A Data Rate Constrained Observer for Discrete Nonlinear Systems

Quentin Voortman, Alexander Yu. Pogromsky, Alexey S. Matveev and Henk Nijmeijer

Abstract— **In this paper, we develop a communication protocol for the observation of discrete time, possibly unstable, dynamical systems over communication channels with limited communication capacity. We develop an observer based on the upper box dimension for one-way communication channels that leads to a certain type of observability. This communication scheme preserves observability under communication losses which makes the communication scheme robust towards communication losses without feedback in the communication channel. Using Lyapunov-like techniques, we provide bounds on the minimum communication rate required to implement this observer. We also use the Lyapunov dimension to provide analytical upper bounds on the communication rate. We compute an analytical upper bound and an exact expression for the Lyapunov dimension of the smoothened Lozi map. This bound is then tested in simulations of the communication protocol for the observation problem of the smoothened Lozi map.**

I. Introduction

In the past two decades, a lot of attention has been given to the problem of controlling dynamical systems with limited data-rates. This problem finds widespread applications in many different real-world systems as communication through wireless technologies is applied more and more. Systems where sensors, controllers and actuators are located at separate locations and connected through channels with limited communication capacity are becoming more common. For some of these decentralized systems, such as cooperative driving of wirelessly connected vehicles, the core features are the presence of one or several channels through which only limited amounts of data can pass, along with one or several sources of uncertainty. The sources of uncertainty can most generally be split up into three categories: uncertainties on the system parameters, uncertainties in the initial condition and stochasticity of the communication channel. Finding efficient communication strategies to minimize the required communication rate is a key component to solving the associated control and estimation problems.

In terms of control theory, there are two main problems: the design of observers and the control of systems with data-rate constraints. Most of the early work (see e.g. [10] and references therein), focused on the linear case for which most variants of the problem have been studied. An overview of the techniques and results that have been obtained for the linear case has been made in [22], [4] and [1].

More recently, the focus shifted towards nonlinear systems. Some early results for specific systems were obtained in [7] and [3]. More general results were obtained in [23] where the concept of feedback entropy was used to obtain specific data-rates and in [17], where, through a generalization of techniques that were initially conceived for linear systems, general data-rates were obtained for nonlinear system. The concept of feedback entropy, which was proven to be essentially equivalent to the invariance entropy [14][6], has been used in many different ways to provide tighter upper- and lower-bounds on the communication rates for dynamical systems (see [20], [15], [26], [18], and [25]) while other papers focus on passivity-based techniques to obtain relevant results [12].

In this paper, we use yet another approach, which focuses on set dimensions. To quantify the dimension of sets, mappings and dynamical systems, there exist many different methods. Two important characteristics for sets are the Hausdorf dimension [8] and the box dimension [11], which is also known as the limit capacity [28]. Both of these characteristics are related to the covering of sets with infinitesimally small balls and can take non-integer values for particular sets such as fractal sets and attractors of dynamical systems. The difficulty of using these dimensions is that it can often not be computed analytically and one has to employ numerical methods to obtain estimates of the dimension (see e.g. [27]). Another solution is to use the Lyapunov dimension which upper bounds the two previous dimensions [13]. The advantage of the Lyapunov dimension resides in the fact that it can be computed analytically by using the second Lyapunov method (see [5] and [16]), which leads to analytical bounds.

We focus on the problem of building an observer in a non-classical sense. We will sometimes refer to this observer as a communication scheme. This observer is made up of a coder, a channel with limited communication capacity, and a decoder. The objective is to design the observer such that it has some robustness towards communication losses. It is often the case in real-world applications that losses occur over the communication channel and the goal of our design is that the communication scheme functions in the presence of losses on one-way channels, i.e. channels without feedback in the communication channel. The main contribution of this paper is an observer which needs limited data-rates and is

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robust towards communication losses, without any feedback in the communication channel.

In Section II, we define the settings of the problem as well as the notion of observability that we want the observer to achieve. In Section III, we present the communication scheme and give a bound on the required communication rate in terms of the largest Lyapunov exponent and the upper box dimension. In Section IV, we provide analytical upper bounds on the communication rate through the Lyapunov dimension. Finally, in Section V, we apply the theory to the smoothened Lozi map, for which we compute the Lyapunov dimension and simulate the communication scheme.

II. Problem Statement

We consider nonlinear discrete-time time-invariant dynamical systems in the following form

$$x(t+1) = \varphi(x(t)), \quad t \geq 0, \quad x(0) \in K, \tag{1}$$

where $x(t)$ is the state, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, and $K \subset \mathbb{R}^n$ is a set of feasible initial states. We define

$$\varphi^t(\cdot) = \underbrace{\varphi(\dots\varphi(\cdot))}_{t \text{ times}}.$$

In the following text, we will sometimes use the following notation to refer to the dynamical system $\{\varphi^t\}_{t\geq 0}$. The objective of the observer, is to construct estimates of the state at a remote location that can only receive limited amounts of information per unit of time. The information is transmitted over a one-way communication channel, characterized by an upper bound on the number of bits that can be sent per unit of time.

Definition 1: For any time interval of arbitrary duration r, at most $b_+(r) < \infty$ bits can be sent over the communication channel. The upper bound on the number of bits that can be sent per unit of time $b_+(r)$ verifies the following property

$$\lim_{r\to\infty} \frac{b_+(r)}{r} = c,$$

where c is called the channel rate.

As a part of the observer, we are interested in designing a coder $\mathscr{C}$ and a decoder $\mathscr{D}$ that guarantee a certain type of observability which we will define later on. Figure 1 shows the system and the observer. The coder $\mathscr{C}$ and decoder $\mathscr{D}$ generate the messages $e(t)$ and estimates $\hat{x}(t)$ in the following way

$$e(t) = \mathscr{C}(x(t), \varepsilon), \qquad \forall t \in \mathbb{N}, \tag{2}$$

$$\hat{x}(t) = \mathscr{D}(\hat{x}(t-1), e(t), \varepsilon), \qquad \forall t \in \mathbb{N}, \tag{3}$$

where ε is the anytime exactness of observation.

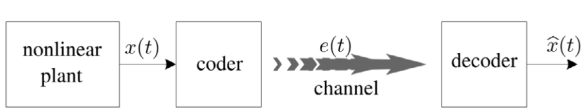

Fig. 1. Observer structure.

To quantify the precision of the estimate, we introduce the notion of anytime exactness of observation.

Definition 2: For the observer (2), (3) $\varepsilon > 0$ is called the anytime exactness of observation of the system (1) if the following holds

$$\|x(t) - \hat{x}(t)\| \leq \varepsilon, \qquad \forall t \in \mathbb{N}.$$

Now that we have introduced the notion of anytime exactness, we define the notion of observability that we will use.

Definition 3: The system (1) is said to be observable via a communication channel with a channel rate as defined in Definition 1 if for any $\varepsilon > 0$, there exists an observer (2)-(3) that observes the system (1) with anytime exactness ε.

III. The observation scheme

In this section, we focus on the design of a communication scheme that guarantees the observability of the dynamical system (1) over a communication channel. We also compute the minimum communication rate required to implement the communication scheme.

Although we interest ourselves to systems that are possibly unstable and highly nonlinear, we must assume some regularity conditions on the set of initial conditions and the mapping itself. We will thus make the following assumption

Assumption 1: The map φ is continuously differentiable. The set K is compact and invariant with respect to the mapping φ $(K = \varphi(K))$.

Please note that for the upcoming results in this document, the three parts of the previous assumption are not always simultaneously necessary. We have however chosen to bundle these together as the goal of the present paper is not to be as general as possible.

For convenience's sake, we also define the following notations

$$A(x) := \frac{\partial \varphi}{\partial x}(x), \qquad A^j(x) := \frac{\partial \varphi^j}{\partial x}(x).$$

In order to compute the communication rate required to use our communication protocol, we will need an upper bound on the largest Lyapunov exponent of the system. The following assumption supposes that such an upper bound exists for our system. In the following assumption, $\Delta v(x) = v(\varphi(x)) - v(x)$.

Assumption 2: There exists a continuous and bounded on K function $v : \mathbb{R}^n \to \mathbb{R}$, constant $\Lambda \geq 0$, and a symmetric positive definite matrix P such that

$$\Delta v(x) + \log_2 \lambda_1(x) \leq \Lambda, \ \forall x \in K \tag{4}$$

where $\log_2 0 := -\infty$ and $\lambda_i(x)$ are the roots of the equation

$$\det\left[A(x)^\top P A(x) - \lambda(x) P\right] = 0, \tag{5}$$

ordered from largest to smallest.
Together with the previously mentioned matrix P, we associate an inner product $\langle x, y\rangle_P = x^\top P y$ and the norms $\|x\|_P = \sqrt{\langle x, x\rangle_P}$, $x \in \mathbb{R}^n$, $\|A\|_P = \sqrt{\max_{\|x\|_P=1} x^\top A^\top P A x}$, $A \in \mathbb{R}^{n\times n}$. We also define $\sigma_i^P(x) := \sqrt{\lambda_i(x)}$, which will be referred to as the P-generalized singular values of the matrix $A(x)$.

The observer that we will introduce is based on the idea of covering the attractor of our dynamical system with balls of radius δ, where δ is chosen carefully. Our first lemma establishes sufficient conditions on δ such that the image of a ball of radius δ will be contained in a ball of radius at most ε after a number of time steps k. In our first lemma, we will use the following quantities

$$\bar{v} := \max_{x\in K} |v(x)|, \qquad \hat{\varepsilon} := \frac{\varepsilon}{2^{\bar{v}+1}},$$

where v is taken from Assumption 2. Next, we define k, which represents an upper bound on the number of time steps that it takes for the mapping φ to map a ball of initial radius δ onto a ball of radius $\hat{\varepsilon}$. For any given $\varepsilon > 0$ and for any δ such that

$$\frac{\hat{\varepsilon}}{2^{\frac{\Lambda}{2}}} \geq \delta > 0, \tag{6}$$

we define

$$k := \frac{2(\log_2 \hat{\varepsilon} - \log_2 \delta)}{\Lambda}, \tag{7}$$

where Λ is taken from the Assumption 2. Note that (6) implies that $\delta < \varepsilon$ and $1 \leq k < \infty$.

Lemma 1: Let Assumption 1 hold for φ. For any given $\varepsilon > 0$, $\exists \delta^* > 0$ such that $\forall \delta : 0 < \delta \leq \delta^*$, $\forall x(t), \hat{x}(t) \in K$ verifying the following condition

$$\|\hat{x}(t) - x(t)\|_P \leq \delta,$$

the following is true

$$\left\|\varphi^i(\hat{x}(t)) - \varphi^i(x(t))\right\|_P \leq \varepsilon, \tag{8}$$

for all $i \leq k$ where k is defined as in (7).
The proof of this lemma will be given in the full version of this paper.

We now introduce our observer which is organized in epochs of $\lfloor k \rfloor + 1$ time steps, where $\lfloor \cdot \rfloor$ refers to the floor function. In the following procedure, l represents the epoch index. For any l and k, we have

$$t_l := l(\lfloor k \rfloor + 1), \tag{9}$$

$$\bar{t}_l := l(\lfloor k \rfloor + 1) + 1, \tag{10}$$

$$\hat{t}_l := l(\lfloor k \rfloor + 1) + \lfloor k \rfloor. \tag{11}$$

We now define the coder/decoder scheme used to communicate over the communication channel.

Procedure 1: The communication scheme is made up of the following steps

1) **Initialization Step** This step is made on the first time period only $(t = 0)$.
   a) For any channel with rate c, and for any desired anytime exactness ε, the coder and the decoder find δ^* that satisfies both Lemma 1 as well as (6). They then both choose an identical $\delta \leq \delta^*$ and construct an identical finite covering of K with balls of radius δ whose centers are $\{q_j\} = Q \subset \mathbb{R}^n$, where j is an index which is identical for identical balls in both coverings.
   b) The epoch index is initialized $l = 0$.
2) **Repeated Step**, This step is repeated for every $t \in \mathbb{N}$.
   If $t = t_l$,
   a) The coder computes the index j of the point closest to $x(t)$ in Q.
   b) The coder sends the index j over the communication channel.
   c) The decoder assigns $\hat{x}(t) = q_j$.
   d) The epoch index is increased by one. $(l = l + 1)$.
   If $t \in [\bar{t}_l, \hat{t}_l]$,
   a) the decoder computes $\hat{x}(t) = \varphi(\hat{x}(t-1))$.

Note that in step 1a of the **Initialization Step**, any δ satisfying $0 < \delta \leq \delta^*$ can be chosen. Lemma 1 and (6) ensures that the communication scheme is always feasible but gives no guarantees on the resulting communication rate. If we are able to find a δ such that the actual data-rate is below a certain threshold c, we will call the observer c-feasible.

Definition 4: The observation scheme defined in Procedure 1 is c-feasible for a system $\{\varphi^t\}_{t\geq 0}$ if the observer leads to observability as defined in Definition 3 of the system over a communication channel with communication rate c.

Our communication scheme possesses some robustness towards communication losses. No feedback in the communication channel is required to recover from a communication loss. If a communication loss occurs, the decoder is able to reconstruct an estimate based only on the next message it receives without needing the previous estimate neither any additional information. This property is important to deal with communication losses since it implies that as soon as the decoder receives a message, we can guarantee δ-closeness between the state and estimate. The recovery procedure thus requires neither feedback in the communication channel nor an increase of the communication rate.

To quantify the communication rate needed to make our coder/decoder scheme feasible, we will need the upper box dimension.

Definition 5: [11] The upper box dimension $\bar{d}_B$ of a subset S of $\mathbb{R}^n$ is given by

$$\bar{d}_B(S) = \limsup_{\delta\to 0} \frac{\log N_\delta(S)}{-\log \delta}, \tag{12}$$

where $N_\delta(S)$ is defined as any of the following equivalent statements:

(i) The smallest number of closed balls of radius δ that cover S.
(ii) The smallest number of cubes of side δ that cover S.
(iii) The number of δ-mesh cubes that intersect S.
(iv) The smallest number of sets of diameter at most δ that cover S.
(v) The largest number of disjoint balls of radius δ with centers in S.

The following theorem establishes a relationship between the system and the rate required to observe the system over a communication channel with the proposed communication procedure. Λ is taken from Assumption 2 and $\bar{d}_B(K)$ is the upper box dimension of the set K. We are now ready to state the first result of this paper.

Theorem 1: Provided that Assumptions 1, and 2 hold for a system $\{\varphi^t\}_{t\geq 0}$, we have that for any communication channel with rate $c > \frac{\Lambda \bar{d}_B(K)}{2}$, the proposed observation scheme is c-feasible for that system.

The proof of this theorem will be provided in the full version of this paper. The result of Theorem 1 can be compared to other bounds that were previously obtained in the literature. Compared to [18], where a rate of nL is required (n being the dimension of the system, L the global Lipschitz constant), our rate is better. Compared to [24] in the case that there is a single unstable Lyapunov exponent, the rate is worse by a factor of $\bar{d}_B(K)$, but our communication protocol presents robustness towards losses in the communication channel, which is not the case for that paper. Finally, we note that in [5], Corollary 6.2.1 provides an estimate for the topological entropy which coincides with our estimate under identical assumptions. We emphasize however that the present observer is an improvement on their result since we also take into account the robustness towards losses.

IV. An analytical upper bound on the communication rate

Now that it is proven that our communication scheme leads to observability with a precisely defined rate, this section is dedicated to providing an analytical upper bound on the communication rate. The unknown component in our rate is the upper box dimension $\bar{d}_B(K)$. Computing the upper box dimension of sets analytically is in general not feasible so we will use results from [13], [5] and [16] to upper bound $\bar{d}_B(K)$ with the Lyapunov dimension. We will first provide definitions for the Lyapunov dimension of a map in a point, a map over a set, and a dynamical system. We will then cite several theorems required to upper bound the upper box dimension with the Lyapunov dimension. Next, we will provide results that allow us to upper bound the Lyapunov dimension analytically and sometimes compute it exactly. The last part of this section is dedicated to providing the analytical bound on the required communication rate. We start by defining the following notation

$$\sigma_i(\varphi^j, x) = \sigma_i\left(\frac{\partial \varphi^j}{\partial x}(x)\right), \qquad x \in K,$$

where $\sigma_i(\cdot)$ refers to the singular values of the matrix ($\sigma_i(A(x))$ are positive numbers, equal to the square root of the eigenvalues of $A(x)^* A(x)$). These singular values are ordered, such that $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$, $\forall x \in K$. We then define the singular value function.

Definition 6: The singular value function of $A^j(x)$ of order $d \in [0,n]$ at $x \in K$ is defined as

$$\omega_d\left(A^j(x)\right) :=
\begin{cases}
1, & d = 0,\\
\sigma_1(\varphi^j,x)\ldots\sigma_d(\varphi^j,x), & d \in \{1,..,n\},\\
\sigma_1(\varphi^j,x)\ldots\sigma_{\lfloor d\rfloor+1}(\varphi^j,x)^{d-\lfloor d\rfloor}, & d \in (0,n)\setminus\{1,..,n\text{-}1\}.
\end{cases}$$

With the definition of the singular value function, we can now define the local Lyapunov dimension of a map. The following three definitions are taken from [16].

Definition 7: The local Lyapunov dimension of a continuously differentiable map $\varphi : \mathbb{R}^n \to R^n$ at the point $x \in K$ is defined as

$$d_L(\varphi, x) := \sup\{d \in [0,n] : \omega_d(A(x)) \geq 1\}.$$

This definition can be extended to the Lyapunov dimension of a mapping of a set.

Definition 8: The Lyapunov dimension of a continuously differentiable map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of a compact set K is defined as

$$d_L(\varphi, K) := \sup_{x\in K}\sup\{d \in [0,n] : \omega_d(A(x)) \geq 1\}.$$

The final definition concerns the Lyapunov dimension of a dynamical system $\{\varphi^t\}_{t\geq 0}$.

Definition 9: The Lyapunov dimension of a dynamical system $\{\varphi^t\}_{t\geq 0}$ with respect to a compact invariant set K is defined as

$$d_L(\{\varphi^t\}_{t\geq 0}, K) := \inf_{t>0}\sup_{x\in K}\sup\{d \in [0,n] : \omega_d\left(A^t(x)\right) \geq 1\}.$$

We have to make one last assumption on the dynamical system, in order to find an analytical upper bound on the communication rate. This assumption, which is similar to Assumption 2, will be used to compute the Lyapunov dimension of the dynamical system. In the next assumption, $\Delta w(x) = w(\varphi(x)) - w(x)$.

Assumption 3: Let $d = j + s \in [1,n]$ where $j = \lfloor d \rfloor \in \{1,\ldots,n\}$ and $s = d - \lfloor d \rfloor \in [0,1)$. There exists a continuous and bounded on K function $w : \mathbb{R}^n \to \mathbb{R}$ and a positive definite symmetric matrix P such that

$$\Delta w(x) + \sum_{i=1}^{j} \log_2 \lambda_i(x) + s \log_2 \lambda_{j+1}(x) < 0, \ \forall x \in K \qquad (13)$$

where the λ_i are solutions of (5).

To upper bound the communication rate, we will use the following theorem and corollary.

Theorem 2: [13] Let Assumption 1 hold. Then

$$\bar{d}_B(K) \leq d_L(\varphi, K).$$

Corollary 1: [13] Under the hypotheses of Theorem 2,

$$\bar{d}_B(K) \leq d_L(\varphi^t, K)$$

for all $t \geq 1$.

The following theorem is a reformulation of Theorem 2 from [16].

Theorem 3: Suppose Assumption 3 holds with $d = j + s$, then for sufficiently large $T > 0$

$$d_L(\{\varphi^t\}_{t\geq 0}, K) \leq d_L(\varphi^T, K) \leq j + s.$$

Due to the lack of space, the proof of this theorem is ommitted from this document. It will be provided in the full version of this paper.

Finally, to obtain an exact analytical expression for the Lyapunov dimension of a dynamical system, the following proposition, which is a reformulation of Proposition 3 and Corollary 3 from [16], is useful.

Proposition 1: Suppose that at one of the equilibrium points of the dynamical system $\{\varphi^t\}_{t\geq 0} : x_{\text{eq}} \equiv \varphi^t(x_{\text{eq}})$ the matrix $A(x_{\text{eq}})$ has the eigenvalues $\lambda_1(x_{\text{eq}}), \ldots, \lambda_n(x_{\text{eq}})$. Suppose there exists a non-singular matrix S such that

$$SA(x_{\text{eq}})S^{-1} = \text{diag}(\lambda_1(x_{\text{eq}}), \ldots, \lambda_n(x_{\text{eq}})) \tag{14}$$

where $|\lambda_1(x_{\text{eq}})| \geq \cdots \geq |\lambda_n(x_{\text{eq}})|$. Let $\varphi_S : w \to S\varphi(S^{-1}w)$ be the discrete mapping after the linear coordinate change. Suppose that for $s+j$ as defined in Assumption 3, we have

$$d_L(\varphi_S, Sx_{\text{eq}}) = s + j$$

then for any invariant set $K \ni x_{\text{eq}}$ of $\{\varphi^t\}_{t\geq 0}$ we have

$$d_L(\{\varphi^t\}_{t\geq 0}, K) = s + j.$$

We are now ready to state the second result of this paper. In the next theorem, Λ is taken from Assumption 2, and j and s are taken from Assumption 3.

Theorem 4: Provided that Assumptions 1, 2, and 3, are verified for a system $\{\varphi^t\}_{t\geq 0}$, we have that for any communication channel with rate $c > \frac{\Lambda(j+s)}{2}$ the observation scheme is c-feasible for that system.

Proof: From Assumption 1, we have that the mapping is C^1 and that K is invariant under the map. We thus apply Corollary 1 to obtain that

$$\bar{d}_B(K) \leq d_L(\varphi^t, K).$$

Since this is true for any t, it also holds for $t = T$ from Theorem 3. We thus obtain

$$\bar{d}_B(K) \leq j + s. \tag{15}$$

Finally, we use Proposition 1 together with (15) to prove the theorem. ∎

V. Application of the Theory on the (Smoothened) Lozi Map

In this section, we will compute the communication rate for our coder/decoder scheme applied to the Lozi map ([19], [9]). The Lozi map, which is a modification of the Hénon map is a discrete-time system $\{\varphi^t\}_{t\geq 0}$, with

$$\varphi : \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \to \begin{pmatrix} 1 - a|x_1(t)| + bx_2(t) \\ x_1(t) \end{pmatrix},$$

where a and b are positive constants. For parameters such that $1 + a - b > 0$, the Lozi map has an equilibrium

$$x_+ = \left(\frac{1}{1+a-b}, \frac{1}{1+a-b} \right).$$

If, in addition, we have $1 - a - b < 0$, there exists a second equilibrium

$$x_- = \left(\frac{1}{1-a-b}, \frac{1}{1-a-b} \right).$$

In this paper, we will assume that the latter is true. In that case, both equilibria are unstable.

The Lozi map is only Lipschitz continuous and not C^1 as required by the Assumption 1. We will thus consider the following C^1 approximation of the map that was first introduced in [2].

$$\varphi_\gamma : \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \to \begin{pmatrix} 1 - af_\gamma(x_1(t)) + bx_2(t) \\ x_1(t) \end{pmatrix}$$

where

$$f_\gamma(x) = \begin{cases} |x|, & \text{if } |x| \geq \gamma; \\ \frac{x^2}{2\gamma} + \frac{\gamma}{2}, & \text{if } |x| < \gamma, \end{cases}$$

and γ is some relatively small constant. To ensure the existence of the two equilibria, one has to include the additional constraint $\gamma < (a+1-b)^{-1}$. We will thus make the following assumption about the parameters.

Assumption 4: The following inequalities hold

$$a, b, \gamma > 0, \quad 1 - a < b < 1, \quad \gamma < (a+1-b)^{-1}.$$

Note that the inequality $b < 1$ is added to ensure that $d_L\{\varphi^t\}_{t\geq 0}, K) < 2$. In this context, we now compute the Lyapunov dimension of the Lozi map.

Theorem 5: Let Assumption 4 hold. Then for any compact invariant set K of the smoothened Lozi map φ_γ, the following inequality holds

$$d_L(\{\varphi_\gamma^t\}_{t\geq 0}, K) \leq 2 - \frac{\log_2 b}{\log_2\left(\sqrt{a^2+4b} - a\right) - 1}.$$

Moreover, if $x_+ \in K$, the following holds

$$d_L(\{\varphi_\gamma^t\}_{t\geq 0}, K) = 2 - \frac{\log_2 b}{\log_2\left(\sqrt{a^2+4b} - a\right) - 1}.$$

The proof of this theorem will be provided in the full version of this paper.

Having found an upper bound for $d_B(K)$, we provide an analytical bound for the communication rate required for our coder/decoder scheme by using an estimate for Λ from [25].

Corollary 2: Suppose that Assumption 4 holds, let K be the compact invariant set of $\{\varphi_\gamma^t\}_{t\geq 0}$ then, for any channel with communication rate

$$c > \frac{\left[2\log_2\left(\frac{\sqrt{a^2+4b}-a}{2}\right) - \log_2(b)\right]\left[\log_2\left(\frac{\sqrt{a^2+4b}+a}{2}\right)\right]}{\log_2\left(\frac{\sqrt{a^2+4b}-a}{2}\right)},$$

the observation scheme is c-feasible for the smoothened Lozi map.

Proof: To prove this, we borrow the result from Theorem 13 from [25] to find Λ for Assumption 2. This gives us that Assumption 2 holds with

$$\Lambda = 2\log_2\left(\sqrt{a^2+4b} + a\right) - 2.$$

We then apply Theorem 4 to obtain the aforementioned bound on c. ∎

We have performed simulations to confirm the theoretical uppper bound for several values of ε, the anytime exactness.

In [21] the following conditions for the presence of a strange attractor in the Lozi system were established

$$0 < b < 1, \tag{16}$$

$$a > 0, \tag{17}$$

$$2a + b < 4, \tag{18}$$

$$b < \frac{a^2 - 1}{2a + 1}, \tag{19}$$

$$a\sqrt{2} > b + 2. \tag{20}$$

Although these conditions do not imply that the *smoothened* Lozi map will have a strange attractor, typical trajectories for small γ display chaotic-like behavior for parameter in those ranges. For the simulation, we have chosen to use the set of parameters $a = 1.7$ and $b = 0.3$, $\gamma = 10^{-5}$, which verify (16)-(20) and Assumption 4. The strange attractor was observed through simulations to be confined to the region $[-1.1, 1.3] \times [-1.1, 1.3]$. For this set of parameters, Proposition 2 implies that the observation scheme is c-feasible for the smoothened Lozi map for any $c > 1.2013$. We have implemented the observation scheme for the following set $\varepsilon \in \{0.5, 0.2, 0.1, 0.05\}$. For each of different ε, we summarize the number of points in the covering N, as well as the resulting communication speed c^* in Table I. We can see that as ε becomes smaller, the actual communication rate becomes higher although for all of these ε we manage to find a covering that results in an effective communication rate very close to or below c.

	$\varepsilon = 0.2$	$\varepsilon = 0.1$	$\varepsilon = 0.075$	$\varepsilon = 0.05$
N [1]	1×10^6	2×10^6	2×10^7	3.5×10^7
c^* [bits/s]	1.0924	1.1499	1.169431	1.212770

TABLE I

RESULTS OF THE SIMULATIONS ON THE SMOOTHENED LOZI SYSTEM.

VI. CONCLUSION

In this paper, we have developed a communication protocol made up of a coder/decoder scheme to observe discrete-time dynamical systems with uncertainties over communication channels with losses. Using the upper box dimension, we have proven that for any rate above $\frac{\Lambda \bar{d}_B}{2}$, we can build such observers. We have also shown that we can provide analytical upper bounds on this rate under certain conditions. For the smoothened Lozi system, we have given an analytical upper bound of the Lyapunov dimension of the dynamical system. We have also provided an analytical upper bound for the communication rate required to observe this system with our coder/decoder scheme. Future extensions of this work include other forms of uncertainty such as parametric uncertainties and noise (both in the dynamical systems as on the communication channel) as well as implementing this communication scheme as a means to reach consensus for several dynamical systems.

REFERENCES

[1] B. Andrievsky, A. Matveev, and A. Fradkov. Control and Estimation under Information Constraints: Toward a Unified Theory of Control, Computation and Communications. *Automation and Remote Control*, 71:572–633, 2010.

[2] M. a. Aziz-Alaoui, C. Robert, and C. Grebogi. Dynamics of a Henon-Lozi-type map. *Chaos Solitons & Fractals*, 12:2323–2341, 2001.

[3] J. Baillieul. Data-rate requirements for nonlinear feedback control. *Proc. 6th IFAC Symp. Nonlinear Control Syst.*, pages 1277–1282, 2004.

[4] J. Baillieul and P. J. Antsaklis. Control and Communcation Challenges in Networked Real-Time Systems. *IEEE Proc.*, 95(1):9–28, 2007.

[5] V. A. Boichenko, G. A. Leonov, and V. Reitmann. *Dimension Theory for Ordinary Differential Equations*. Teubner Verlag, 2005.

[6] F. Colonius, C. Kawan, and G. Nair. A note on topological feedback entropy and invariance entropy. *Systems and Control Letters*, 62(5):377–381, 2013.

[7] C. De Persis. A Note on Stabilization via Communication Channel in the presence of Input Constraints. *Proceedings of the 42nd IEEE Conference on Decision and Control*, pages 187–192, 2003.

[8] A. Douady and J. Oesterle. Dimension de Hausdorff des attracteurs. *Comptes Rendus de l'Academie des Sciences Serie A*, 290:1135–1138, 1980.

[9] Z. Elhadj. *Lozi Mappings: Theory and Applications*. CRC Press, Boca Raton, FL, 2013.

[10] N. Elia and S. Mitter. Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, 46(9):1384–1400, 2001.

[11] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, 1997.

[12] A. L. Fradkov, B. Andrievsky, and R. J. Evans. Synchronization of nonlinear systems under information constraints. *Chaos*, 18(3):037109–8, 2008.

[13] B. R. Hunt. Maximum local Lyapunov dimension bounds the box dimension of chaotic attractors. *Nonlinearity*, 9:845–852, 1996.

[14] C. Kawan. *Invariance entropy for deterministic control systems*. Springer Cham Heidelberg New York Dordrecht London, 2009.

[15] C. Kawan. State estimation , entropy and Lyapunov exponents. *Systems & Control Letters*, 2017.

[16] N. V. Kuznetsov. The Lyapunov dimension and its estimation via the Leonov method. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 380(25-26):2142–2149, 2016.

[17] D. Liberzon and J. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Transactions on Automatic Control*, 50(6):910–915, 2005.

[18] D. Liberzon and S. Mitra. Entropy and minimal data rates for state estimation and model detection. *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, pages 247–256, 2016.

[19] R. Lozi. Un Attracteur Étrange du Type de Hénon. *Journal de Physique Colloques*, 39(C5):C5–9–C5–10, 1978.

[20] A. S. Matveev and A. V. Savkin. *Estimation and Control over Communication Networks*. Birkhauser Boston, 2009.

[21] M. Misiurewicz. Strange attractors for the Lozi mappings. *Annals of the New York Academy of Sciences*, pages 348–358, 1980.

[22] G. Nair, F. Fagnani, S. Zampieri, and R. Evans. Feedback Control Under Data Rate Constraints: An Overview. *Proceedings of the IEEE*, 95(1):108–137, 2007.

[23] G. N. Nair, R. J. Evans, I. M. Mareels, and W. Moran. Topological feedback entropy and nonlinear stabilization. *IEEE Transactions on Automatic Control*, 49(9):1585–1597, 2004.

[24] A. Pogromsky and A. Matveev. Estimation of the topological entropy via the direct Lyapunov method. *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 4(PART 1):45–50, 2010.

[25] A. Pogromsky and A. Matveev. Data rate limitations for observability of nonlinear systems. *IFAC-PapersOnLine*, 49(14):119–124, 2016.

[26] H. Sibai and S. Mitra. Optimal Data Rate for State Estimation of Switched Nonlinear Systems. *Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control*, pages 71–80, 2017.

[27] S. Siegmund and P. Taraba. Approximation of box dimension of attractors using the subdivision algorithm. *Dynamical Systems*, 21(1):1–24, 2006.

[28] F. Takens. Detecting strange attractors in turbulence. In *Rand D., Young LS. (eds) Dynamical Systems and Turbulence, Warwick*, pages 366–381. Springer, Berlin, Heidelberg, 1980.