum* of the signal.

The need for a local frequency spectrum arises in other disciplines, too. It arises in music, for instance, where a signal is usually described not by a time function nor by the Fourier transform of that function, but by its *musical score*; indeed, when a composer writes a score, he prescribes the frequencies of the tones that should be present at a certain moment. It arises also in mechanics, where the position and the momentum of a particle are given simultaneously, leading to a description of mechanical phenomena in a *phase space*.

In this section we introduce a strong candidate for the local frequency spectrum of a signal, viz. the *Wigner distribution function*, and consider its relation to other signal representations.

2.1. Definition of the Wigner distribution function

The Wigner distribution function [Wig-32, Mor-62, Bru-73, Bas-78,80b,81b,84b, Fri-80, Cla-80, Szu-80] $F(x,u)$ of the signal $\varphi(x)$ is defined by

$$F(x,u) = \int \varphi(x + \tfrac{1}{2}x')\varphi^*(x - \tfrac{1}{2}x') \exp[-iux']dx'; \tag{2.1-1}$$

an asterisk will denote throughout complex conjugation. A distribution function according to definition (2.1-1) was first introduced by E. Wigner in mechanics [Wig-32] and provided a description of mechanical phenomena in a phase space.

The Wigner distribution function is a function that may act as a local frequency spectrum of the signal; indeed, with x as a parameter, the integral in its definition represents a Fourier transformation (with frequency variable u) of the product $\varphi(x + \frac{1}{2}x')\varphi^*(x - \frac{1}{2}x')$. Instead of the definition in the space domain, there exists an equivalent definition in the frequency domain, reading

$$F(x,u) = \frac{1}{2\pi} \int \bar{\varphi}(u + \tfrac{1}{2}u')\bar{\varphi}^*(u - \tfrac{1}{2}u') \exp[iu'x]du'. \tag{2.1-2}$$

The Wigner distribution function $F(x,u)$ represents the signal in space and frequency simultaneously. It thus forms an intermediate signal description between the pure space representation $\varphi(x)$ and the pure frequency representation $\bar{\varphi}(u)$. Furthermore, this simultaneous space-frequency description closely resembles the ray concept in geometrical optics, where the position and direction of a ray are also given simultaneously. In a way, $F(x,u)$ is the amplitude of a ray, passing through the point x and having a frequency (i.e., direction) u.

Properties of the Wigner distribution function will be studied in subsequent sections; we only mention here that it is a *real* function, and that it is almost a complete description of the signal. Indeed, from the inverse relations that correspond to definitions (2.1-1) and (2.1-2),

$$\varphi(x_1)\varphi^*(x_2) = \frac{1}{2\pi} \int F[\tfrac{1}{2}(x_1 + x_2), u] \exp[iu(x_1 - x_2)]du \tag{2.1-3}$$

and

$$\bar{\varphi}(u_1)\bar{\varphi}^*(u_2) = \int F[x, \tfrac{1}{2}(u_1 + u_2)] \exp[-i(u_1 - u_2)x]dx, \tag{2.1-4}$$

we conclude that the signal can be reconstructed from its Wigner distribution function up to a constant phase factor; such a phase factor, however, is often less important.

A local frequency spectrum, such as the Wigner distribution function, describes the signal in space x and frequency u, simultaneously. It is thus a function of *two* variables, derived, however, from a function of *one* variable. Therefore, it must satisfy certain restrictions, or, to put it another way: not any function of two variables is a local frequency spectrum. The restrictions that a local frequency spectrum must satisfy correspond to *Heisenberg's uncertainty principle* in mechanics, which states the impossibility of a too accurate determination of both position and momentum of a particle.

It is not difficult to derive the *necessary and sufficient conditions* that a function of two variables must satisfy in order to be a Wigner distribution function. A real function $F(x,u)$ is a Wigner distribution function if and only if it satisfies the condition [Gro-72]

$$F(a+\tfrac{1}{2}x, b+\tfrac{1}{2}u)F(a-\tfrac{1}{2}x, b-\tfrac{1}{2}u) =$$

$$= \frac{1}{2\pi}\iint F(a+\tfrac{1}{2}x_o, b+\tfrac{1}{2}u_o)F(a-\tfrac{1}{2}x_o, b-\tfrac{1}{2}u_o)\exp[-i(ux_o-u_ox)]dx_odu_o \qquad (2.1\text{-}5)$$

for any a and b; a proof of this statement can be found in Appendix A.

2.2. Relatives of the Wigner distribution function

The Wigner distribution function is a representative of a rather broad class of space-frequency functions [Coh-66,89, Cla-80], which are related to each other by linear transformations. Some well-known space-frequency representations – like Woodward's ambiguity function [Woo-53, Pap-74,77, Szu-80], Rihaczek's complex energy density function [Rih-68], and Mark's physical spectrum [Mar-70] – belong to this class. Woodward's *ambiguity function* $A(x',u')$, which is defined by

$$A(x',u') = \int \varphi(x+\tfrac{1}{2}x')\varphi^*(x-\tfrac{1}{2}x')\exp[-iu'x]dx, \qquad (2.2\text{-}1)$$

is related to the Wigner distribution function through a double Fourier transformation:

$$A(x,u) = \frac{1}{2\pi}\iint \exp[i(u_ox-ux_o)]F(x_o,u_o)dx_odu_o. \qquad (2.2\text{-}2)$$

Rihaczek's *complex energy density function* $C(x,u)$, which is defined by

$$C(x,u) = \varphi(x)\bar{\varphi}^*(u)\exp[-iux], \qquad (2.2\text{-}3)$$

is related to the Wigner distribution function through the convolution

$$C(x,u) = \frac{1}{2\pi}\iint 2\exp[-2i(u-u_o)(x-x_o)]F(x_o,u_o)dx_odu_o. \qquad (2.2\text{-}4)$$

The real part $R(x,u)$ of the complex energy density function is related to the Wigner distribution function via the convolution

$$R(x,u) = \frac{1}{2\pi} \iint 2\cos[2(u-u_o)(x-x_o)] F(x_o,u_o) dx_o du_o, \tag{2.2-5}$$

where the realness of the Wigner distribution function has been used. Mark's *physical spectrum*, which is defined as the squared modulus of the *cross-ambiguity function* $S_w(x,u)$ of the signal $\varphi(x)$ and a *window function* $w(x)$,

$$S_w(x,u) = \int \varphi(y) w^*(y-x) \exp[-iuy] dy, \tag{2.2-6}$$

is related to the Wigner distribution function via the relation

$$|S_w(x,u)|^2 = \frac{1}{2\pi} \iint F_w(x_o - x, u_o - u) F(x_o,u_o) dx_o du_o, \tag{2.2-7}$$

in which $F_w(x,u)$ represents the Wigner distribution function of the window function $w(x)$; note that the function $S_w(x,u)$ resembles the *short-time Fourier transform* known in speech processing [Rab-78, Opp-78].

The space-frequency functions mentioned above belong to a broad class of space-frequency functions known as the Cohen class [Coh-66]. Any function of this class is described by the general formula

$$\frac{1}{2\pi} \iiint \varphi(y + \tfrac{1}{2}x') \varphi^*(y - \tfrac{1}{2}x') k(x,u,x',u') \exp[-i(ux' - u'x + u'y)] dy dx' du', \tag{2.2-8}$$

and the choice of the kernel $k(x,u,x',u')$ selects one particular function of the Cohen class. The Wigner distribution function, for instance, arises for $k(x,u,x',u') = 1$, whereas $k(x,u,x',u') = 2\pi\delta(x-x')\delta(u-u')$ yields the ambiguity function. In this contribution we shall confine ourselves to the Wigner distribution function and we shall not consider other members out of this general class of space-frequency functions.

2.3. Extension to stochastic signals

The Wigner distribution function depends *quadratically* upon the signal. This implies that a linear combination of signals leads to all kinds of *cross Wigner distribution functions* and that the resulting Wigner distribution function is not just the linear combination of the Wigner distribution functions of the respective signals. While this may be a disadvantage in some cases, a quadratical dependence can be advantageous, too. Indeed, the quadratical behaviour of the Wigner distribution function enables us to extend the theory to *stochastic signals* instead of deterministic signals, without increasing the dimensionality of the formulas. Let us consider this point in more detail.

Suppose that we are dealing with a stochastic signal $\varphi(x)$ that can be described by the *ensemble average* of the product $\varphi(x_1)\varphi^*(x_2)$, known as the *correlation function* $\Gamma(x_1,x_2)$ [Pap-77]:

$$\Gamma(x_1,x_2) = \mathrm{E}\{\varphi(x_1)\varphi^*(x_2)\}. \tag{2.3-1}$$

The Wigner distribution function of such a stochastic signal can then be defined by [Bas-78,80b,81b]

$$F(x,u) = \int \Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x') \exp[-iux']dx', \tag{2.3-2}$$

which definition is equivalent to the original definition (2.1-1) for a deterministic signal, but with the product $\varphi(x+\frac{1}{2}x')\varphi^*(x-\frac{1}{2}x')$ replaced by the correlation function $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x')$. We remark that the latter definition (2.3-2) is similar to the definition of Walther's *generalized radiance* [Wal-68, Wol-78, Fri-79]. Such an extension of the theory allows us to describe partially coherent light by means of a Wigner distribution function, as well [Bas-78,81b,86a]. We shall study this subject in more detail in Section 7.

3. EXAMPLES OF WIGNER DISTRIBUTION FUNCTIONS

We shall illustrate the concept of the Wigner distribution function by some examples from Fourier optics. For the time being, we confine ourselves to *time-harmonic optical signals* of the form $\tilde{\varphi}(x,t) = \varphi(x)\exp[-i\omega t]$. Since the time dependence is known a priori, the *complex amplitude* $\varphi(x)$ serves as an adequate description of the signal, and the time dependence can be omitted from the formulas. For convenience, we restrict ourselves to one-dimensional space functions $\varphi(x)$ to denote the complex amplitude; the extension to more dimensions is straightforward.

3.1. Point source

A point source located at the position x_o can be described by the *impulse signal* $\varphi(x) = \delta(x - x_o)$. Its Wigner distribution function takes the form $F(x,u) = \delta(x - x_o)$. At one point $x = x_o$ all frequencies are present, whereas there is no contribution at other points. This is exactly what we expect as the local frequency spectrum of a point source.

3.2. Plane wave

As a second example we consider a plane wave, described in the frequency domain by the *frequency impulse* $\bar{\varphi}(u) = 2\pi\delta(u - u_o)$, or, equivalently, in the space domain by the *harmonic signal* $\varphi(x) = \exp[iu_o x]$. A plane wave and a point source are *dual* to each other, i.e., the Fourier transform of one function has the same form as the other function. Due to this duality, the Wigner distribution function of a plane wave will be the same as the one of the point source, but rotated in the space-frequency domain through 90 degrees. Indeed, the Wigner distribution function of the plane wave takes the form $F(x,u) = 2\pi\delta(u - u_o)$. At all points, only one frequency $u = u_o$ manifests itself, which is exactly what we expect as the local frequency spectrum of a plane wave.

3.3. Quadratic-phase signal

The quadratic-phase signal $\varphi(x) = \exp[i\frac{1}{2}\alpha x^2]$ represents, at least for small x, i.e., in the paraxial approximation, a *spherical wave* whose *curvature* is equal to α. The Wigner distribution function of such a signal is $F(x,u) = 2\pi\delta(u-\alpha x)$, and we conclude that at any point x only one frequency $u = \alpha x$ manifests itself. This corresponds exactly to the ray picture of a spherical wave.

3.4. Smooth-phase signal

The Wigner distribution function of the smooth-phase signal $\varphi(x) = \exp[i\gamma(x)]$, where $\gamma(x)$ is a sufficiently smooth function of x, takes the form $F(x,u) \simeq 2\pi\delta(u - d\gamma/dx)$. We remark that at any point x only one frequency $u = d\gamma/dx$ manifests itself. Note that the latter expression for the Wigner distribution function becomes an equality when $\gamma(x)$ varies only linearly or quadratically in x (see Examples 3.2 and 3.3). The concept of a smooth-phase signal and its Wigner distribution function may be useful, for instance, in the geometrical theory of aberrations [Bor-75] and in describing Bryngdahl's geometrical transformations [Bry-74] and geometric-optical systems (see Example 5.6 and Section 8.2).

3.5. Gaussian signal

Let us consider the Gaussian signal

$$\varphi(x) = \left(\frac{2}{\rho^2}\right)^{\frac{1}{4}} \exp\left[-\frac{\pi}{\rho^2}(x-x_o)^2 + iu_o x\right], \tag{3.5-1}$$

where ρ is a positive quantity. The Wigner distribution function of this Gaussian signal reads

$$F(x,u) = 2\exp\left[-\left(\frac{2\pi}{\rho^2}(x-x_o)^2 + \frac{\rho^2}{2\pi}(u-u_o)^2\right)\right]. \tag{3.5-2}$$

Note that it is a function that is Gaussian in both x and u, centered on the space-frequency point (x_o, u_o). The effective widths in the x- and the u-direction follow readily from the normalized second-order central moments $\frac{1}{2}(\rho^2/2\pi)$ and $\frac{1}{2}(2\pi/\rho^2)$ in the respective directions.

When we consider *Gaussian beams*, we have to deal with a Gaussian signal that is multiplied by a quadratic-phase signal, e.g.,

$$\varphi(x) = \left(\frac{2}{\rho^2}\right)^{\frac{1}{4}} \exp\left[-\frac{\pi}{\rho^2}x^2 + i\tfrac{1}{2}\alpha x^2\right]. \tag{3.5-3}$$

The Wigner distribution function of such a signal takes the form

$$F(x,u) = 2\exp\left[-\left(\frac{2\pi}{\rho^2}x^2 + \frac{\rho^2}{2\pi}(u-\alpha x)^2\right)\right]. \tag{3.5-4}$$

It may be convenient to consider the Gaussian beam as a quadratic-phase signal having a *complex* curvature $\alpha + i(2\pi/\rho^2)$ [Des-72, Bas-79d]; this complex curvature sometimes behaves like the ordinary curvature of a quadratic-phase signal, as we shall see later on in Section 8.5.

3.6. Hermite functions

As a final example we shall consider the Hermite functions $\psi_m(\xi)$ $(m = 0, 1, ...)$ that are defined, for instance, by means of the generating function [Bru-67, Abr-70, Jan-81b]

$$\exp[\pi\xi^2 - 2\pi(\eta - \xi)^2] = 2^{-\frac{1}{4}} \sum_{m=0}^{\infty} (m!)^{-\frac{1}{2}} (4\pi)^{\frac{1}{2}m} \eta^m \psi_m(\xi); \tag{3.6-1}$$

the Hermite functions can be expressed in the form [Bru-67]

$$\psi_m(\xi) = 2^{\frac{1}{4}} 2^{-\frac{1}{2}m} (m!)^{-\frac{1}{2}} \exp[-\pi\xi^2] H_m(\sqrt{2\pi}\xi) \quad (m = 0, 1, ...), \tag{3.6-2}$$

where $H_m(\xi)$ $(m = 0, 1, ...)$ are the *Hermite polynomials* [Abr-70]. Note that $\psi_0(\xi)$ is just a Gaussian function, and that, consequently, $\psi_0(x/\rho)$ is exactly the Gaussian signal that we considered in Example 3.5 with $x_o = u_o = 0$. We remark that the Hermite functions satisfy the orthonormality relation

$$\int \psi_m(\xi)\psi_n^*(\xi) d\xi = \delta_{m-n} \quad (m, n = 0, 1, ...). \tag{3.6-3}$$

It can be shown [Jan-81b] that the Wigner distribution function of the signal $\psi_m(x/\rho)$ $(m = 0, 1, ...)$ takes the form

$$F(x, u) = 2(-1)^m \exp\left[-\left(\frac{2\pi}{\rho^2}x^2 + \frac{\rho^2}{2\pi}u^2\right)\right] L_m\left[2\left(\frac{2\pi}{\rho^2}x^2 + \frac{\rho^2}{2\pi}u^2\right)\right], \tag{3.6-4}$$

where $L_m(\xi)$ $(m = 0, 1, ...)$ are the *Laguerre polynomials* [Erd-53]. We shall use the Hermite functions and their Wigner distribution functions later on in this contribution.

4. PROPERTIES OF THE WIGNER DISTRIBUTION FUNCTION

In this section we list some properties of the Wigner distribution function. Other properties can be found elsewhere [Mor-62, Bru-67,73, Cla-80, Szu-80, Bre-83].

4.1. Inversion formulas

The inverse relations that correspond to the definitions of the Wigner distribution function have already been presented by the relations (2.1-3) and (2.1-4). In fact, these inverse relations formulate the conditions that a function of two variables must satisfy in order to be a Wigner distribution function: a function of x and

u is a Wigner distribution function if and only if the right-hand integral in relation (2.1-3) or (2.1-4) is separable in the form of the left-hand side of that relation. From relations (2.1-3) and (2.1-4) we conclude that the signal $\varphi(x)$ and its frequency spectrum $\bar{\varphi}(u)$ can be reconstructed from the Wigner distribution function up to a constant phase factor.

4.2. Realness

It follows immediately from the definitions of the Wigner distribution function that it is a *real* function. Unfortunately, the Wigner distribution function is not necessarily nonnegative, which prohibits a direct interpretation of this function as an energy density function.

4.3. Space and frequency limitation

It follows directly from the definitions that, if the signal $\varphi(x)$ is limited to a certain *space interval,* the Wigner distribution function is limited to the same interval. Similarly, if the frequency spectrum $\bar{\varphi}(u)$ is limited to a certain *frequency interval,* the Wigner distribution function is limited to the same interval.

4.4. Space and frequency shift

It follows immediately from the definitions that a *space shift* of the signal $\varphi(x)$ yields the same shift for the Wigner distribution function. Similarly, a *frequency shift* of the frequency spectrum $\bar{\varphi}(u)$, which corresponds to a *modulation* of the signal $\varphi(x)$, yields the same shift for the Wigner distribution function. We have already met these space and frequency shifts when we considered the Gaussian signal in Example 3.5.

4.5. Some equalities and inequalities

Several integrals of the Wigner distribution function have clear physical meanings. The integral over the frequency variable u, for instance,

$$\frac{1}{2\pi}\int F(x,u)du = |\varphi(x)|^2, \tag{4.5-1}$$

represents the *intensity of the signal,* whereas the integral over the space variable x,

$$\int F(x,u)dx = |\bar{\varphi}(u)|^2, \tag{4.5-2}$$

is equal to the *intensity of the frequency spectrum*. These integrals are evidently nonnegative. The same holds for the integral over the entire space-frequency domain,

$$\frac{1}{2\pi}\iint F(x,u)dxdu = \int |\varphi(x)|^2 dx = \frac{1}{2\pi}\int |\bar{\varphi}(u)|^2 du, \tag{4.5-3}$$

which represents the *total energy* of the signal. It is not difficult to recognize Parseval's theorem in relation (4.5-3).

The integral in relation (4.5-3) represents in fact the zero-order moment of the Wigner distribution function. The normalized first-order moment of the Wigner distribution function in the x-direction,

$$m_x = \frac{\frac{1}{2\pi}\iint xF(x,u)dxdu}{\frac{1}{2\pi}\iint F(x,u)dxdu} = \frac{\int x|\varphi(x)|^2 dx}{\int |\varphi(x)|^2 dx}, \tag{4.5-4}$$

is equal to the *center of gravity* [Pap-77] of the intensity $|\varphi(x)|^2$ of the signal; a similar relation holds for the normalized first-order moment in the u-direction,

$$m_u = \frac{\frac{1}{2\pi}\iint uF(x,u)dxdu}{\frac{1}{2\pi}\iint F(x,u)dxdu} = \frac{\frac{1}{2\pi}\int u|\bar{\varphi}(u)|^2 du}{\frac{1}{2\pi}\int |\bar{\varphi}(u)|^2 du}, \tag{4.5-5}$$

which is equal to the center of gravity of the intensity $|\bar{\varphi}(u)|^2$ of the frequency spectrum. The normalized second-order central moment in the x-direction,

$$m_{xx} = \frac{\frac{1}{2\pi}\iint (x-m_x)^2 F(x,u)dxdu}{\frac{1}{2\pi}\iint F(x,u)dxdu} = \frac{\int (x-m_x)^2|\varphi(x)|^2 dx}{\int |\varphi(x)|^2 dx} = d_x^2, \tag{4.5-6}$$

is equal to the square of the *duration* [Pap-77] or *effective width* of the signal $\varphi(x)$; a similar relation holds for the normalized second-order central moment in the u-direction

$$m_{uu} = \frac{\frac{1}{2\pi}\iint (u-m_u)^2 F(x,u)dxdu}{\frac{1}{2\pi}\iint F(x,u)dxdu} = \frac{\frac{1}{2\pi}\int (u-m_u)^2|\bar{\varphi}(u)|^2 du}{\frac{1}{2\pi}\int |\bar{\varphi}(u)|^2 du} = d_u^2, \tag{4.5-7}$$

which is related to the effective width of the frequency spectrum $\bar{\varphi}(u)$. Note that, again, these second-order moments are nonnegative. We can also define the mixed second-order moment

$$m_{ux} = m_{xu} = \frac{\frac{1}{2\pi}\iint (x-m_x)(u-m_u)F(x,u)dxdu}{\frac{1}{2\pi}\iint F(x,u)dxdu}. \tag{4.5-8}$$

The *real symmetric matrix* of second-order moments

$$M = \begin{bmatrix} m_{xx} & m_{xu} \\ m_{ux} & m_{uu} \end{bmatrix} \tag{4.5-9}$$

is nonnegative definite, as we shall prove in Section 8.5.

Instead of the *global* moments like in relations (4.5-3)-(4.5-8), where the integration is over both the space and the frequency variable, we can also consider *local* moments, like in relations (4.5-1) and (4.5-2), where we integrate over one variable only. The normalized first-order local moment with respect to the frequency variable,

$$U(x) = \frac{\frac{1}{2\pi}\int uF(x,u)dxdu}{\frac{1}{2\pi}\int F(x,u)dxdu} = \operatorname{Im}\left\{\frac{d}{dx}\ln\varphi(x)\right\}, \tag{4.5-10}$$

can be interpreted as the *average frequency* of the Wigner distribution function at position x. When we represent the signal $\varphi(x)$ by its absolute value $|\varphi(x)|$ and its phase $\arg\varphi(x)$, we obtain the relation

$$U(x) = \frac{d}{dx}\arg\varphi(x), \tag{4.5-11}$$

from which we conclude that the average frequency $U(x)$ is equal to the derivative of the phase of the signal. Other interesting local moments can be found elsewhere [Cla-80].

Several integrals of the Wigner distribution function can be interpreted as *radiometric quantities*. We already found the intensity of the signal [see equation (4.5-1)] and the intensity of the frequency spectrum [see equation (4.5-2)]; the latter is, apart from a usual factor $\cos^2\theta$ (where θ is the angle of observation in radiometry), proportional to the *radiant intensity* [Car-77, Wol-78]. The integral

$$J_z(x) = \frac{1}{2\pi}\int\frac{\sqrt{k^2-u^2}}{k}F(x,u)du \tag{4.5-12}$$

(where $k = 2\pi/\lambda = \omega/c$ is the usual *wave number*) is proportional to the *radiant emittance* [Car-77, Wol-78]; when we combine the latter integral with the integral

$$J_x(z) = \frac{1}{2\pi}\int\frac{u}{k}F(x,u)du, \tag{4.5-13}$$

we can construct the two-dimensional vector

$$J = (J_x,\, J_z), \tag{4.5-14}$$

which is proportional to the *geometrical vector flux* [Win-79]. The total *radiant flux* [Car-77] follows from integrating the radiant emittance over the space variable.

An important relationship between the Wigner distribution functions of two signals and the signals themselves has been formulated by Moyal [Moy-49]; it reads

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)dxdu = \left|\int\varphi_1(x)\varphi_2^*(x)dx\right|^2 = \left|\frac{1}{2\pi}\int\bar{\varphi}_1(u)\bar{\varphi}_2^*(u)du\right|^2. \tag{4.5-15}$$

This relationship has an application in averaging one Wigner distribution function with another Wigner distribution function. The result, unlike the Wigner distribution function itself, is always nonnegative. A direct consequence of *Moyal's formula* is the property that if two functions are orthonormal then their corresponding Wigner distribution functions are orthonormal, as well. We shall use this property later on, when we consider modal expansions of partially coherent light.

Relations (4.5-3) and (4.5-15), together with Schwarz' inequality, yield the relationship

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)dxdu \le \left(\frac{1}{2\pi}\iint F_1(x,u)dxdu\right)\left(\frac{1}{2\pi}\iint F_2(x,u)dxdu\right), \tag{4.5-16}$$

which can be considered as *Schwarz' inequality* for Wigner distribution functions.

Another important inequality, which has been formulated by De Bruijn [Bru-67], reads

$$\frac{1}{2\pi}\iint \left(\frac{2\pi}{\rho^2}(x-x_o)^2 + \frac{\rho^2}{2\pi}(u-u_o)^2\right)^n F(x,u)dxdu \ge n!\frac{1}{2\pi}\iint F(x,u)dxdu, \tag{4.5-17}$$

where n is a nonnegative integer. For the special case $n=1$, and choosing $x_o = m_x$ and $u_o = m_u$, this inequality reduces to

$$\frac{2\pi}{\rho^2}d_x^2 + \frac{\rho^2}{2\pi}d_u^2 \ge 1, \tag{4.5-18}$$

from which we can directly derive the *uncertainty principle* [Pap-68,77]

$$2d_x d_u \ge 1 \tag{4.5-19}$$

by choosing $\rho^2 = 2\pi(d_x/d_u)$. Note that the equality sign in relation (4.5-19) holds if and only if the signal is Gaussian, as in Example 3.5; for all other signals, the product of the effective widths in the space and the frequency directions is larger. We thus conclude that the Gaussian Wigner distribution function occupies the smallest possible area in the space-frequency plane.

5. RAY SPREAD FUNCTION OF AN OPTICAL SYSTEM

It is not difficult to derive how the Wigner distribution function is propagated through a linear system. A linear system that transforms a signal φ_i in the input plane into a signal φ_o in the output plane, can be described in four different ways, depending on whether we describe the input and the output signal in the space or in the frequency domain. We thus have four equivalent input-output relationships,

$$\varphi_o(x_o) = \int h_{xx}(x_o, x_i)\varphi_i(x_i)dx_i, \tag{5.0-1}$$

$$\bar{\varphi}_o(u_o) = \int h_{ux}(u_o, x_i)\varphi_i(x_i)dx_i, \tag{5.0-2}$$

$$\varphi_o(x_o) = \frac{1}{2\pi}\int h_{xu}(x_o,u_i)\bar{\varphi}_i(u_i)du_i, \tag{5.0-3}$$

$$\bar{\varphi}_o(u_o) = \frac{1}{2\pi}\int h_{uu}(u_o,u_i)\bar{\varphi}_i(u_i)du_i, \tag{5.0-4}$$

in which the four *system functions* h_{xx}, h_{ux}, h_{xu}, and h_{uu} are completely determined by the system. Relation (5.0-1) is the usual system representation in the space domain by means of the *impulse response* $h_{xx}(x_o,x_i)$, which is also known as the (coherent) *point spread function* in Fourier optics; the function $h_{xx}(x,x_i)$ is the space domain response of the system at point x due to the input impulse signal $\varphi_i(x)=\delta(x-x_i)$. Relation (5.0-4) is a similar system representation in the frequency domain; the function $h_{uu}(u,u_i)$ is the frequency domain response of the system at frequency u due to the input impulse signal $\bar{\varphi}_i(u)=2\pi\delta(u-u_i)$, which is the Fourier transform of the harmonic input signal $\varphi_i(x)=\exp[iu_ix]$. In Fourier optics such a harmonic signal is a representation of the space dependence of a uniform, obliquely incident, time-harmonic plane wave; in this context we might call $h_{uu}(u_o,u_i)$ the *wave spread function* of the system. Relations (5.0-2) and (5.0-3) are hybrid system representations, since the input and the output signal are described in different domains.

There is a similarity between the four system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu} and the four *Hamilton characteristics* [Bor-75] that can be used to describe geometric-optical systems. Indeed, for a geometric-optical system the *point characteristic* is nothing but the phase of the point spread function [Bas-79a]; similar relations hold between the *angle characteristic* and the wave spread function, and between the *mixed characteristics* and the hybrid system representations.

Unlike the *four* system representations (5.0-1)-(5.0-4), there is only *one* system representation when we describe the input and the output signal by their Wigner distribution functions. Combining the system representations (5.0-1)-(5.0-4) with the definitions (2.1-1) and (2.1-2) of the Wigner distribution function results in the relationship [Bas-78,80b]

$$F_o(x_o,u_o) = \frac{1}{2\pi}\iint K(x_o,u_o,x_i,u_i)F_i(x_i,u_i)dx_idu_i, \tag{5.0-5}$$

in which the Wigner distribution functions of the input and the output signal are related through a superposition integral.

The function $K(x_o,u_o,x_i,u_i)$ in the input-output relationship (5.0-5) is completely determined by the system and can be expressed in terms of the four system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu}, by combining the system representations (5.0-1)-(5.0-4) with the definitions (2.1-1) and (2.1-2) of the Wigner distribution function; we find

$$K(x_o,u_o,x_i,u_i) =$$

$$= \iint h_{xx}(x_o+\tfrac{1}{2}x_o', x_i+\tfrac{1}{2}x_i')h^*_{xx}(x_o-\tfrac{1}{2}x_o', x_i-\tfrac{1}{2}x_i')\exp[-iu_ox_o'+iu_ix_i']dx_o'dx_i', \tag{5.0-6}$$

$$= \frac{1}{2\pi}\iint h_{ux}(u_o+\tfrac{1}{2}u_o', x_i+\tfrac{1}{2}x_i')h^*_{ux}(u_o-\tfrac{1}{2}u_o', x_i-\tfrac{1}{2}x_i')\exp[iu_o'x_o+iu_ix_i']du_o'dx_i', \tag{5.0-7}$$

$$= \frac{1}{2\pi} \iint h_{xu}(x_o + \tfrac{1}{2}x'_o, u_i + \tfrac{1}{2}u'_i) h^*_{xu}(x_o - \tfrac{1}{2}x'_o, u_i - \tfrac{1}{2}u'_i) \exp[-iu_o x'_o - iu'_i x_i] dx'_o du'_i, \quad (5.0\text{-}8)$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint h_{uu}(u_o + \tfrac{1}{2}u'_o, u_i + \tfrac{1}{2}u'_i) h^*_{uu}(u_o - \tfrac{1}{2}u'_o, u_i - \tfrac{1}{2}u'_i) \exp[iu'_o x_o - iu'_i x_i] du'_o du'_i. \quad (5.0\text{-}9)$$

Relations (5.0-6)-(5.0-9) can be considered as the definitions of a *double* Wigner distribution function; hence, the function $K(x_o, u_o, x_i, u_i)$ has all the properties of a Wigner distribution function, for instance the property of realness. Physical constraints that can be imposed upon a system, can be expressed in terms of this double Wigner distribution function. *Losslessness* of a system [But-77,81], for instance, with which we mean that the total energies of the input and the output signal are equal, can thus be expressed as

$$\frac{1}{2\pi} \iint K(x_o, u_o, x_i, u_i) dx_o du_o = 1. \quad (5.0\text{-}10)$$

In a formal way, the function $K(x, u, x_i, u_i)$ is the space-frequency domain response of the system at space-frequency point (x, u) due to $F_i(x, u) = 2\pi\delta(x - x_i)\delta(u - u_i)$. We emphasize that this is in a formal way only, since there does not exist an actual signal whose Wigner distribution function has the form $2\pi\delta(x - x_i)\delta(u - u_i)$. Nevertheless, thinking in optical terms, such an input signal could be considered to represent a *single ray*, entering the system at the point x_i with a frequency (direction) u_i. Hence, we might call the function $K(x_o, u_o, x_i, u_i)$ the *ray spread function* of the system.

It is not difficult to express the ray spread function of a *cascade* of two systems in terms of the respective ray spread functions $K_1(x_o, u_o, x_i, u_i)$ and $K_2(x_o, u_o, x_i, u_i)$. The ray spread function of the overall system has the form

$$K(x_o, u_o, x_i, u_i) = \frac{1}{2\pi} \iint K_2(x_o, u_o, x, u) K_1(x, u, x_i, u_i) dx du. \quad (5.0\text{-}11)$$

Some examples of ray spread functions of elementary Fourier-optical systems [But-77,81] might elucidate the concept of the ray spread function.

5.1. Thin lens; spreadless system

A thin lens having a *focal distance* f can be described by the point spread function

$$h_{xx}(x_o, x_i) = \exp\left[-i\frac{k}{2f}x_o^2\right] \delta(x_o - x_i); \quad (5.1\text{-}1)$$

its input-output relation thus reads $\varphi_o(x) = \exp[-i(k/2f)x^2]\varphi_i(x)$. The corresponding ray spread function takes the form

$$K(x_o, u_o, x_i, u_i) = 2\pi\delta(x_i - x_o)\delta\left(u_i - u_o - \frac{k}{f}x_o\right), \quad (5.1\text{-}2)$$

and the input-output-relationship (5.0-5) for a thin lens reduces to

$$F_o(x,u) = F_i\left(x, u + \frac{k}{f}x\right). \tag{5.1-3}$$

Relation (5.1-2) represents exactly the geometric-optical behaviour of a thin lens: if a single ray is incident on a thin lens, it leaves the lens from the same position but its direction is changed as a function of the position.

The thin lens is a special kind of a *spreadless system*, which can generally be described by the input-output relation $\varphi_o(x) = m(x)\varphi_i(x)$. In this general case the input-output relation (5.0-5) reduces to

$$F_o(x,u) = \frac{1}{2\pi}\int F_m(x, u-u_i)F_i(x,u_i)du_i \tag{5.1-4}$$

and has the form of a mere multiplication in the x-direction and a convolution in the u-direction of the input Wigner distribution function $F_i(x,u)$ and the Wigner distribution function $F_m(x,u)$ of the *modulation function* $m(x)$.

5.2. Free space in the Fresnel approximation; shift-invariant system

The point spread function of a section of free space having a *length* z has, in the Fresnel approximation, the form

$$h_{xx}(x_o, x_i) = \sqrt{\frac{k}{2\pi i z}}\exp\left[i\frac{k}{2z}(x_o - x_i)^2\right], \tag{5.2-1}$$

while the corresponding wave spread function reads

$$h_{uu}(u_o, u_i) = \exp\left[-i\frac{z}{2k}u_o^2\right] 2\pi\delta(u_i - u_o); \tag{5.2-2}$$

its input-output relation thus reads $\bar{\varphi}_o(u) = \exp[-i(z/2k)u^2]\bar{\varphi}_i(u)$. The similarity between the wave spread function of free space and the point spread function of a lens shows that these two systems are duals of one another. This becomes apparent also from the ray spread function, which for free space takes the form

$$K(x_o, u_o, x_i, u_i) = 2\pi\delta\left(x_i - x_o + \frac{z}{k}u_o\right)\delta(u_i - u_o): \tag{5.2-3}$$

we conclude that the frequency behaviour of one system is similar to the space behaviour of the other system. The input-output relationship (5.0-5) for a section of free space reduces to

$$F_o(x,u) = F_i\left(x - \frac{z}{k}u, u\right). \tag{5.2-4}$$

Relation (5.2-3) again represents exactly the geometric-optical behaviour of a section of free space: if a single ray propagates through free space, its direction remains the same but its position changes according to the actual direction.

Free space in the Fresnel approximation is a special kind of a *shift-invariant system*, which can generally be described by the input-output relationship $\varphi_o(x) = \int m(x - x_i)\varphi_i(x_i)dx_i$, or, equivalently, by $\bar{\varphi}_o(u) = \bar{m}(u)\bar{\varphi}_i(u)$. In this general case the input-output relation (5.0-5) reduces to

$$F_o(x,u) = \int F_m(x - x_i, u)F_i(x_i, u)dx_i, \tag{5.2-5}$$

and has the form of a mere multiplication in the u-direction and a convolution in the x-direction of the input Wigner distribution function $F_i(x,u)$ and the Wigner distribution function $F_m(x,u)$ of the modulating function $\bar{m}(u)$.

5.3. Fourier transformer

An optical Fourier transformation can easily be achieved between the two focal planes of a lens. For a Fourier transformer whose point spread function reads

$$h_{xx}(x_o, x_i) = \sqrt{\frac{\beta}{2\pi i}}\exp[-i\beta x_o x_i] \tag{5.3-1}$$

and whose input-output relation can most easily be expressed in the hybrid form $\varphi_o(x) = \sqrt{\beta/2\pi i}\,\bar{\varphi}_i(\beta x)$, the ray spread function takes the form

$$K(x_o, u_o, x_i, u_i) = 2\pi\delta\left(x_i + \frac{u_o}{\beta}\right)\delta(u_i - \beta x_o), \tag{5.3-2}$$

and the input-output relationship (5.0-5) reduces to

$$F_o(x,u) = F_i\left(-\frac{u}{\beta}, \beta x\right). \tag{5.3-3}$$

We conclude that the space and frequency domains are *interchanged*, as can be expected for a Fourier transformer.

5.4. Magnifier

A magnification can easily be achieved in optics with the help of a lens and two appropriate sections of free space. Let a magnifier be represented by the point spread function

$$h_{xx}(x_o, x_i) = \sqrt{m}\,\delta(mx_o - x_i); \tag{5.4-1}$$

its input-output relation thus reads $\varphi_o(x) = \sqrt{m}\varphi_i(mx)$. Then its ray spread function will read

$$K(x_o, u_o, x_i, u_i) = 2\pi\delta(x_i - mx_o)\delta\left(u_i - \frac{u_o}{m}\right), \tag{5.4-2}$$

and the input-output relationship (5.0-5) reduces to

$$F_o(x,u) = F_i\left(mx, \frac{u}{m}\right). \tag{5.4-3}$$

We note that the space and frequency domains are merely *scaled*, as can be expected for a magnifier.

5.5. First-order optical systems

Examples 5.1, 5.2, 5.3, and 5.4 are special cases of Luneburg's *first-order optical systems* [Lun-66]. A first-order optical system can, of course, be characterized by its system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu}: they all have a constant absolute value, and their phases vary quadratically in the pertinent variables. Note that a Dirac function can be considered as a limiting case of such a quadratically varying function: indeed,

$$\lim_{\alpha\to\infty}\sqrt{\frac{\alpha}{\pi i}}\exp[i\alpha x^2]=\delta(x). \tag{5.5-1}$$

A system representation in terms of Wigner distribution functions, however, is far more elegant: the ray spread function of a first-order optical system has the form [Bas-79d]

$$K(x_o,u_o,x_i,u_i)=2\pi\gamma\delta(x_i-Ax_o-Bu_o)\delta(u_i-Cx_o-Du_o), \tag{5.5-2}$$

and its input-output relationship (5.0-5) reduces to

$$F_o(x,u)=\gamma F_i(Ax+Bu,Cx+Du). \tag{5.5-3}$$

The constant γ in these equations is nonnegative; it equals unity if the system is lossless [Bas-79d], i.e., if for any input signal the total energy of the output signal equals that of the input signal [cf. equation (5.0-10)]. The four real constants A, B, C, and D constitute a matrix

$$T=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{5.5-4}$$

which is *symplectic* [Lun-66, Des-72, Bas-79d]; for a 2x2 matrix, symplecticity can be expressed by the condition $AD-BC=1$.

From relation (5.5-2) we conclude that a *single* input ray entering a first-order system at the point x_i with a frequency u_i, yields a *single* output ray leaving the system at the point x_o with a frequency u_o, where x_i, u_i, x_o, and u_o are related by

$$\begin{bmatrix} x_i \\ u_i \end{bmatrix}=\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_o \\ u_o \end{bmatrix}. \tag{5.5-5}$$

Relation (5.5-5) is a well-known geometric-optical matrix description of a first-order optical system [Lun-66]; the $ABCD$-matrix (5.5-4) is known as the *ray transformation matrix* [Des-72]. As examples we give the ray transformation matrices of the four basic systems – lens, free space, Fourier transformer, and magnifier – considered in Sections 5.1 through 5.4, respectively:

$$\begin{bmatrix} 1 & 0 \\ \dfrac{k}{f} & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & -\dfrac{z}{k} \\ 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 0 & -\dfrac{1}{\beta} \\ \beta & 0 \end{bmatrix},\quad \begin{bmatrix} m & 0 \\ 0 & \dfrac{1}{m} \end{bmatrix}. \tag{5.5-6}$$

If two rays, with positions x_1 and x_2, and directions u_1 and u_2, propagate through a first-order system, then the quantity $x_1u_2-u_1x_2$ remains invariant. This quantity is known as the *Lagrange invariant* [Lun-66,Des-72] and is a generalization of the *Smith-Helmholtz invariant* [Bor-75]; the latter invariant applies to an ideal imaging system, with one of the rays passing through the origin. The invariance of the quantity $x_1u_2 - u_1x_2$ is, in fact, equivalent to the property that a first-order system is symplectic.

Quadratic-phase signals (see Example 3.3) fit very well to a first-order optical system, since their general character remains unchanged when they propagate through such a system. We recall that a quadratic-phase signal is completely described by its curvature α. Let α_i be the input curvature, then the output curvature α_o is related to α_i by the *bilinear relation* [Bas-79d]

$$\alpha_i = \frac{C + D\alpha_o}{A + B\alpha_o}, \tag{5.5-7}$$

which follows immediately from relation (5.5-3). For the special first-order systems considered in Sections 5.1 through 5.4, the bilinear relations reduce to

$$\begin{aligned}
\alpha_i &= \frac{k}{f} + \alpha_o, && (5.5\text{-}8)\\
\frac{1}{\alpha_i} &= -\frac{z}{k} + \frac{1}{\alpha_o}, && (5.5\text{-}9)\\
\alpha_i &= -\frac{\beta^2}{\alpha_o}, && (5.5\text{-}10)\\
\alpha_i &= \frac{\alpha_o}{m^2}, && (5.5\text{-}11)
\end{aligned}$$

respectively, which follows directly from substituting the respective ray transformation matrix elements [see expressions (5.5-6)] into the bilinear relation (5.5-7). In fact, the bilinear relation (5.5-7) also applies to Gaussian beams, if we describe such a beam formally by a *complex* curvature [Bas-79d], cf. Example 3.5; we will discuss this point in more detail in Section 8.5.

5.6. Coordinate transformer

As a final example we consider Bryngdahl's *coordinate transformer* [Bry-74], whose point spread function reads as

$$h_{xx}(x_o, x_i) = \sqrt{\frac{\beta}{2\pi i}} \exp[i\gamma(x_i) - i\beta x_o x_i], \tag{5.6-1}$$

with $\gamma(x)$ a slowly varying function of x. The ray spread function of such a coordinate transformer thus takes the form

$$K(x_o, u_o, x_i, u_i) \simeq 2\pi\delta\left(x_i + \frac{u_o}{\beta}\right)\delta\left(u_i - \beta x_o + \frac{d\gamma}{dx_i}\right). \tag{5.6-2}$$

We note that for low-frequency input signals (i.e., $u_i \simeq 0$), relation (5.6-2) directly represents the desired coordinate transformation $\beta x_o = d\gamma/dx_i$.

6. TRANSPORT EQUATIONS FOR THE WIGNER DISTRIBUTION FUNCTION

In the previous section we studied, in Example 5.2, the propagation of the Wigner distribution function through free space, by considering a section of free space as an optical system. It is possible, however, to find the propagation of the Wigner distribution function through free space directly from the differential equation that the signal must satisfy. To show this, we let the space variable z enter into the formulas, and we thus express the signal by $\varphi(x;z)$ and its Wigner distribution function by $F(x,u;z)$; for convenience, we recall the definition of the Wigner distribution function

$$F(x,u;z) = \int \varphi(x+\tfrac{1}{2}x';z)\varphi^*(x-\tfrac{1}{2}x';z)\exp[-iux']dx'. \tag{6.0-1}$$

In the Fresnel approximation of free space, the signal $\varphi(x;z)$ satisfies a differential equation which has the form of a diffusion equation of the parabolic type [Pap-68]:

$$-i\frac{\partial\varphi}{\partial z} = \left(k + \frac{1}{2k}\frac{\partial^2}{\partial x^2}\right)\varphi. \tag{6.0-2}$$

The propagation of the Wigner distribution function is described by a *transport equation* [Bre-73,79 Bes-76, McC-76, Bas-79b,79c], which in this case takes the form

$$\frac{u}{k}\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} = 0. \tag{6.0-3}$$

[Relation (6.0-3) is a special case of the transport equation (6.0-6) that corresponds to the more general differential equation (6.0-5), which will be studied in the next paragraph.] The transport equation (6.0-3) has the solution

$$F(x,u;z) = F\left(x - \frac{z}{k}u, u;0\right), \tag{6.0-4}$$

which is equivalent to the previously derived relation (5.2-4).

The differential equation (6.0-2) is a special case of the more general equation

$$-i\frac{\partial\varphi}{\partial z} = L\left(x, -i\frac{\partial}{\partial x};z\right)\varphi, \tag{6.0-5}$$

where L is some explicit function of the space variables x and z, and of the partial derivative of φ contained in the operator $\partial/\partial x$. The transport equation that corresponds to this differential equation reads

$$-\frac{\partial F}{\partial z} = 2\ \mathrm{Im}\left\{L\left(x+\tfrac{1}{2}i\frac{\partial}{\partial u}, u-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}F; \tag{6.0-6}$$

a derivation of this formula can be found in Appendix B. In the elegant, symbolic notation of Besieris and Tappert [Bes-76], the transport equation (6.0-6) takes the form

$$-\frac{\partial F}{\partial z} = 2\ \mathrm{Im}\left\{L(x,u;z)\exp\left[\tfrac{1}{2}i\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial u} - \frac{\overleftarrow{\partial}}{\partial u}\frac{\overrightarrow{\partial}}{\partial x}\right)\right]\right\}F, \tag{6.0-7}$$

where, depending on the directions of the arrows, the differential operators on the right-hand side operate on $L(x,u;z)$ or $F(x,u;z)$. In the *Liouville approximation* (or *geometric-optical approximation*) the transport equation (6.0-7) reduces to

$$-\frac{\partial F}{\partial z} = 2\ \mathrm{Im}\left\{L(x,u;z)\left[1+\tfrac{1}{2}i(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial u}-\frac{\overleftarrow{\partial}}{\partial u}\frac{\overrightarrow{\partial}}{\partial x})\right]\right\}F, \tag{6.0-8}$$

which is a *linearized* version of relation (6.0-7). In the usual notation this linearized transport equation reads

$$-\frac{\partial F}{\partial z} = 2\ (\mathrm{Im}\,L)\ F + \frac{\partial\,\mathrm{Re}\,L}{\partial x}\frac{\partial F}{\partial u} - \frac{\partial\,\mathrm{Re}\,L}{\partial u}\frac{\partial F}{\partial x}. \tag{6.0-9}$$

Relation (6.0-9) is a first-order partial differential equation, which can be solved by the method of characteristics [Cou-60]: along a path described in a parameter notation by $x = x(s)$, $z = z(s)$, and $u = u(s)$, and defined by the differential equations

$$\frac{dx}{ds} = \frac{\partial\,\mathrm{Re}\,L}{\partial u},\quad \frac{dz}{ds} = 1,\quad \frac{du}{ds} = \frac{\partial\,\mathrm{Re}\,L}{\partial x}, \tag{6.0-10}$$

the *partial* differential equation (6.0-9) reduces to the *ordinary* differential equation

$$\frac{dF}{ds} + 2\ (\mathrm{Im}\,L)\ F = 0. \tag{6.0-11}$$

In the special case that $L(x,u;z)$ is a *real* function of x, u, and z, relation (6.0-11) implies that along the path defined by relations (6.0-10) the Wigner distribution function has a *constant* value (see also, for instance, [Fri-81]).

Let us consider some examples.

6.1. Free space in the Fresnel approximation

In free space in the Fresnel approximation the signal is governed by equation (6.0-2), and the function $L(x,u;z)$ reads $L(x,u;z) = k - u^2/2k$. The corresponding transport equation (6.0-3) and its solution (6.0-4) have already been mentioned in the introductory paragraph of this section.

6.2. Free space

In free space (but not necessarily in the Fresnel approximation) the signal $\varphi(x;z)$ must satisfy the Helmholtz equation, which we write in the form

$$-i\frac{\partial\varphi}{\partial z} = \sqrt{k^2+\frac{\partial^2}{\partial x^2}}\varphi. \tag{6.2-1}$$

The function $L(x,u;z)$ reads

$$L(x,u;z) = \sqrt{k^2-u^2}, \tag{6.2-2}$$

and the linearized transport equation takes the form

$$\frac{u}{k}\frac{\partial F}{\partial x} + \frac{\sqrt{k^2 - u^2}}{k}\frac{\partial F}{\partial z} = 0. \tag{6.2-3}$$

This linearized transport equation can again be solved explicitly; the solution reads

$$F(x, u; z) = F\left(x - \frac{u}{\sqrt{k^2 - u^2}}z, u; 0\right). \tag{6.2-4}$$

Note that in the Fresnel approximation the relations (6.2-1), (6.2-3), and (6.2-4) reduce to (6.0-2), (6.0-3), and (6.0-4), respectively.

When we integrate the linearized transport equation (6.2-3) over the frequency variable u and we use the definitions (4.5-12)-(4.5-14), we find

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_z}{\partial z} = 0, \tag{6.2-5}$$

which shows that the geometrical vector flux J has zero divergence [Win-79].

6.3. Weakly inhomogeneous medium

In a weakly inhomogeneous medium the differential equation that the signal must satisfy, and the function $L(x, u; z)$, are described by relations (6.2-1) and (6.2-2), respectively, but now with $k = k(x, z)$. The linearized transport equation now takes the form

$$\frac{u}{k}\frac{\partial F}{\partial x} + \frac{\sqrt{k^2 - u^2}}{k}\frac{\partial F}{\partial z} + \frac{\partial k}{\partial x}\frac{\partial F}{\partial u} = 0, \tag{6.3-1}$$

which, in general, cannot be solved explicitly. With the method of characteristics we conclude that along the path defined by

$$\frac{dx}{ds} = \frac{u}{k}, \quad \frac{dz}{ds} = \frac{\sqrt{k^2 - u^2}}{k}, \quad \frac{du}{ds} = \frac{\partial k}{\partial x}, \tag{6.3-2}$$

the Wigner distribution function has a constant value. When we eliminate the frequency variable u from equations (6.3-2), we are immediately led to

$$\frac{d}{ds}\left(k\frac{dx}{ds}\right) = \frac{\partial k}{\partial x}, \quad \frac{d}{ds}\left(k\frac{dz}{ds}\right) = \frac{\partial k}{\partial z}, \tag{6.3-3}$$

which are the equations for an optical light ray in geometrical optics [Bor-75]. Note that in a homogeneous medium (i.e., $\partial k/\partial x \equiv \partial k/\partial z \equiv 0$) the linearized transport equation (6.3-1) reduces to (6.2-3), and that the ray paths become straight lines.

6.4. Higher-dimensional Wigner distribution function

Until now the space variables x and z in the Wigner distribution function were treated differently: a frequency variable u was associated with the space variable x, but there was not such a frequency variable associated with the space variable z. This different treatment of the space variables corresponds to the fact that the signal $\varphi(x;z)$ may be chosen arbitrarily, for instance, in a plane $z = constant$; the z-dependence then follows from the properties of the medium through which the signal is propagating. We can, however, define a *higher-dimensional Wigner distribution function*, treating the space variables x and z in like manner, by

$$\mathrm{F}(x,u,z,w) = \iint \varphi(x+\tfrac{1}{2}x';z+\tfrac{1}{2}z')\varphi^*(x-\tfrac{1}{2}x';z-\tfrac{1}{2}z')\exp[-i(ux'+wz')]dx'dz'. \quad (6.4\text{-}1)$$

We can always regain the original Wigner distribution function $F(x,u;z)$ from the higher-dimensional one through the relation

$$F(x,u;z) = \frac{1}{2\pi}\int \mathrm{F}(x,u,z,w)dw. \quad (6.4\text{-}2)$$

The use of the higher-dimensional Wigner distribution function may lead to some mathematical elegance, as we shall show in this section.

The differential equation (6.0-5) is a special case of the more general equation

$$L\left(x,-i\frac{\partial}{\partial x},z,-i\frac{\partial}{\partial z}\right)\varphi = 0. \quad (6.4\text{-}3)$$

The corresponding equation for the Wigner distribution function $\mathrm{F}(x,u,z,w)$ can easily be derived to read

$$2\ \mathrm{Im}\left\{L(x,u,z,w)\exp\left[\tfrac{1}{2}i(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial u}-\frac{\overleftarrow{\partial}}{\partial u}\frac{\overrightarrow{\partial}}{\partial x}+\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial w}-\frac{\overleftarrow{\partial}}{\partial w}\frac{\overrightarrow{\partial}}{\partial z})\right]\right\}\mathrm{F} = 0. \quad (6.4\text{-}4)$$

As an example we shall study again the propagation in free space, which is governed by the Helmholtz equation

$$\frac{1}{k}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\varphi = 0. \quad (6.4\text{-}5)$$

The function $L(x,u,z,w)$ reads $L(x,u,z,w) = k-(u^2+w^2)/k$, and the higher-order Wigner distribution function satisfies the partial differential equation

$$\frac{u}{k}\frac{\partial \mathrm{F}}{\partial x}+\frac{w}{k}\frac{\partial \mathrm{F}}{\partial z} = 0, \quad (6.4\text{-}6)$$

which resembles relation (6.2-3), but, unlike the latter one, is exact.

7. WIGNER DISTRIBUTION FUNCTION FOR PARTIALLY COHERENT LIGHT

As we mentioned before in Section 2.3, the theory of the Wigner distribution function can easily be extended from deterministic signals (completely coherent light) to stochastic signals (partially coherent light), without increasing the dimensionality of the formulas. In Section 7.1 we shall first give a description of partially coherent light and define the Wigner distribution function in this case. Some examples of partially coherent light with their corresponding Wigner distribution function are considered in Section 7.2. In Section 7.3 we will introduce modal expansions for partially coherent light and we will find some new properties for the Wigner distribution function. An overall degree of coherence for partially coherent light will be introduced in Section 7.4.

7.1. Description of partially coherent light

Let partially coherent light be described by a temporally stationary stochastic process $\tilde{\varphi}(x,t)$; the ensemble average of the product $\tilde{\varphi}(x_1,t_2)\tilde{\varphi}^*(x_2,t_2)$ is then only a function of the time difference $t_1 - t_2$:

$$\mathrm{E}\{\tilde{\varphi}(x_1,t_1)\tilde{\varphi}^*(x_2,t_2)\} = \tilde{\Gamma}(x_1,x_2,t_1-t_2). \tag{7.1-1}$$

The function $\tilde{\Gamma}(x_1,x_2,\tau)$ is known as the (mutual) *coherence function* [Wol-54,55, Pap-68, Bas-77] of the stochastic process. The (mutual) *power spectrum* [Pap-68, Bas-77] or *cross-spectral density function* [Man-76] $\Gamma(x_1,x_2,\omega)$ is defined as the temporal Fourier transform of the coherence function:

$$\Gamma(x_1,x_2,\omega) = \int \tilde{\Gamma}(x_1,x_2,\tau)\exp[i\omega\tau]d\tau. \tag{7.1-2}$$

The basic property [Man-76, Bas-77] of the power spectrum is that it is a *nonnegative definite Hermitian function* of x_1 and x_2, i.e.,

$$\Gamma(x_1,x_2,\omega) = \Gamma^*(x_2,x_1,\omega) \tag{7.1-3}$$

and

$$\iint g(x_1,\omega)\Gamma(x_1,x_2,\omega)g^*(x_2,\omega)dx_1dx_2 \geq 0 \tag{7.1-4}$$

for any function $g(x,\omega)$.

Instead of describing a stochastic process in a space domain by means of its power spectrum $\Gamma(x_1,x_2,\omega)$, we can represent it equally well in a spatial-frequency domain by means of the spatial Fourier transform $\bar{\Gamma}(u_1,u_2,\omega)$ of the power spectrum:

$$\bar{\Gamma}(u_1,u_2,\omega) = \iint \Gamma(x_1,x_2,\omega)\exp[-i(u_1x_1-u_2x_2)]dx_1dx_2. \tag{7.1-5}$$

Unlike the power spectrum $\Gamma(x_1,x_2,\omega)$, which expresses the coherence of the light at two different positions, its Fourier transform $\bar{\Gamma}(u_1,u_2,\omega)$ expresses the coherence of the light in two different directions. Therefore we call $\Gamma(x_1, x_2, \omega)$ the *positional* power spectrum [Bas-81b,86a] and $\bar{\Gamma}(u_1, u_2, \omega)$ the *directional* power spectrum [Bas-81b,86a] of the light. It is evident that the directional power spectrum $\bar{\Gamma}(u_1, u_2, \omega)$ is a nonnegative definite Hermitian function of u_1 and u_2.

Apart from the pure space representation of a stochastic process by means of its positional power spectrum or the pure spatial-frequency representation by means of its directional power spectrum, we can describe a stochastic process in space and spatial frequency simultaneously. In this section we therefore use the Wigner distribution function. Since in the present discussion the explicit temporal-frequency dependence is of no importance, we shall, for the sake of convenience, omit the temporal-frequency variable ω from the formulas in the remainder of this section.

The Wigner distribution function of a stochastic process can be defined in terms of the positional power spectrum by

$$F(x, u) = \int \Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x') \exp[-iux']dx' \tag{7.1-6}$$

or, equivalently, in terms of the directional power spectrum by

$$F(x, u) = \frac{1}{2\pi} \int \bar{\Gamma}(u + \tfrac{1}{2}u', u - \tfrac{1}{2}u') \exp[iu'x]du'. \tag{7.1-7}$$

A distribution function according to definitions (7.1-6) and (7.1-7) was first introduced in optics by Walther [Wal-68,78], who called it the *generalized radiance*.

7.2. Examples of Wigner distribution functions

Let us consider some simple examples.

(1) Spatially *incoherent* light can be described by its positional power spectrum, which reads as $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x') = p(x)\delta(x')$, where the 'intensity' $p(x)$ is a nonnegative function. The corresponding Wigner distribution function takes the form $F(x, u) = p(x)$; note that it is a function only of the space variable x and that it does not depend on u.

(2) As a second example, we consider light that is *dual* to incoherent light, i.e., light whose frequency behaviour is similar to the space behaviour of incoherent light and vice versa. Such light is known as *spatially stationary* light. The positional power spectrum of spatially stationary light takes the form $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x') = s(x')$; its directional power spectrum thus reads as $\bar{\Gamma}(u + \frac{1}{2}u', u - \frac{1}{2}u') = \bar{s}(u)\delta(u')$, where the nonnegative function $\bar{s}(u)$ is the Fourier transform of $s(x')$. Note that, indeed, the directional power spectrum of spatially stationary light has a form that is similar to the positional power spectrum of incoherent light. The duality between incoherent light and spatially stationary light is, in fact, the Van Cittert-Zernike theorem.

The Wigner distribution function of spatially stationary light reads as $F(x, u) = \bar{s}(u)$; note that it is a function only of the frequency variable u and that it does not depend on x. It thus has the same form as the Wigner distribution function of incoherent light, except that it is rotated through 90 degrees in the space-frequency plane.

(3) Incoherent light and spatially stationary light are special cases of so-called *quasi-homogeneous* light [Car-77, Wol-78, Bas-81b]. Such quasi-homogeneous light can be locally considered as spatially stationary, having, however, a slowly varying intensity. It can be represented by a positional power spectrum such as $\Gamma(x + \frac{1}{2}x', x - \frac{1}{2}x') \simeq p(x)s(x')$, where p is a slowly varying function compared with s. The Wigner distribution function of quasi-homogeneous light takes the form of a product: $F(x, u) \simeq p(x)\bar{s}(u)$; both $p(x)$ and $\bar{s}(u)$ are nonnegative, which implies that the Wigner distribution function is nonnegative. The special case of incoherent light arises for $\bar{s}(u) = 1$, whereas for spatially stationary light we have $p(x) = 1$.

We remark that – except in some very special cases, like the case of partially coherent Gaussian light treated in the next example, and the obvious cases of spatially incoherent light and spatially stationary light treated in the previous examples – the product forms that arise in the power spectrum and in the Wigner distribution function of quasi-homogeneous light do not hold exactly. One of the problems is that we cannot formulate precisely what should be understood by 'p is a slowly varying function compared with s.' Nevertheless, although in general not physically realizable, the concept of quasi-homogeneous light is often used in optics and may yield useful results. But it should be applied with care!

(4) Let us consider *Gaussian* light, also known as *Gaussian Schell-model* light [Sch-61, Gor-80], whose positional power spectrum reads as

$$\Gamma(x_1, x_2) = \frac{\sqrt{2\sigma}}{\rho} \exp\left[-\left(\frac{\pi}{2\rho^2}\right) \left(\sigma(x_1 + x_2)^2 + \frac{1}{\sigma}(x_1 - x_2)^2 \right) \right] \quad (0 < \sigma \leq 1); \qquad (7.2\text{-}1)$$

the quantity σ is a measure of the coherence of the Gaussian light. The nonnegative definiteness of the power spectrum requires that σ be bounded by 0 and 1; $\sigma = 1$ leads to Gaussian light that is completely coherent (see Example 3.5, with $x_o = u_o = 0$), whereas $\sigma \to 0$ leads to the incoherent limit. The Wigner distribution function of such Gaussian light takes the form

$$F(x, u) = 2\sigma \exp\left[-\sigma \left(\frac{2\pi}{\rho^2} x^2 + \frac{\rho^2}{2\pi} u^2 \right) \right] \quad (0 < \sigma \leq 1), \qquad (7.2\text{-}2)$$

which is again Gaussian both in x and in u. Note that Gaussian light is a special case of quasi-homogeneous light, and that in this special case the product forms hold exactly.

(5) Completely *coherent* light is our final example. Its positional power spectrum $\Gamma(x_1, x_2) = q(x_1)q^*(x_2)$ has the form of a product of a function with its complex-conjugate version [Bas-77]. The Wigner distribution function of coherent light thus reads

$$F(x, u) = \int q(x + \tfrac{1}{2}x')q^*(x - \tfrac{1}{2}x') \exp[-iux']dx', \qquad (7.2\text{-}3)$$

and has the same form as the Wigner distribution function for deterministic signals [see definition (2.1-1)].

7.3. Modal expansions

The properties of the Wigner distribution function that we derived in Section 4 hold for the Wigner distribution function of partially coherent light, as well (see also [Ped-82, Bre-84]). Of course, we must replace the intensity $|\varphi(x)|^2$ of the signal by the *positional intensity* $\Gamma(x,x)$ and the intensity $|\bar{\varphi}(u)|^2$ of the frequency spectrum by the *directional intensity* $\bar{\Gamma}(u,u)$.

Moyal's formula needs some special attention. It now takes the form

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)dxdu = \iint \Gamma_1(x_1,x_2)\Gamma_2^*(x_1,x_2)dx_1dx_2 =$$

$$= \left(\frac{1}{2\pi}\right)^2 \iint \bar{\Gamma}_1(u_1,u_2)\bar{\Gamma}_2^*(u_1,u_2)du_1du_2. \tag{7.3-1}$$

But it still has the property that the integrals yield a nonnegative result, as we shall show in this section.

To derive some inequalities for the Wigner distribution function of partially coherent light, we introduce *modal expansions* for the power spectrum and the Wigner distribution function. We represent the positional power spectrum $\Gamma(x_1,x_2)$ by its modal expansion (see, for instance, [Wol-82], and also [Gam-64, Gor-80] where a modal expansion of the nonnegative definite Hermitian mutual intensity $\tilde{\Gamma}(x_1,x_2,0)$ is introduced)

$$\Gamma(x_1,x_2) = \frac{1}{\rho}\sum_{m=0}^{\infty}\lambda_m q_m\left(\frac{x_1}{\rho}\right)q_m^*\left(\frac{x_2}{\rho}\right); \tag{7.3-2}$$

a similar expansion holds for the directional power spectrum. For the mathematical subtleties of this expansion, we refer to the standard mathematical literature [Cou-53, Rie-55]. In the modal expansion (7.3-2), the functions q_m $(m=0,1,...)$ are the *eigenfunctions*, and the numbers λ_m $(m=0,1,...)$ are the *eigenvalues* of the integral equation

$$\int \Gamma(x_1,x_2)q_m\left(\frac{x_2}{\rho}\right)dx_2 = \lambda_m q_m\left(\frac{x_1}{\rho}\right) \quad (m=0,1,...); \tag{7.3-3}$$

the positive factor ρ is a mere scaling factor. Since the kernel $\Gamma(x_1,x_2)$ is *Hermitian* and under the assumption of *discrete* eigenvalues, the eigenfunctions can be made *orthonormal*:

$$\int q_m(\xi)q_n^*(\xi)d\xi = \delta_{m-n} \quad (m,n=0,1,...). \tag{7.3-4}$$

Moreover, since the kernel $\Gamma(x_1,x_2)$ is *nonnegative definite* Hermitian, the eigenvalues are nonnegative. Note that the light is completely coherent if there is only *one* nonvanishing eigenvalue. As a matter of fact, the modal expansion (7.3-2) expresses the partially coherent light as a superposition of coherent modes.

When we substitute the modal expansion (7.3-2) into the definition (7.1-6) of the Wigner distribution function, the Wigner distribution function can be expressed as

$$F(x,u) = \sum_{m=0}^{\infty} \lambda_m f_m \left(\frac{x}{\rho}, \rho u \right), \tag{7.3-5}$$

where

$$f_m(\xi,\eta) = \int q_m(\xi + \tfrac{1}{2}\xi') q_m^*(\xi - \tfrac{1}{2}\xi') \exp[-i\eta\xi'] d\xi' \quad (m = 0, 1, ...) \tag{7.3-6}$$

are the Wigner distribution functions of the eigenfunctions q_m. By applying Moyal's generalized formula (7.3-1) and using the orthonormality property (7.3-4), it can easily be seen that the Wigner distribution functions f_m satisfy the *orthonormality relation*

$$\frac{1}{2\pi} \iint f_m(\xi,\eta) f_n(\xi,\eta) d\xi d\eta = \left| \int q_m(\xi) q_n^*(\xi) d\xi \right|^2 = \delta_{m-n} \quad (m, n = 0, 1, ...). \tag{7.3-7}$$

As an example, we remark that the eigenvalues λ_m of the Gaussian light (7.2-1) take the form [Gor-80, Sta-82a, Bas-83b]

$$\lambda_m = \frac{2\sigma}{1+\sigma} \left(\frac{1-\sigma}{1+\sigma} \right)^m \quad (0 < \sigma \le 1,\ m = 0, 1, ...), \tag{7.3-8}$$

whereas the eigenfunctions q_m are just the Hermite functions described in Example 3.6; the Wigner distribution functions f_m thus take the form

$$f_m(\xi,\eta) = 2(-1)^m \exp\left[-\left(2\pi\xi^2 + \frac{\eta^2}{2\pi} \right) \right] L_m \left[2 \left(2\pi\xi^2 + \frac{\eta^2}{2\pi} \right) \right] \quad (m = 0, 1, ...), \tag{7.3-9}$$

cf. Example 3.6. Note that for $\sigma = 1$, the eigenvalue λ_0 is the only nonvanishing eigenvalue and that the Gaussian light is completely coherent.

The modal expansion (7.3-2) allows us to reestablish some inequalities for the Wigner distribution function of partially coherent light, which we have already mentioned in Section 4.5.

(1) Using the modal expansion, it is easy to see that *De Bruijn's inequality* (4.5-17) holds not only in the completely coherent (or deterministic) case, but also for the Wigner distribution function of partially coherent light. The uncertainty relation $2d_x d_u \ge 1$, derived in Section 4.5, also holds for partially coherent light; a more sophisticated uncertainty principle for partially coherent light, which will take into account the overall degree of coherence of the light, will be derived in Section 8.7.

(2) If we use *Moyal's generalized formula* (7.3-1) and we expand the power spectra $\Gamma_1(x_1, x_2)$ and $\Gamma_2(x_1, x_2)$ in the form (7.3-2), it can readily be shown that

$$\frac{1}{2\pi} \iint F_1(x,u) F_2(x,u) dx du \ge 0. \tag{7.3-10}$$

Thus, as we remarked before, averaging one Wigner distribution function with another one always yields a nonnegative result. In particular, the averaging with the

Wigner distribution function of completely coherent Gaussian light is of some practical importance [Bru-67, Mar-70, Jan-81a], since the coherent Gaussian Wigner distribution function occupies the smallest possible area in the space-frequency domain, as we concluded before.

(3) An upper bound for the expression that arises in relation (7.3-10) can be found by applying Schwarz' inequality, which yields the relationship

$$\frac{1}{2\pi}\iint F_1(x,u)F_2(x,u)dxdu \leq$$

$$\leq \left(\frac{1}{2\pi}\iint F_1^2(x,u)dxdu\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\iint F_2^2(x,u)dxdu\right)^{\frac{1}{2}}. \tag{7.3-11}$$

The right-hand side of relation (7.3-11) again has an upper bound, which leads to *Schwarz' inequality* (4.5-16); indeed,we have the important inequality

$$\frac{1}{2\pi}\iint F^2(x,u)dxdu \leq \left(\frac{1}{2\pi}\iint F(x,u)dxdu\right)^2. \tag{7.3-12}$$

To prove this inequality, we first remark that, by using the modal expansion (7.3-5), the identity

$$\frac{1}{2\pi}\iint F(x,u)dxdu = \sum_{m=0}^{\infty}\lambda_m \tag{7.3-13}$$

holds. Secondly, we observe the identity

$$\frac{1}{2\pi}\iint F^2(x,u)dxdu = \sum_{m=0}^{\infty}\lambda_m^2, \tag{7.3-14}$$

which can be easily proved by applying the modal expansion (7.3-5) and by using the orthonormality property (7.3-7). Finally, we remark that, since all eigenvalues λ_m are nonnegative, the inequality

$$\sum_{m=0}^{\infty}\lambda_m^2 \leq \left(\sum_{m=0}^{\infty}\lambda_m\right)^2 \tag{7.3-15}$$

holds, which completes the proof of relation (7.3-12). Note that the equality sign in relation (7.3-15), and hence in relation (7.3-12), holds if there is only one nonvanishing eigenvalue, i.e., in the case of complete coherence. The quotient of the two expressions that arise in relation (7.3-12) or relation (7.3-15) can therefore serve as a measure of the *overall degree of coherence* of the light. We shall explore the overall degree of coherence in the next section.

7.4. Overall degree of coherence

To measure the overall degree of coherence of partially coherent light, we introduce quantities that are based on the eigenvalues λ_m of the light. As long as they are based on the eigenvalues λ_m and *not* on the eigenfunctions q_m, they have an interesting property. Since a lossless system does not alter the eigenvalues of the power spectrum [Bas-81b], an overall degree of coherence that is based on these eigenvalues remains invariant when the light propagates through such a system.

For the sake of convenience, we define normalized versions ν_m of the eigenvalues λ_m by

$$\nu_m = \frac{\lambda_m}{\sum_{n=0}^{\infty} \lambda_n} \quad (m = 0, 1, ...); \tag{7.4-1}$$

we have, of course, the relation

$$\sum_{m=0}^{\infty} \nu_m = 1. \tag{7.4-2}$$

As a measure for the overall degree of coherence, we then define the class of quantities μ_p (with parameter p) by [Bas-84a,86b]

$$\mu_p = \left(\sum_{m=0}^{\infty} \nu_m^p \right)^{\frac{1}{p-1}} \quad (p > 1). \tag{7.4-3}$$

Note that for the parameter value $p = 2$, we have in fact the case already mentioned in Section 7.3: μ_2 is just the quotient of the expressions that arise in relation (7.3-15). Note also that, as a consequence of relations (7.3-13) and (7.3-14), we do not have to calculate any eigenvalues when we want to determine μ_2. In this section we shall study some properties of the quantities μ_p.

It can readily be seen that μ_p is bounded by 0 and 1, and that the case $\mu_p = 1$ corresponds to complete coherence, in which case there is only one nonvanishing eigenvalue. Moreover, we remark that μ_p is independent of the parameter p, if there are M *identical* nonvanishing eigenvalues; in that case μ_p takes the value $\mu_p = 1/M$.

Considered as a function of p, μ_p has the following properties [Bas-86b]:

$$\mu_\infty = \lim_{p \to \infty} \mu_p = \nu_{\max}, \tag{7.4-4}$$

$$\mu_1 = \lim_{p \downarrow 1} \mu_p = \exp\left[\sum_{m=0}^{\infty} \nu_m \ln \nu_m \right], \tag{7.4-5}$$

$$\frac{d\mu_p}{dp} \geq 0 \quad (p > 1). \tag{7.4-6}$$

Property (7.4-4) is evident. Property (7.4-5) can be proved as follows. We write μ_p in the form

$$\mu_p = \exp\left[\frac{f(p)}{g(p)}\right], \quad \text{with} \quad f(p) = \ln\sum_{m=0}^{\infty} \nu_m^p \quad \text{and} \quad g(p) = p - 1;$$

hence,

$$f'(p) = \frac{df}{dp} = \frac{\sum_{m=0}^{\infty} \nu_m^p \ln \nu_m}{\sum_{m=0}^{\infty} \nu_m^p} \quad \text{and} \quad g'(p) = \frac{dg}{dp} = 1.$$

We remark that $\lim_{p\downarrow 1} f(p) = \lim_{p\downarrow 1} g(p) = 0$, and with L'Hospital's rule [Abr-70] we conclude that

$$\lim_{p\downarrow 1} \frac{f(p)}{g(p)} = \lim_{p\downarrow 1} \frac{f'(p)}{g'(p)} = \sum_{m=0}^{\infty} \nu_m \ln \nu_m.$$

From the continuity of the exp-function, we finally get property (7.4-5). Property (7.4-6) can be proved as follows. Let $q > p > 1$; we then have the (in)equality

$$\mu_p = \left[\sum_{m=0}^{\infty} \nu_m^p\right]^{\frac{1}{p-1}} = \left[\sum_{m=0}^{\infty} \left(\nu_m^q\right)^{\frac{p-1}{q-1}} (\nu_m)^{\frac{q-p}{q-1}}\right]^{\frac{1}{p-1}} \leq$$

$$\leq \left[\left(\sum_{m=0}^{\infty} \nu_m^q\right)^{\frac{p-1}{q-1}} \left(\sum_{m=0}^{\infty} \nu_m\right)^{\frac{q-p}{q-1}}\right]^{\frac{1}{p-1}} = \left[\sum_{m=0}^{\infty} \nu_m^q\right]^{\frac{1}{q-1}} = \mu_q,$$

where use has been made of Hölder's (in)equality [Abr-70] and relation (7.4-2). We thus conclude that μ_p is a nondecreasing function of the parameter p. We remark that $1/\mu_\infty$ is the 'effective number N of uncorrelated random variables representing a signal' as defined by Starikov and that $1/\mu_2$ is used as an upper bound for this number (see [Sta-82b], Eqs. (20) and (26), respectively).

From property (7.4-6) we conclude that μ_1 is the smallest overall degree of coherence out of the class μ_p. In a certain sense it is the *best* overall degree of coherence, because it exhibits best the departure of partially coherent light from the completely coherent case. Note that μ_1 is related to Shannon's *informational entropy* [O'N-61,63, Gam-64], defined by the expression

$$-\sum_{m=0}^{\infty} \nu_m \ln \nu_m.$$

As an example we consider again partially coherent Gaussian light, described by relation (7.2-1), with eigenvalues described by relation (7.3-8). Substituting from relation (7.3-8) into the definition (7.4-3) of μ_p yields (for $p \downarrow 1$, $p = 2$, and $p \to \infty$)

$$\mu_1 = \frac{2\sigma}{1+\sigma}\left(\frac{1-\sigma}{1+\sigma}\right)^{\frac{1-\sigma}{2\sigma}}, \tag{7.4-7}$$

$$\mu_2 = \sigma, \tag{7.4-8}$$

$$\mu_\infty = \frac{2\sigma}{1+\sigma}, \tag{7.4-9}$$

with $0 < \sigma \le 1$. We remark that for $0 < \sigma \ll 1$, we have $\mu_1 \simeq (2/e)\sigma \simeq 0$ and $\mu_\infty \simeq 2\sigma \simeq 0$, while for $\sigma = 1 - 2\epsilon$ with $0 < \epsilon \ll 1$, we have $\mu_1 \simeq \epsilon^\epsilon \simeq 1$ and $\mu_\infty \simeq 1 - \epsilon \simeq 1$; furthermore, we note that $\mu_1 \ge (2/e)\sigma$ and that $\mu_\infty \ge \sigma$. We finally observe the property $\mu_1 \le \mu_2 \le \mu_\infty$, in accordance with relation (7.4-6).

8. APPLICATIONS OF THE WIGNER DISTRIBUTION FUNCTION IN OPTICS

We have already considered a number of simple applications of the Wigner distribution function in the previous sections of this contribution: in fact, any example that we have considered represents such an application. In this section we shall study some more advanced applications. We begin with the description of some optical set-ups for the optical generation of the Wigner distribution function in Section 8.1. We then focus on three main categories of optical problems in which the concept of the Wigner distribution function can be applied usefully: first, the application to geometric-optical systems (Sections 8.2, 8.3, and 8.4) and, especially, first-order optical systems (Section 8.5); second, the application to problems in which the signals appear quadratically (Section 8.6); and third, the application to problems where properties of the signal are studied in space and frequency simultaneously (Section 8.7).

8.1. Optical generation of the Wigner distribution function

We will first consider a category of applications to problems that are not really optical but in which optics can be very helpful. In acoustics it often appears to be advantageous to describe the one-dimensional acoustical time signal in time and frequency simultaneously, leading to the musical score of the signal; as we have noted before, the musical score is the acoustical counterpart of the ray concept in geometrical optics. Many applications occur where the Wigner distribution function might be a valuable means to extract relevant information from the acoustical signal. We mention speech recognition and speaker identification [Bar-80a,b, Pre-83], acoustical perception [Pre-82], under-water acoustics [Szu-82], etc. Special attention should be paid to the quality evaluation of loud-speakers [Jan-83]; it appears that what you hear when you listen to a loud-speaker, can directly be observed in the Wigner distribution function of the impulse response of the loud-speaker! Hence, it makes sense to generate the musical score of one-dimensional time signals. In order to visualize the musical score, the two-dimensional Wigner distribution function of the one-dimensional signal could be generated by optical means and displayed in the output plane of an optical system. We shall describe some optical set-ups that generate the Wigner distribution function.

The Wigner distribution function of a one-dimensional signal can easily be displayed by optical means. In principle, we can use the optical arrangements that are designed to display the ambiguity function [Sai-73, Mar-77,79, Rho-81]; it suffices to rotate one part of these arrangements around the optical axis through 90 degrees, thus displaying the Wigner distribution function instead of the ambiguity function [Bas-80a]. This can readily be understood by observing that both the Wigner distribution function and the ambiguity function can be considered as Fourier transforms of the product $\varphi(x+\frac{1}{2}y)\varphi^*(x-\frac{1}{2}y)$: the former as a Fourier transform with respect to y, and the latter as a Fourier transform with respect to x.

If we do not have to work in real time and if the signal is real, then the generation of the Wigner distribution function by optical means is rather easy. A simple way to generate the Wigner distribution function is the following [Eic-82]. We fabricate two identical transparencies that are constant in one dimension and that vary in the other dimension according to the signal's amplitude. These two transparencies are placed in cascade – rotated relatively to each other – and illuminated by a plane wave of coherent laser light. We thus generate the product $\varphi(x+\frac{1}{2}y)\varphi(x-\frac{1}{2}y)$ [cf. the definition (2.1-1) of the Wigner distribution function]. An astigmatic optical system follows the two transparencies, and performs a Fourier transformation in the y-direction and an ideal imaging in the x-direction. In the output plane we then have the Wigner distribution function.

It is not difficult to generate the Wigner distribution function, using only *one* transparency [Mar-79, Bas-80a, Bre-82]. In the set-up described by Brenner and Lohmann [Bre-82], the product $\varphi(x+\frac{1}{2}y)\varphi(x-\frac{1}{2}y)$ is generated by imaging the rotated transparency onto itself in such a way that the y-coordinate in the image is inverted; a roof top prism is used to realize this coordinate inversion. After the second passage through the transparency, the light is Fourier-transformed in the y-direction and ideally imaged in the x-direction, as described before.

In the case of complex signals, the production of the Wigner distribution function is not as simple as in the real case. The reason for this becomes apparent from the definitions (2.1-1) and (2.1-2) of the Wigner distribution function, which shows that we need the complex conjugate version of the signal, too. To generate the Wigner distribution function in this case, Brenner and Lohmann [Bre-82] have described two ways. One way uses almost the same set-up as before, but with the transparency of the signal replaced by a hologram of the signal; the hologram contains the required conjugate version of the signal, as well. With some suitable masks, we select the wanted term, and the Wigner distribution function of the complex signal appears in the output plane. The second way mentioned by Brenner and Lohmann starts with the complex signal itself and generates its complex conjugate version by means of a nonlinear material. This second set-up can, in principle, be used for real-time generation of the Wigner distribution function.

One possible application to the processing of one-dimensional signals is obvious. We can generate the two-dimensional Wigner distribution function of the signal, then perform an optical filtering in the Wigner distribution function domain, and derive again a one-dimensional signal from the filtered Wigner distribution function. The overall processing will be a *shift-variant* filtering of the one-dimensional signal. It is not clear whether this processing technique offers advantages over other shift-variant processing techniques.

The Wigner distribution function of a two-dimensional signal is four-dimensional. It is difficult, however, to display such a four-dimensional function. Nevertheless, several set-ups for the optical generation of the Wigner distribution function of two-dimensional signals have been proposed already. Bamler and Glünder [Bam-83] display the Wigner distribution function as two-dimensional slices of the four-dimensional Wigner distribution function, using techniques that are similar to the ones described above for the one-dimensional case. Easton et al. [Eas-84] use a different approach; they use the Radon transformation to generate a Fourier transform and display the one-dimensional Wigner distribution function as one-dimensional slices of the four-dimensional Wigner distribution function. Again, the Wigner distribution function of the two-dimensional signal might be a valuable tool to extract relevant information from the signal, for instance in distinguishing patterns with known texture direction.

8.2. Geometric-optical systems

Let us start by studying a *modulator* (cf. Example 5.1) described by the coherent input-output relationship $\varphi_o(x) = m(x)\varphi_i(x)$; for partially coherent light, the input-output relationship reads as $\Gamma_o(x_1, x_2) = m(x_1)\Gamma_i(x_1, x_2)m^*(x_2)$. The input and output Wigner distribution functions are related by the relationship

$$F_o(x, u_o) = \frac{1}{2\pi}\int F_i(x, u_i)du_i \int m(x + \tfrac{1}{2}x')m^*(x - \tfrac{1}{2}x') \exp[-i(u_o - u_i)x']dx'. \qquad (8.2\text{-}1)$$

This input-output relationship can be written in two distinct forms. On the one hand we can represent it in a *differential* format reading as follows:

$$F_o(x, u) = m\left(x + \tfrac{1}{2}i\frac{\partial}{\partial u}\right) m^*\left(x - \tfrac{1}{2}i\frac{\partial}{\partial u}\right) F_i(x, u). \qquad (8.2\text{-}2)$$

On the other hand, we can represent it in an *integral* format that reads as follows [cf. relation (5.1-4)]:

$$F_o(x, u) = \frac{1}{2\pi}\int F_m(x, u - u_i)F_i(x, u_i)du_i, \qquad (8.2\text{-}3)$$

where $F_m(x, u)$ is the Wigner distribution function of the modulation function $m(x)$. Which of these two forms is superior depends on the problem.

We now confine ourselves to the case of a *pure phase* modulation function $m(x) = \exp[i\gamma(x)]$. We then get

$$m(x+\tfrac{1}{2}x')m^*(x-\tfrac{1}{2}x') = \exp\left[i\sum_{k=0}^{\infty}\frac{2}{(2k+1)!}\gamma^{(2k+1)}(x)(\tfrac{1}{2}x')^{2k+1}\right], \tag{8.2-4}$$

where the expression $\gamma^{(n)}(x)$ denotes the nth derivative of $\gamma(x)$. If we consider only the first-order derivative in relation (8.2-4), we arrive at the following expressions:

$$m\left(x+\tfrac{1}{2}i\frac{\partial}{\partial u}\right)m^*\left(x-\tfrac{1}{2}i\frac{\partial}{\partial u}\right) \simeq \exp\left[-\frac{d\gamma}{dx}\frac{\partial}{\partial u}\right], \tag{8.2-5}$$

$$F_m(x,u) \simeq 2\pi\delta\left(u-\frac{d\gamma}{dx}\right), \tag{8.2-6}$$

and the input-output relationship of the pure phase modulator becomes

$$F_o(x,u) \simeq F_i\left(x, u-\frac{d\gamma}{dx}\right), \tag{8.2-7}$$

which is a mere coordinate transformation. We conclude that a single input ray yields a single output ray.

The ideas described above have been applied to the design of optical coordinate transformers [Bry-74, Jia-84] and to the theory of aberrations [Loh-83]. Now, if the first-order approximation is not sufficiently accurate, i.e., if we have to take into account higher-order derivatives in relation (8.2-4), the Wigner distribution function allows us to overcome this problem. Indeed, we still have the exact input-output relationships (8.2-2) and (8.2-3), and we can take into account as many derivatives in relation (8.2-4) as necessary. We thus end up with a more general differential form [Fra-82] than expression (8.2-5) or a more general integral form [Jan-82] than expression (8.2-6). The latter case, for instance, will yield an Airy function [Abr-70] instead of a Dirac function, when we take not only the first but also the third derivative into account.

From expression (8.2-7) we concluded that a single input ray yields a single output ray. This may also happen in more general – not just modulation-type – systems; we call such systems *geometric-optical systems*. These systems have the simple input-output relationship

$$F_o(x,u) \simeq F_i[g_x(x,u), g_u(x,u)], \tag{8.2-8}$$

where the $\simeq$ sign becomes an $=$ sign in the case of linear functions g_x and g_u, i.e., in the case of Luneburg's first-order optical systems, which we have considered in Section 5.5. There appears to be a close relationship to the description of such geometric-optical systems by means of the Hamilton characteristics [Bas-79a].

Instead of the black-box approach of a geometric-optical system, which leads to the input-output relationship (8.2-8), we can also consider the system as a continuous medium and formulate transport equations, as we did in Section 6. For geometric-optical systems, this transport equation takes the form of a first-order partial differential equation [Mar-80], which can be solved by the method of characteristics. In Section 6 we reached the general conclusion that these characteristics represent the geometric-optical ray paths and that along these ray paths the Wigner distribution function has a constant value.

The use of the transport equation is not restricted to *deterministic* media; Bremmer [Bre-79] has applied it to *stochastic* media. Neither is the transport equation restricted to the *scalar* treatment of wave fields; Bugnolo and Bremmer [Bug-83] have applied it to study the propagation of *vectorial* wave fields. In the vectorial case, the concept of the Wigner distribution function leads to a Hermitian matrix rather than to a scalar function and permits the description of *nonisotropic* media as well.

We have already considered some examples of geometric-optical systems in Section 6; two more advanced examples are studied in the next two sections. Other examples are described by Ojeda-Castañeda and Sicre [Oje-85].

8.3. Flux transport through free space

Let us consider, in the $z = 0$ plane, a quasi-homogeneous planar Lambertian source [Car-77, Wol-78], whose positional intensity is uniform in the interval $|x| < x_{\max}$ and vanishes outside that interval, and whose radiant intensity has the directional dependence $\cos\theta$ in the interval $|\theta| < \theta_{\max}$ and vanishes outside that interval; as usual in radiometry, θ is the observation angle with respect to the z axis. The Wigner distribution function of such a source is given by

$$F(x,u) = \frac{\pi}{2x_{max} k \sin\theta_{max}} \operatorname{rect}\left(\frac{x}{2x_{max}}\right) \operatorname{rect}\left(\frac{u}{2k\sin\theta_{max}}\right) \frac{k}{\sqrt{k^2-u^2}}; \tag{8.3-1}$$

for convenience, we have normalized the total radiant flux [Car-77] to unity:

$$\frac{1}{2\pi}\iint F(x,u)\frac{\sqrt{k^2-u^2}}{k}dxdu = 1. \tag{8.3-2}$$

We wish to determine the radiant flux through an aperture with width $2x_{max}$, parallel to the source plane, and symmetrically located around the z axis at a distance z_o from the source plane. In the geometric-optical approximation, the Wigner distribution function at the $z = z_o$ plane reads as $F(x + z_o u/\sqrt{k^2-u^2}, u)$, and the radiant flux through the aperture follows readily from the integral

$$\frac{1}{2\pi}\int_{-x_{max}}^{x_{max}} dx \int du\; F\left(x + \frac{u}{\sqrt{k^2-u^2}} z_o, u\right) \frac{\sqrt{k^2-u^2}}{k} =$$

$$= \frac{2x_{max}\sin\gamma - z_o(1-\cos\gamma)}{2x_{max}\sin\theta_{max}}, \tag{8.3-3}$$

where $\gamma = \min[\theta_{max}, \arctan(2x_{max}/z_o)]$.

Similar techniques can be applied in the more general case when the source and the aperture have different widths and when the optical axes of the source and the aperture planes are translated or even rotated with respect to each other. Such problems arise, for instance, when two optical fibers are not ideally connected to each other and we want to determine the energy transfer from one fiber to the other [Ett-85].

8.4. Rotationally symmetric fiber

As our last example of a geometric-optical system, let us consider – by way of exception – a *two*-dimensional one. In an *optical fiber* that extends along the z axis, the signal depends, at a certain z value, on the two transverse space variables x and y; its Wigner distribution function depends on these space variables and on the two frequency variables u and v. The transport equation in the fiber now has the form

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+\sqrt{k^2-(u^2+v^2)}\frac{\partial F}{\partial z}+k\frac{\partial k}{\partial x}\frac{\partial F}{\partial u}+k\frac{\partial k}{\partial y}\frac{\partial F}{\partial v}=0, \tag{8.4-1}$$

which is the two-dimensional analogue of the transport equation (6.3-1). We now assume that the index of refraction has a *rotationally symmetric* profile; hence, $k=k(\sqrt{x^2+y^2})$. When we apply the coordinate transformation

$$x=r\cos\varphi,\quad y=r\sin\varphi,\quad h=ux-vy,\quad k^2=u^2+v^2+w^2, \tag{8.4-2}$$

we arrive at the transport equation

$$\sqrt{k^2-w^2-\frac{h^2}{r^2}}\frac{\partial F}{\partial r}+\frac{h}{r^2}\frac{\partial F}{\partial\varphi}+w\frac{\partial F}{\partial z}=0. \tag{8.4-3}$$

We remark that the derivatives of the Wigner distribution function with respect to the *ray invariants* h and w do not enter the transport equation (8.4-3). From the definition of the characteristics

$$w\frac{dh}{dz}=0,\quad w\frac{dw}{dz}=0,\quad w\frac{dr}{dz}=\sqrt{k^2-w^2-\frac{h^2}{r^2}},\quad w\frac{d\varphi}{dz}=\frac{h}{r^2}, \tag{8.4-4}$$

we conclude that $dh/dz=dw/dz=0$, and h and w are, indeed, invariant along a ray.

8.5. Gaussian beams and first-order optical systems

The propagation of the first- and second-order moments of the Wigner distribution function through a first-order optical system, described by a ray transformation matrix (cf. Section 5.5)

$$T=\begin{bmatrix} A & B\\ C & D\end{bmatrix}, \tag{8.5-1}$$

can be phrased in an easy way [Bas-79d]. For the first-order moments m_x and m_u of the input and the output signal we find

$$\begin{bmatrix} m_x^i\\ m_u^i\end{bmatrix}=\begin{bmatrix} A & B\\ C & D\end{bmatrix}\begin{bmatrix} m_x^o\\ m_u^o\end{bmatrix}, \tag{8.5-2}$$

whereas for the second-order moments m_{xx}, m_{xu}, m_{ux}, and m_{uu} we have

$$\begin{bmatrix} m^i_{xx} & m^i_{xu} \\ m^i_{ux} & m^i_{uu} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} m^o_{xx} & m^o_{xu} \\ m^o_{ux} & m^o_{uu} \end{bmatrix} \begin{bmatrix} A & C \\ B & D \end{bmatrix}. \tag{8.5-3}$$

Relation (8.5-3) yields a number of relationships between the second-order moments of the Wigner distribution functions of the input and the output signal at a first-order system. These relations are equivalent to the relationships formulated by Papoulis [Pap-74] for the second derivatives of the ambiguity functions of these signals. This can easily be understood: since the ambiguity function and the Wigner distribution function are related through a Fourier transformation [see relation (2.2-2)], moments of one function are equal to derivatives of the other, and vice versa.

Relation (8.5-3) provides a proof for the nonnegative definiteness of the symmetric matrix M [see relation (4.5-9)] of second-order moments. Indeed, from relation (8.5-3) we conclude that the quadratic form $A^2 m^o_{xx} + AB(m^o_{xu} + m^o_{ux}) + B^2 m^o_{uu}$ is nonnegative for arbitrary values of A and B. Nonnegative definiteness of the symmetric matrix M implies the following inequalities:

$$m_{xx} \geq 0, \tag{8.5-4}$$

$$m_{uu} \geq 0, \tag{8.5-5}$$

$$m_{xx} m_{uu} - m^2_{xu} \geq 0. \tag{8.5-6}$$

We shall now find a different representation for the propagation of the second-order moments through a first-order system. First we note that as a direct consequence of the symplecticity of the ray transformation matrix T, the determinant of the matrix M is invariant when the light propagates through a first-order system. To find a further propagation law, we first consider the second-order moments of a general Gaussian signal.

If a Gaussian signal whose Wigner distribution function has the form (7.2-2) is the input signal of a first-order optical system described by the ray transformation matrix (8.5-1), then the output Wigner distribution function has the form

$$F(x,u) = 2\sigma \exp\left[-\sigma\left(\frac{2\pi}{\rho^2}(Ax+Bu)^2 + \frac{\rho^2}{2\pi}(Cx+Du)^2\right)\right]. \tag{8.5-7}$$

Since the ray transformation matrix is symplectic, this output Wigner distribution function can be expressed in the form

$$F(x,u) = 2\sigma \exp\left[-\sigma\left(\beta x^2 + \frac{1}{\beta}(u-\alpha x)^2\right)\right] \quad (\beta > 0,\ 0 < \sigma \leq 1), \tag{8.5-8}$$

where the parameters α and β are related to $A,\ B,\ C,\ D$, and ρ through the formulas

$$\frac{1}{\beta} = \frac{2\pi}{\rho^2} B^2 + \frac{\rho^2}{2\pi} D^2, \tag{8.5-9}$$

$$-\frac{\alpha}{\beta} = \frac{2\pi}{\rho^2} AB + \frac{\rho^2}{2\pi} CD. \tag{8.5-10}$$

The Wigner distribution function of the form (8.5-8) is in fact the Wigner distribution function of a cross section through a Gaussian beam [cf. relation (3.5-4)]. When this beam propagates through a first-order optical system, the *beam parameters* α and β change, but the general form (8.5-8) of the Wigner distribution function is preserved. To be more specific, we calculate the matrix of second-order moments for a signal of the form (8.5-8),

$$M = \frac{1}{2\sigma}\begin{bmatrix} \dfrac{1}{\beta} & \dfrac{\alpha}{\beta} \\ \dfrac{\alpha}{\beta} & \dfrac{\alpha^2+\beta^2}{\beta} \end{bmatrix} \quad (\beta > 0,\ 0 < \sigma \leq 1), \tag{8.5-11}$$

and substitute it in the propagation law (8.5-3). We find that if a Gaussian beam with beam parameters α_i, β_i, and σ_i forms the input of a first-order optical system with the ray transformation matrix (8.5-1), then the Gaussian beam at the output of the system has parameters α_o, β_o, and σ_o, where $\sigma_o = \sigma_i$ and where α_o and β_o are related to α_i and β_i by the relations

$$\frac{\beta_o}{\beta_i} = (A + B\alpha_o)^2 + B^2\beta_o^2, \tag{8.5-12}$$

$$\alpha_i\frac{\beta_o}{\beta_i} = (A + B\alpha_o)(C + D\alpha_o) + BD\beta_o^2. \tag{8.5-13}$$

When we introduce the *complex beam parameter* γ through $\gamma = \alpha + i\beta$, relations (8.5-12) and (8.5-13) can be combined into one relation that has the bilinear form (5.5-7)

$$\gamma_i = \frac{C + D\gamma_o}{A + B\gamma_o}; \tag{8.5-14}$$

hence, the complex beam parameter γ behaves like the curvature of a quadratic-phase signal.

Now, since any nonnegative definite symmetrix matrix M can be expressed in the form (8.5-11), the propagation law (8.5-14) also holds for the second-order moments of an arbitrary signal. If we express the matrix M in the form (8.5-11), the parameters α, β, and σ are determined by the second-order moments through the relations

$$\alpha = \frac{m_{xu}}{m_{xx}}, \tag{8.5-15}$$

$$\beta = \frac{\sqrt{m_{xx}m_{uu} - m_{xu}^2}}{m_{xx}} = \frac{m}{m_{xx}}, \tag{8.5-16}$$

$$\sigma = \frac{1}{2\sqrt{m_{xx}m_{uu} - m_{xu}^2}} = \frac{1}{2m}, \tag{8.5-17}$$

where we have introduced the quantity $m = \sqrt{m_{xx}m_{uu} - m_{xu}^2}$. Since m^2 is equal to the determinant of M, it remains invariant when the light propagates through a first-order optical system. From this invariance and the definition of the quantity

m, we directly conclude that $m_{xx}m_{uu}$ takes its minimum value m^2 when $m_{xu} = 0$. In terms of the beam parameters, this situation occurs when $\alpha = 0$; if we were dealing with Gaussian beams, this would mean that we are at the *waist* of the beam. From the uncertainty principle $2d_x d_u \geq 1$, we conclude that the quantity m is larger than $\frac{1}{2}$, which corresponds to the condition $\sigma \leq 1$ that arises in relation (8.5-11).

8.6. Quadratic signal dependence

An important category of applications of the Wigner distribution function is formed by those problems in which the signal enters the theory quadratically. The most important application in this category is to the theory of partial coherence. We have already discussed this application in great detail in Section 7 (see also the applications mentioned in Section 8).

Another obvious example in this category is the treatment of so-called *bilinear systems*, as studied by Ojeda-Castañeda and Sicre [Oje-84]. Such a bilinear system is described by the input-output relationship

$$|\varphi_o(x_o)|^2 = \iint g(x_o;x_1,x_2)\varphi_i(x_1)\varphi_i^*(x_2)dx_1dx_2, \tag{8.6-1}$$

in which the system kernel is Hermitian, $g(x_o;x_1,x_2) = g^*(x_o;x_2,x_1)$, and nonnegative definite. With

$$F_g(x_o;x,u) = \int g(x_o;x+\tfrac{1}{2}x', x-\tfrac{1}{2}x')\exp[-iux']dx', \tag{8.6-2}$$

we can write the input-output relationship (8.6-1) in the form

$$|\varphi_o(x_o)|^2 = \frac{1}{2\pi}\iint F_g(x_o;x,u)F_i(x,u)dxdu. \tag{8.6-3}$$

Suppose now that we may identify F_g with the projection of a so-called *bilinear ray spread function* $G(x_o,u_o;x,u)$:

$$F_g(x_o;x,u) = \frac{1}{2\pi}\int G(x_o,u_o;x,u)du_o. \tag{8.6-4}$$

We may then formulate the input-output relationship in terms of Wigner distribution functions as

$$F_o(x_o,u_o) = \frac{1}{2\pi}\iint G(x_o,u_o;x_i,u_i)F_i(x_i,u_i)dx_idu_i, \tag{8.6-5}$$

and the output intensity follows by means of the identity

$$|\varphi_o(x_o)|^2 = \frac{1}{2\pi}\int F(x_o,u_o)du_o. \tag{8.6-6}$$

Note the similarity between relations (8.6-5) and (5.0-5). Of course, the bilinear ray spread function G cannot be fully determined from the projection relationship (8.6-4). However, the link between the system kernel g and the bilinear ray spread function G can be very instructive, as was shown by Ojeda-Castañeda and Sicre [Oje-84].

8.7. Uncertainty principle and informational entropy

The ordinary uncertainty principle $2d_x d_u \geq 1$ tells us that the product of the effective widths of the intensity functions in the space and the frequency domain has a lower bound and that this lower bound is reached when the light is completely coherent and Gaussian. We found the same uncertainty relation for partially coherent light in Section 7.3. In this section we derive more advanced uncertainty principles [Bas-84a,86b], by taking into account the overall degree of coherence of the light. As a matter of fact, what we expect from an uncertainty principle for partially coherent light is that the product of the effective widths still has a lower bound but that this lower bound depends on the overall degree of coherence of the light: the better the coherence, the smaller the lower bound. Hence we need a measure of the overall degree of coherence, for which the quantity μ_p defined in Section 7.4 will turn out to be a good choice.

We start the derivation of the uncertainty principle with the important identity (see, for instance, [Jan-81b], Sec.3.2, Ex.(viii), case $\mu = 1$)

$$\frac{2\pi}{\rho^2} d_x^2 + \frac{\rho^2}{2\pi} d_u^2 = \sum_{m=0}^{\infty} \nu_m \sum_{n=0}^{\infty} (2n+1) \left| \int q_m(\xi) \psi_n^*(\xi) d\xi \right|^2 . \tag{8.7-1}$$

The expression $(2\pi/\rho^2)d_x^2 + (\rho^2/2\pi)d_u^2$ is thus described in terms of the normalized eigenvalues ν_m and the inner products of the eigenfunctions q_m with the Hermite functions ψ_m. Note that due to the orthonormality of the eigenfunctions [relation (7.3-4)] and the Hermite functions [relation (3.6-3)], the inner products satisfy the orthonormality property

$$\sum_{n=0}^{\infty} \left(\int q_l(\xi) \psi_n^*(\xi) d\xi \right) \left(\int q_m(\xi) \psi_n^*(\xi) d\xi \right)^* = \delta_{l-m} \quad (l, m = 0, 1, ...). \tag{8.7-2}$$

If we assume (without loss of generality) that the eigenvalues are ordered in decreasing order, $\nu_0 \geq \nu_1 \geq ... \geq 0$, then the right-hand side of relation (8.7-1) has the lower bound [Bas-83a,84a]

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (2n+1) \left| \int q_m(\xi) \psi_n^*(\xi) d\xi \right|^2 \geq \sum_{m=0}^{\infty} \nu_m (2m+1); \tag{8.7-3}$$

the proof of this (in)equality can be found in Appendix C. The equality sign in relation (8.7-3) holds if $| \int q_m(\xi) \psi_n^*(\xi) d\xi | = \delta_{m-n}$, which means that the eigenfunctions q_m are just the Hermite functions ψ_m. Combining relations (8.7-1) and (8.7-3), and choosing $\rho^2 = 2\pi(d_x/d_u)$, we are led to the basic relationship

$$2d_x d_u \geq \sum_{m=0}^{\infty} \nu_m (2m+1). \tag{8.7-4}$$

To find an uncertainty principle for partially coherent light, we have to find a lower bound for the right-hand side of the (in)equality (8.7-4); this will be the subject of this section.

The (in)equality

$$\sum_{m=0}^{\infty} \nu_m(2m+1) \geq \sum_{m=0}^{\infty} \nu_m = 1, \tag{8.7-5}$$

in which the equality sign holds if $\nu_0 = 1$ and $\nu_1 = \nu_2 = \ldots = 0$, is evident; it leads to the ordinary uncertainty relation $2d_x d_u \geq 1$, in which the equality sign holds for completely coherent Gaussian light. A less evident uncertainty relation follows from the (in)equality [Bas-84a]

$$\sum_{m=0}^{\infty} \nu_m(2m+1) \geq 2\left(2-\frac{1}{p}\right)^{-\frac{p}{p-1}} \left(\sum_{m=0}^{\infty} \nu_m^p\right)^{-\frac{1}{p-1}} \quad (p>1), \tag{8.7-6}$$

where the equality sign holds for $p \to \infty$ and identical nonvanishing eigenvalues ν_m; note that the overall degree of coherence μ_p [cf. relation (7.4-3)] has entered the formula. Relation (8.7-6) together with relation (8.7-4) leads to the uncertainty relation

$$2d_x d_u \geq \frac{c(p)}{\mu_p} \quad (p>1), \tag{8.7-7}$$

where the function $c(p)$ (whose values are in the order of 1) depends on the parameter p through the relation

$$c(p) = 2\left(2-\frac{1}{p}\right)^{-\frac{p}{p-1}} \quad (p>1). \tag{8.7-8}$$

Note that $c(1) = 2/e \simeq 0.736$, $c(2) = 8/9 \simeq 0.889$, and $c(\infty) = 1$.

To formulate an even more sophisticated uncertainty relation for partially coherent light, we proceed in the following way [Bas-86b]. We choose a measure for the overall degree of coherence of the light, for which we use the quantity μ_1, based on Shannon's informational entropy (see Section 7.4)

$$-\sum_{m=0}^{\infty} \nu_m \ln \nu_m,$$

and then solve the following problem: among all partially coherent wave fields having the same informational entropy, find the wave field that minimizes the product $d_x d_u$. To solve this problem, we have to find the distribution of normalized eigenvalues ν_m for which the right-hand side of relation (8.7-4) takes its minimum value under the constraints

$$\nu_0 \geq \nu_1 \geq \ldots \geq 0, \quad \sum_{m=0}^{\infty} \nu_m = 1, \quad \text{and} \quad -\sum_{m=0}^{\infty} \nu_m \ln \nu_m = \mathit{constant}.$$

Using standard variation techniques, it is not difficult to show that the minimum of this right-hand side occurs when the normalized eigenvalues take the form of relation (7.3-8) and that this minimum takes the value $1/2\sigma$; the quantity σ is related to the informational entropy through the formula

$$\exp\left[\sum_{m=0}^{\infty} \nu_m \ln \nu_m\right] = \frac{2\sigma}{1+\sigma}\left(\frac{1-\sigma}{1+\sigma}\right)^{\frac{1-\sigma}{2\sigma}} \quad (0 < \sigma \leq 1) \tag{8.7-9}$$

[cf. relations (7.4-5) and (7.4-7)]. We thus conclude that an uncertainty principle for partially coherent light reads as

$$2d_x d_u \geq \frac{1}{\sigma}, \tag{8.7-10}$$

where σ is related to the informational entropy through relation (8.7-9) and where the equality sign holds if the light is Gaussian (but not necessarily coherent). Note that, since $\mu_1 \geq c(1)\sigma = (2/e)\sigma$ (cf. the final remarks in Section 7.4), the right-hand side of relation (8.7-10) is larger than the right-hand side of relation (8.7-7) for $p = 1$, which shows that the final uncertainty relation (8.7-10) is indeed the most restrictive one.

9. APPENDICES

9.1. Appendix A. Proof of condition (2.1-5)

The proof that relation (2.1-5) is a *necessary* condition can easily be given by showing that a Wigner distribution function indeed satisfies this relation. The proof that relation (2.1-5) is a *sufficient* condition is slightly more difficult. We first note that a real function $F(x, u)$ can always be expressed in the form

$$F(x, u) = \int \Gamma(x + \tfrac{1}{2}x', x - \tfrac{1}{2}x') \exp[-iux']dx', \tag{A-1}$$

with $\Gamma(x_1, x_2)$ a Hermitian function, i.e., $\Gamma(x_1, x_2) = \Gamma^*(x_2, x_1)$. We shall now prove the sufficiency of relation (2.1-5) by showing that if a real function $F(x, u)$ of the form (A-1) satisfies relation (2.1-5), the function $\Gamma(x_1, x_2)$ factorizes in the form

$$\Gamma(x_1, x_2) = \varphi(x_1)\varphi^*(x_2), \tag{A-2}$$

and the function $F(x, u)$ is thus a Wigner distribution function. We therefore substitute the form (A-1) into relation (2.1-5). The *left-hand side* of relation (2.1-5) can then be written as

$$\iint \Gamma(a + \tfrac{1}{2}x + \tfrac{1}{2}x', a + \tfrac{1}{2}x - \tfrac{1}{2}x')\Gamma^*(a - \tfrac{1}{2}x + \tfrac{1}{2}x'', a - \tfrac{1}{2}x - \tfrac{1}{2}x'')$$
$$\times \exp[-i(b + \tfrac{1}{2}u)x' + i(b - \tfrac{1}{2}u)x'')]dx'dx'';$$

after the change of variables $x' = \xi + \frac{1}{2}\xi'$, $x'' = \xi - \frac{1}{2}\xi'$, we arrive at

$$\iint \Gamma(a + \tfrac{1}{2}x + \tfrac{1}{2}\xi + \tfrac{1}{4}\xi', a + \tfrac{1}{2}x - \tfrac{1}{2}\xi - \tfrac{1}{4}\xi')\Gamma^*(a - \tfrac{1}{2}x + \tfrac{1}{2}\xi - \tfrac{1}{4}\xi', a - \tfrac{1}{2}x - \tfrac{1}{2}\xi + \tfrac{1}{4}\xi')$$
$$\times \exp[-i(b\xi' + u\xi)]d\xi d\xi'. \tag{A-3}$$

The *right-hand side* of relation (2.1-5) takes the form

$$\frac{1}{2\pi}\iiiint \Gamma(a+\tfrac{1}{2}x_o+\tfrac{1}{2}x', a+\tfrac{1}{2}x_o-\tfrac{1}{2}x')\Gamma^*(a-\tfrac{1}{2}x_o+\tfrac{1}{2}x'', a-\tfrac{1}{2}x_o-\tfrac{1}{2}x'')$$

$$\times\exp[-i(ux_o-u_ox)-i(b+\tfrac{1}{2}u_o)x'+i(b-\tfrac{1}{2}u_o)x'']dx'dx''dx_odu_o,$$

which, after the change of variables $x'=\xi+\frac{1}{2}\xi'$, $x''=\xi-\frac{1}{2}\xi'$, can be written as

$$\frac{1}{2\pi}\iiiint \Gamma(a+\tfrac{1}{2}x_o+\tfrac{1}{2}\xi+\tfrac{1}{4}\xi', a+\tfrac{1}{2}x_o-\tfrac{1}{2}\xi-\tfrac{1}{4}\xi')$$

$$\times\Gamma^*(a-\tfrac{1}{2}x_o+\tfrac{1}{2}\xi-\tfrac{1}{4}\xi', a-\tfrac{1}{2}x_o-\tfrac{1}{2}\xi+\tfrac{1}{4}\xi')\exp[iu_o(x-\xi)-i(b\xi'+ux_o)]d\xi d\xi'dx_odu_o;$$

integrating over u_o and ξ, and then changing the variable x_o into ξ results in

$$\iint \Gamma(a+\tfrac{1}{2}\xi+\tfrac{1}{2}x+\tfrac{1}{4}\xi', a+\tfrac{1}{2}\xi-\tfrac{1}{2}x-\tfrac{1}{4}\xi')\Gamma^*(a-\tfrac{1}{2}\xi+\tfrac{1}{2}x-\tfrac{1}{4}\xi', a-\tfrac{1}{2}\xi-\tfrac{1}{2}x+\tfrac{1}{4}\xi')$$

$$\times\exp[-i(b\xi'+u\xi)]d\xi d\xi'. \tag{A-4}$$

Equality of the expressions (A-3) and (A-4) implies

$$\Gamma(x_1,x_2)\Gamma^*(x_3,x_4)=\Gamma(x_1,x_3)\Gamma^*(x_2,x_4) \tag{A-5}$$

for all x_1, x_2, x_3, and x_4. From relation (A-5) we conclude that the function $\Gamma(x_1,x_2)$ factorizes in the form (A-2), and that the corresponding function $F(x,u)$ is thus a Wigner distribution function.

9.2. Appendix B. Derivation of the transport equation (6.0-6)

Starting from the differential equation

$$\left\{i\frac{\partial}{\partial z}+L\left(x,-i\frac{\partial}{\partial x};z\right)\right\}\varphi=0, \tag{B-1}$$

we can formulate the relation

$$\left\{i\frac{\partial}{\partial z}+L\left(x_1,-i\frac{\partial}{\partial x_1};z\right)-L^*\left(x_2,-i\frac{\partial}{\partial x_2};z\right)\right\}\varphi(x_1,z)\varphi^*(x_2,z)=0 \tag{B-2}$$

for the product $\varphi(x_1,z)\varphi^*(x_2,z)$. Expressing this product in terms of the Wigner distribution function $F(x,u;z)$ through the inversion formula (2.1-3) yields

$$\frac{1}{2\pi}\int\left\{(i\frac{\partial}{\partial z}+L\left(x_1,-i\frac{\partial}{\partial x_1};z\right)-L^*\left(x_2,-i\frac{\partial}{\partial x_2};z\right)\right\}$$

$$\times F[\tfrac{1}{2}(x_1+x_2),u_o;z]\exp[iu_o(x_1-x_2)]du_o=0, \tag{B-3}$$

which can be expressed as

$$\frac{1}{2\pi}\int\left\{i\frac{\partial}{\partial z}+L\left(x+\tfrac{1}{2}x',u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)-L^*\left(x-\tfrac{1}{2}x',u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}$$

$$\times F(x,u_o;z)\exp[iu_ox']du_o=0. \tag{B-4}$$

Multiplication of relation (B-4) by $\exp[-iux']$, and writing the integral over x' yields

$$\frac{1}{2\pi}\iint\left\{i\frac{\partial}{\partial z}+L\left(x+\tfrac{1}{2}x',u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)-L^*\left(x-\tfrac{1}{2}x',u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}$$

$$\times F(x,u_o;z)\exp[i(u_o-u)x']du_odx'=0, \tag{B-5}$$

which can be expressed as

$$\frac{1}{2\pi}\iint\left\{i\frac{\partial}{\partial z}+L\left(\tfrac{1}{2}i\frac{\partial}{\partial u},u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)-L^*\left(\tfrac{1}{2}i\frac{\partial}{\partial u},u_o-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}$$

$$\times F(x,u_o;z)\exp[i(u_o-u)x']du_odx'=0. \tag{B-6}$$

Carrying out the integrations in relation (B-6) leads to

$$\left\{i\frac{\partial}{\partial z}+L\left(x+\tfrac{1}{2}i\frac{\partial}{\partial u},u-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)-L^*\left(x+\tfrac{1}{2}i\frac{\partial}{\partial u},u-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}F=0, \tag{B-7}$$

which is equivalent to the desired result (6.0-6)

$$-\frac{\partial F}{\partial z}=2\ \mathrm{Im}\left\{L\left(x+\tfrac{1}{2}i\frac{\partial}{\partial u},u-\tfrac{1}{2}i\frac{\partial}{\partial x};z\right)\right\}F.$$

9.3. Appendix C. Proof of (in)equality (8.7-3)

Let the sequence of numers b_m $(m=0,1,...)$ be defined by

$$b_m=\sum_{n=0}^{\infty}|a_{mn}|^2\gamma_n\quad(m=0,1,...), \tag{C-1}$$

where the numbers γ_n $(n=0,1,...)$ are ordered according to $\gamma_0\le\gamma_1\le...$ and the coefficients a_{mn} satisfy the orthonormality condition [cf. relation (8.7-2)]

$$\sum_{n=0}^{\infty}a_{ln}a_{mn}^*=\delta_{l-m}\quad(l,m=0,1,...). \tag{C-2}$$

We now consider the numbers b_m $(m=0,1,...,M)$ as the diagonal entries of an $(M+1)$-square Hermitian matrix $H=[h_{ij}]_{ij=0}^{M}$ with

$$h_{ij}=\sum_{n=0}^{\infty}a_{in}a_{jn}^*\gamma_n\quad(i,j=0,1,...,M\text{ and }M\ge 0). \tag{C-3}$$

Let the eigenvalues λ_m $(m = 0, 1, ..., M)$ of this Hermitian matrix be ordered according to $\lambda_0 \le \lambda_1 \le ... \le \lambda_M$. From Cauchy's inequalities [Mar-65] for the eigenvalues of a submatrix of a Hermitian matrix, we conclude that $\lambda_m \ge \gamma_m$ $(m = 0, 1, ..., M)$, and hence

$$\sum_{m=0}^{M} b_m = \sum_{m=0}^{M} h_{mm} = \sum_{m=0}^{M} \lambda_m \ge \sum_{m=0}^{M} \gamma_m. \tag{C-4}$$

Furthermore, with the numbers ν_m $(m = 0, 1, ...)$ satisfying the property $\nu_0 \ge \nu_1 \ge ...$, we can formulate the relation

$$\sum_{m=0}^{M} \nu_m \sum_{n=0}^{\infty} |a_{mn}|^2 \gamma_n = \sum_{m=0}^{M} \nu_m b_m = \nu_0 b_0 + \sum_{m=1}^{M} \nu_m b_m =$$

$$= \nu_0 b_0 + \sum_{m=1}^{M} \nu_m \left(\sum_{n=0}^{m} b_n - \sum_{n=0}^{m-1} b_n \right) = \nu_0 b_0 + \sum_{m=1}^{M} \nu_m \sum_{n=0}^{m} b_n - \sum_{m=0}^{M-1} \nu_{m+1} \sum_{n=0}^{m} b_n =$$

$$= \sum_{m=0}^{M-1} \nu_m \sum_{n=0}^{m} b_n + \nu \sum_{n=0}^{M} b_n - \sum_{m=0}^{M-1} \nu_{m+1} \sum_{n=0}^{m} b_n = \nu_M \sum_{n=0}^{M} b_n + \sum_{m=0}^{M-1} (\nu_m - \nu_{m+1}) \sum_{n=0}^{m} b_n \ge$$

$$\ge \nu_M \sum_{n=0}^{M} \gamma_n + \sum_{m=0}^{M-1} (\nu_m - \nu_{m+1}) \sum_{n=0}^{m} \gamma_n = \sum_{m=0}^{M} \nu_m \gamma_m. \tag{C-5}$$

On choosing $\gamma_n = 2n + 1$ $(n = 0, 1, ...)$ and taking the limit $M \to \infty$, we arrive at the (in)equality

$$\sum_{m=0}^{\infty} \nu_m \sum_{n=0}^{\infty} |a_{mn}|^2 (2n + 1) \ge \sum_{m=0}^{\infty} \nu_m (2m + 1). \tag{C-6}$$

The equality sign in this (in)equality holds if $|a_{mn}| = \delta_{m-n}$. When we choose

$$a_{mn} = \int q_m(\xi) \psi_n^*(\xi) d\xi, \tag{C-7}$$

relation (C-6) becomes identical to relation (8.7-4), and the proof of the latter (in)equality is complete.

10. REFERENCES

[Abr-70] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions* (Dover, New York, 1970).

[Bam-83] R. Bamler and H. Glünder, "The Wigner distribution function of two-dimensional signals: coherent-optical generation and display," Opt. Acta **30**, 1789-1803 (1983).

[Bar-80a] H.O. Bartelt and K.-H. Brenner, "The Wigner distribution function: an alternate signal representation in optics," Israel J. Techn. **18**, 260-262 (1980).

[Bar-80b] H.O. Bartelt, K.-H. Brenner, and A.W. Lohmann, "The Wigner distribution function and its optical production," Opt. Commun. **32**, 32-38 (1980).

[Bas-77] M.J. Bastiaans, "A frequency-domain treatment of partial coherence," Opt. Acta **24**, 261-274 (1977).

[Bas-78] M.J. Bastiaans, "The Wigner distribution function applied to optical signals and systems," Opt. Commun. **25**, 26-30 (1978).

[Bas-79a] M.J. Bastiaans, "The Wigner distribution function and Hamilton's characteristics of a geometric-optical system," Opt. Commun. **30**, 321-326 (1979).

[Bas-79b] M.J. Bastiaans, "Transport equations for the Wigner distribution function," Opt. Acta **26**, 1265-1272 (1979).

[Bas-79c] M.J. Bastiaans, "Transport equations for the Wigner distribution function in an inhomogeneous and dispersive medium," Opt. Acta **26**, 1333-1344 (1979).

[Bas-79d] M.J. Bastiaans, "Wigner distribution function and its application to first-order optics," J. Opt. Soc. Am. **69**, 1710-1716 (1979).

[Bas-80a] M.J. Bastiaans, "Wigner distribution function display: a supplement to Ambiguity function display using a single 1-D input," Appl. Opt. **19**, 192 (1980).

[Bas-80b] M.J. Bastiaans, "The Wigner distribution function and its applications in optics," in *Optics in Four Dimensions-1980*, AIP Conf. Proc. **65**, M.A. Machado and L.M. Narducci, eds. (American Institute of Physics, New York, 1980), pp. 292-312.

[Bas-81a] M.J. Bastiaans, "Signal description by means of a local frequency spectrum," in *Transformations in Optical Signal Processing*, W.T. Rhodes, J.R. Fienup, and B.E.A. Saleh, eds., Proc. Soc. Photo-Opt. Instrum. Eng. **373**, 49-62 (1981).

[Bas-81b] M.J. Bastiaans, "The Wigner distribution function of partially coherent light," Opt. Acta **28**, 1215-1224 (1981).

[Bas-83a] M.J. Bastiaans, "Uncertainty principle for partially coherent light," J. Opt. Soc. Am. **73**, 251-255 (1983).

[Bas-83b] M.J. Bastiaans, "Lower bound in the uncertainty principle for partially coherent light," J. Opt. Soc. Am. **73**, 1320-1324 (1983).

[Bas-84a] M.J. Bastiaans, "New class of uncertainty relations for partially coherent light," J. Opt. Soc. Am. **A 1**, 711-715 (1984).

[Bas-84b] M.J. Bastiaans, "Use of the Wigner distribution function in optical problems," in *1984 European Conference on Optics, Optical Systems and Applications*, Proc. Soc. Phot-Opt. Instrum. Eng. **492**, 251-262 (1984).

[Bas-86a] M.J. Bastiaans, "Application of the Wigner distribution function to partially coherent light," J. Opt. Soc. Am. **A 3**,1227-1238 (1986).

[Bas-86b] M.J. Bastiaans, "Uncertainty principle and informational entropy for partially coherent light," J. Opt. Soc. Am. **A 3**, 1243-1246 (1986).

[Bes-76] I.M. Besieris and F.D. Tappert, "Stochastic wave-kinetic theory in the

Liouville approximation," J. Math. Phys. **17**, 734-743 (1976).
[Bor-75] M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1975).
[Bre-73] H. Bremmer, "General remarks concerning theories dealing with scattering and diffraction in random media," Radio Sci. **8**, 511-534 (1973).
[Bre-79] H. Bremmer, "The Wigner distribution and transport equations in radiation problems," J. Appl. Science Eng. **A 3**, 251-260.
[Bre-82] K.-H. Brenner and A.W. Lohmann, "Wigner distribution function display of complex 1D signals," Opt. Commun. **42**, 310-314 (1982).
[Bre-83] K.-H. Brenner, *Phasenraumdarstellungen in Optik und Signalverarbeitung*, Ph.D. dissertation, (Universität Erlangen-Nürnberg, Erlangen, 1983).
[Bre-84] K.-H. Brenner and J. Ojeda-Castañeda, "Ambiguity function and Wigner distribution function applied to partially coherent imagery," Opt. Acta **31**, 213-223 (1984).
[Bru-67] N.G. de Bruijn, "Uncertainty principles in Fourier analysis," in *Inequalities*, O. Shisha, ed. (Academic, New York, 1967), pp. 57-71.
[Bru-73] N.G. de Bruijn, "A theory of generalized functions, with applications to Wigner distribution and Weyl correspondence," Nieuw Arch. Wiskunde (3) **21**, 205-280 (1973).
[Bry-74] O. Bryngdahl, "Geometrical transformations in optics," J. Opt. Soc. Am. **64**, 1092-1099 (1974).
[Bug-83] D.S. Bugnolo and H. Bremmer, "The Wigner distribution matrix for the electric field in a stochastic dielectric with computer simulation," Adv. Electr. Phys. **61**, 299-389 (1983).
[But-77] H.J. Butterweck, "General theory of linear, coherent-optical data-processing systems," J. Opt. Soc. Am. **67**, 60-70 (1977).
[But-81] H.J. Butterweck, "Principles of optical data-processing," in *Progress in Optics*, E. Wolf, ed. (North-Holland, Amsterdam, 1981), Vol. **19**, pp. 211-280.
[Car-77] W.H. Carter and E. Wolf, "Coherence and radiometry with quasi-homogeneous planar sources," J. Opt. Soc. Am. **67**, 785-796 (1977).
[Cla-80] T.A.C.M. Claasen and W.F.G. Mecklenbräuker, "The Wigner distribution – a tool for time-frequency signal analysis," Part I: Continuous-time signals; Part II: Discrete-time signals; Part III: Relations with other time-frequency signal transformations; Philips J. Res. **35**, 217-250, 276-300, 372-389 (1980).
[Coh-66] L. Cohen, "Generalized phase-space distribution functions," J. Math. Phys. **7**, 781-786 (1966).
[Coh-89] L. Cohen, "Time-frequency distributions – A review," Proc. IEEE **77**, 941-981 (1989).
[Cou-53] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953), Vol. **1**.
[Cou-60] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1960), Vol. **2**.
[Des-72] G.A. Deschamps, "Ray techniques in electromagnetics," Proc. IEEE **60**,

1022-1035 (1972).

[Eas-84] R.L. Easton Jr, A.J. Ticknor, and H.H. Barrett, "Application of the Radon transform to optical production of the Wigner distribution function," Opt. Eng. **23**, 738-744 (1984).

[Eic-82] G. Eichmann and B.Z. Dong, "Two-dimensional optical filtering of 1-D signals," Appl. Opt. **21**, 3152-3156 (1982).

[Erd-53] A. Erdélyi, ed., *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. **2**, Chap. 10.

[Ett-85] W. van Etten, W. Lambo, and P. Simons, "Loss in multimode fiber connections with a gap," Appl. Opt. **24**, 970-976 (1985).

[Fra-82] S. Frankenthal, M.J. Beran, and A.M. Whitman, "Caustic correction using coherence theory," J. Acoust. Soc. Am. **71**, 348-358 (1982).

[Fri-79] A.T. Friberg, "On the existence of a radiance function for finite planar sources of arbitrary states of coherence," J. Opt. Soc. Am. **69**, 192-198 (1979).

[Fri-80] A.T. Friberg, "Phase-space methods for partially coherent wave fields," in *Optics in Four Dimensions-1980*, AIP Conf. Proc. **65**, M.A. Machado and L.M. Narducci, eds. (American Institute of Physics, New York, 1980), pp. 313-331.

[Fri-81] A.T. Friberg, "On the generalized radiance associated with radiation from a quasihomogeneous planar source," Opt. Acta **28**, 261-277 (1981).

[Gam-64] H. Gamo, "Matrix treatment of partial coherence," in *Progress in Optics*, E. Wolf, ed. (North-Holland, Amsterdam, 1964), Vol. **3**, pp. 187-332.

[Gor-80] F. Gori, "Collett-Wolf sources and multimode lasers," Opt. Commun. **34**, 301-305 (1980).

[Gro-72] S.R. de Groot and L.G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972), Chap. 6.

[Jan-81a] A.J.E.M. Janssen, "Weighted Wigner distributions vanishing on lattices," J. Math. Anal. Appl. **80**, 156-167 (1981).

[Jan-81b] A.J.E.M. Janssen, "Positivity of weighted Wigner distributions," SIAM J. Math. Anal. **12**, 752-758 (1981).

[Jan-82] A.J.E.M. Janssen, "On the locus and spread of pseudo-density functions in the time-frequency plane," Philips J. Res. **37**, 79-110 (1982).

[Jan-83] C.P. Janse and A.J.M. Kaizer, "Time-frequency distributions of loudspeakers: the application of the Wigner distribution," J. Audio Eng. Soc. **31**, 198-223 (1983).

[Jia-84] J.-Z. Jiao, B. Wang, and H. Liu, "Wigner distribution function and optical geometrical transformation," Appl. Opt. **23**, 1249-1254 (1984).

[Loh-83] A. W. Lohmann, J. Ojeda-Castañeda, and N. Streibl, "The influence of wave aberrations on the Wigner distribution," Opt. Appl. **13**, 465-471 (1983).

[Lun-66] R.K. Luneburg, *Mathematical Theory of Optics* (U. California Press, Berkeley, Calif., 1966).

[Man-76] L. Mandel and E. Wolf, "Spectral coherence and the concept of cross-spectral purity," J. Opt. Soc. Am. **66**, 529-535 (1976).

[Mar-65] M. Marcus and H. Minc, *Introduction to Linear Algebra* (MacMillan, New York, 1965), Sec. 3.5, Theorem 5.9.

[Mar-70] W.D. Mark, "Spectral analysis of the convolution and filtering of non-stationary stochastic processes," J. Sound Vib. **11**, 19-63 (1970).

[Mar-77] R.J. Marks II, J.F. Walkup, and T.F. Krile, "Ambiguity function display: An improved coherent processor," Appl. Opt. **16**, 746-750 (1977); addendum **16**, 1777 (1977).

[Mar-79] R.J. Marks II and M.W. Hall, "Ambiguity function display using a single 1-D input," Appl. Opt. **18**, 2539-2540 (1979).

[Mar-80] N. Marcuvitz, "Quasiparticle view of wave propagation," Proc. IEEE **68**, 1380-1395 (1980).

[McC-76] J.J. McCoy and M.J. Beran, "Propagation of beamed signals through inhomogeneous media: a diffraction theory," J. Acoust. Soc. Am. **59**, 1142-1149 (1976).

[Mor-62] H. Mori, I. Oppenheim, and J. Ross, "Some topics in quantum statistics: The Wigner function and transport theory," in *Studies in Statistical Mechanics*, J. de Boer and G.E. Uhlenbeck, eds. (North-Holland, Amsterdam, 1962), Vol. **1**, pp. 213-298.

[Moy-49] J.E. Moyal, "Quantum mechanics as a statistical theory," Proc. Cambridge Philos. Soc. **45**, 99-132 (1949).

[Oje-84] J. Ojeda-Castañeda and E.E. Sicre, "Bilinear optical systems: Wigner distribution function and ambiguity function representations," Opt. Acta **31**, 255-260 (1984).

[Oje-85] J. Ojeda-Castañeda and E.E. Sicre, "Quasi ray-optical approach to longitudinal periodicities of free and bounded wavefields," Opt. Acta **32**, 17-26 (1985).

[O'N-61] E.L. O'Neill and T. Asakura, "Optical image formation in terms of entropy transformations," J. Phys. Soc. Jpn. **16**, 301-308 (1961).

[O'N-63] E.L. O'Neill, *Introduction to Statistical Optics* (Addison-Wesley, Reading, Mass., 1963), Chap. 8.

[Opp-78] A.V. Oppenheim, "Digital processing of speech," in *Applications of Digital Signal Processing*, A.V. Oppenheim, ed. (Prentice-Hall, Englewood Cliffs, 1978), Chap. 6.

[Pap-68] A. Papoulis, *Systems and Transforms with Applications in Optics* (McGraw-Hill, New York, 1968).

[Pap-74] A. Papoulis, "Ambiguity function in Fourier optics," J. Opt. Soc. Am. **64**, 779-788 (1974).

[Pap-77] A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977).

[Ped-82] H.M. Pedersen, "Radiometry and coherence for quasi-homogeneous scalar wavefields," Opt. Acta **29**, 877-892 (1982).

[Pre-82] D. Preis, "Phase distortion and phase equalization in audio signal processing – a tutorial review," J. Audio Eng. Soc. **30**, 774-794 (1982).

[Pre-83] W. Press, *Wigner distribution function as a speech spectrogram with unlimited resolution simultaneously in time and frequency*, r83-106-

01, The Mitre Corporation, July 1983.

[Rab-78] L.R. Rabiner and R.W. Schafer, *Digital Processing of Speech Signals* (Prentice-Hall, Englewood Cliffs, 1978), Chap. 6.

[Rho-81] W.T. Rhodes, "Acousto-optical signal processing: Convolution and correlation," Proc. IEEE **69**, 65-79 (1981).

[Rie-55] F. Riesz and B.Sz.-Nagy, *Functional Analysis* (Ungar, New York, 1955).

[Rih-68] A.W. Rihaczek, "Signal energy distribution in time and frequency," IEEE Trans. Inf. Theory **IT-14**, 369-374 (1968).

[Sai-73] R.A.K. Said and D.C. Cooper, "Crosspath real-time optical correlator and ambiguity function processor," Proc. Inst. Elec. Eng. **120**, 423-428 (1973).

[Sch-61] A.C. Schell, *The Multiple Plate Antenna*, Ph.D. dissertation (Massachusetts Institute of Technology, Cambridge, Mass., 1961), Sec. 7.5.

[Sta-82a] A. Starikov and E. Wolf, "Coherent-mode representation of Gaussian Schell-model sources and of their radiation fields," J. Opt. Soc. Am. **72**, 923-928 (1982).

[Sta-82b] A. Starikov, "Effective number of degrees of freedom of partially coherent sources," J. Opt. Soc. Am. **72**, 1538-1544 (1982).

[Szu-80] H.H. Szu and J.A. Blodgett, "Wigner distribution and ambiguity function," in *Optics in Four Dimensions-1980*, AIP Conf. Proc. **65**, M.A. Machado and L.M. Narducci, eds. (American Institute of Physics, New York, 1980), pp. 355-381.

[Szu-82] H.H. Szu, "Two-dimensional optical processing of one-dimensional acoustic data," Opt. Eng. **21**, 804-813 (1982).

[Wal-68] A. Walther, "Radiometry and coherence," J. Opt. Soc. Am. **58**, 1256-1259 (1968).

[Wal-78] A. Walther, "Propagation of the generalized radiance through lenses," J. Opt. Soc. Am. **68**, 1606-1610 (1978).

[Wig-32] E. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749-759 (1932).

[Win-79] R. Winston and W.T. Welford, "Geometrical vector flux and some new nonimaging concentrators," J. Opt. Soc. Am. **69**, 532-536 (1979).

[Wol-54] E. Wolf, "A macroscopic theory of interference and diffraction of light from finite sources. I. Fields with a narrow spectral range," Proc. R. Soc. London Ser. A **225**, 96-111 (1954).

[Wol-55] E. Wolf, "A macroscopic theory of interference and diffraction of light from finite sources. II. Fields with a spectral range of arbitrary width," Proc. R. Soc. London Ser. A **230**, 246-265 (1955).

[Wol-78] E. Wolf, "Coherence and radiometry," J. Opt. Soc. Am. **68**, 6-17 (1978).

[Wol-82] E. Wolf, "New theory of partial coherence in the space-frequency domain. I. Spectra and cross spectra of steady state sources," J. Opt. Soc. Am. **72**, 343-351 (1982).

[Woo-53] P.M. Woodward, *Probability and Information Theory with Applications to Radar* (Pergamon, London, 1953), Chap. 7.

List of important symbols and their first occurrences

A	Eq. (5.5-2)	element of T
$A(x,u)$	Eq. (2.2-1)	ambiguity function
B	Eq. (5.5-2)	element of T
C	Eq. (5.5-2)	element of T
$C(x,u)$	Eq. (2.2-3)	complex energy density function
D	Eq. (5.5-2)	element of T
d_u	Eq. (4.5-7)	effective width of $\bar{\varphi}(u)$
d_x	Eq. (4.5-6)	effective width of $\varphi(x)$
$F(x,u)$	Eq. (2.1-1)	Wigner distribution function
$F(x,u;z)$	Eq. (6.0-1)	Wigner distribution function with z-dependence
$\mathrm{F}(x,u,z,w)$	Eq. (6.4-1)	higher-dimensional Wigner distribution function
$f_m(\xi,\eta)$	Eq. (7.3-5)	Wigner distribution function of the eigenfunction $q_m(\xi)$
$G(x_o,u_o,x_i,u_i)$	Eq. (8.6-4)	bilinear ray spread function
$g(x_o;x_1,x_2)$	Eq. (8.6-1)	bilinear system kernel
$H_m(\xi)$	Eq. (3.6-2)	m-th Hermite polynomial
$h_{uu}(u_o,u_i)$	Eq. (5.0-4)	wave spread function
$h_{ux}(u_o,x_i)$	Eq. (5.0-2)	hybrid spread function
$h_{xu}(x_o,u_i)$	Eq. (5.0-3)	hybrid spread function
$h_{xx}(x_o,x_i)$	Eq. (5.0-1)	point spread function
J	Eq. (4.5-14)	geometrical vector flux
J_x	Eq. (4.5-13)	element of J
J_z	Eq. (4.5-12)	element of J; radiant emittance
$K(x_o,u_o,x_i,u_i)$	Eq. (5.0-5)	ray spread function
k	Eq. (4.5-12)	wave number
$k(x,u,x',u')$	Eq. (2.2-8)	Cohen kernel
$L(x,u;z)$	Eq. (6.0-5)	
$L(x,u,z,w)$	Eq. (6.4-3)	
$L_m(\xi)$	Eq. (3.6-4)	m-th Laguerre polynomial
M	Eq. (4.5-9)	matrix of second-order moments
m	Eq. (8.5-16)	square root of the determinant of M
m_u	Eq. (4.5-5)	center of gravity of $\lvert\bar{\varphi}(u)\rvert^2$
m_x	Eq. (4.5-4)	center of gravity of $\lvert\varphi(x)\rvert^2$
m_{uu}	Eq. (4.5-7)	second-order moment of $F(x,u)$
m_{ux}	Eq. (4.5-8)	second-order moment of $F(x,u)$
m_{xu}	Eq. (4.5-8)	second-order moment of $F(x,u)$
m_{xx}	Eq. (4.5-6)	second-order moment of $F(x,u)$
$\bar{m}(u)$	Section 5.2	modulation function in the frequency domain
$m(x)$	Section 5.1	modulation function in the space domain
$p(x)$	Section 7.2	intensity function in the space domain
$q_m(\xi)$	Eq. (7.3-2)	m-th eigenfunction of $\Gamma(x_1,x_2)$
$R(x,u)$	Eq. (2.2-5)	real part of $C(x,u)$
$\bar{s}(u)$	Section 7.2	intensity function in the frequency domain
T	Eq. (5.5-4)	ray transformation matrix

$U(x)$	Eq. (4.5-10)	average frequency
α	Section 3.3	curvature
α	Eq. (8.5-8)	real part of γ
β	Section 5.3	Fourier transformation constant
β	Eq. (8.5-8)	imaginary part of γ
$\Gamma(x_1, x_2)$	Eq. (2.3-1)	correlation function
$\tilde{\Gamma}(x_1, x_2, \tau)$	Eq. (7.1-1)	coherence function
$\Gamma(x_1, x_2, \omega)$	Eq. (7.1-2)	positional power spectrum
$\bar{\Gamma}(u_1, u_2, \omega)$	Eq. (7.1-5)	directional power spectrum
γ	Eq. (8.5-14)	complex beam parameter
θ	Section 4.5	observation angle
λ_m	Eq. (7.3-2)	m-th eigenvalue of $\Gamma(x_1, x_2)$
μ_p	Eq. (7.4-3)	overall degree of coherence
ν_m	Eq. (7.4-1)	m-th normalized eigenvalue
ρ	Eq. (3.5-1)	spatial width constant
σ	Eq. (7.2-1)	measure of the overall degree of coherence
σ	Eq. (8.5-11)	real beam parameter
$\varphi(x)$	Section 1	space signal; complex amplitude
$\varphi(x; z)$	Section 6	space signal with z-dependence
$\bar{\varphi}(u)$	Section 1	frequency spectrum
$\psi_m(\xi)$	Eq. (3.6-1)	m-th Hermite function

Contents