Continuous Time Observers of Nonlinear Systems with Data-Rate Constraints ⋆

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Abstract: In this paper, we present the design of a non-classical observer that constructs estimates of the state of continuous nonlinear systems at remote locations. The system is connected to the remote location through a communication channel which can only transmit limited amounts of data per unit of time. The observer is designed with the objective that it is as data-efficient as possible whilst possessing a certain robustness towards communication losses without any feedback in the communication channel. We provide bounds for the sufficient data-rate to implement the observer. We apply the theory to the problem of observing the Lorenz system over a channel with minimal data-rates.

Keywords: Observers, State estimation, Limited data rate, Continuous systems, Nonlinear systems

1. INTRODUCTION

During the past twenty years, there has been a focus on data-rate problems in the systems and control field. This broad class of problems finds many real-world applications such as systems connected through wireless technologies or microelectromechanical systems. The general configuration involves systems where the sensors, actuators and controllers are placed at locations that are remote from one another. These components are connected through data-limited communication channels, and are subjected to various uncertainties. The main sources of uncertainty include: noise, uncertainty in the system parameters, and uncertainty about the initial conditions. In this paper, we focus on the latter case. One of the key issues in the concerned area is finding data-efficient communication strategies that are robust against the uncertainties.

Recent interest of control community in the issue of limited data-rate was mostly concentrated on state estimation problems and control problems, with a focus on stabilization. At the beginning of the 2000's, most of the work dealt with linear systems; as a result, this case has been studied rather comprehensively (we refer the reader to Elia and Mitter (2001), Nair et al. (2007), Baillieul and Antsaklis (2007) and Andrievsky et al. (2010) for extended surveys).

Among the early control-oriented results in the nonlinear case, there are those obtained in De Persis (2003) and Baillieul (2004). General results were obtained in Nair et al. (2004), where the notion of feedback entropy was introduced and used as a means to obtain bounds on the data-rates required for stabilization. Another important early result was established in Liberzon and Hespanha (2005), which adapted previously developed techniques for linear systems to the nonlinear case in order to obtain sufficient data-rate bounds. Later, other types of entropy such as invariance entropy (Kawan (2009), Colonius et al. (2013)) were introduced as a means to provide upper and lower bounds on the data-rates required for either observation or control (see Matveev and Savkin (2009), Kawan (2017), Sibai and Mitra (2017), Liberzon and Mitra (2016), and Pogromsky and Matveev (2016)). Finally, some papers relied on passivity-based methods to provide bounds (Fradkov et al. (2008)).

In this paper, we use both the Lyapunov exponent theory as well as the dimension theory to quantify the data-rate. There are many approaches for the definition and computation of the dimension of sets and dynamical systems. For sets, two important samples are the Hausdorf dimension (Douady and Oesterle (1980)), and the box dimension (Falconer (1997)), which is sometimes referred to as the limit capacity (Takens (1980)). These dimensions are concerned with the number of balls required to cover a particular set. Both of them suffer a common problem: in general, they cannot be computed analytically and one must thus rely on numerical methods to compute them

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(see e.g. Siegmund and Taraba (2006)). To overcome this drawback, a solution is to use other dimensions for which analytical expressions exist, to upper-bound these dimensions. In this context, we remark that according to Hunt (1996), the Lyapunov dimension of a dynamical system over a particular set upper bounds the upper box dimension. Meanwhile, the Lyapunov dimension can be computed analytically through the second Lyapunov method (see Boichenko et al. (2005) and Kuznetsov (2016)) thus providing analytical upper bounds.

The objective of this paper is to build an observer (also referred to as a communication scheme) in a non-traditional scenario for a dynamical system whose state evolves over time. The observer is made up of a quantizer, coder, data-rate constrained channel and a decoder which all interact to provide an estimate of the state at a certain remote site. The objective of the observer is to provide estimates in real time with an as high as desired precision at the remote location whilst using as little data as possible. The design of the observer should also be robust against communication losses. This property is extremely valuable for real-world applications where, due to internal and external factors, losses are a fairly common occurrence. Unlike many papers in the area, this issue of robustness is treated in the absence of a feedback communication channel via which an information about the outcome of the "feedforward" transmission (loss of data or success) becomes available to the coder.

The body of the paper is organized as follows. In Section 2, we define the problem statement and the type of observability to be ensured by the observer. In Section 3, we introduce the observer itself and provide a bound on the communication bit-rate required to implement it. In Section 4, we focus on providing analytical bounds on the channel rate. Finally, in Section 5, we study the problem of observing the Lorenz system for which we analytically compute the theoretical rate.

2. PROBLEM STATEMENT

We consider autonomous continuous-time nonlinear systems in the following form.

$$\dot{x}(t) = f(x(t))), \ x(0) \in K, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, and $K \subseteq \mathbb{R}^n$ is a set of feasible initial states. The objective is to build an observer which consists of four parts: the quantizer $\mathcal{Q}$, coder $\mathcal{C}$, communication channel, and decoder $\mathcal{D}$, as displayed in Figure 1. The overall objective is to produce at time t an estimate $\hat{x}(t)$ of the state at the site of the decoder. The worst-case precision of the estimate is called the anytime exactness of observation of the system.

Definition 1. For the observer (3)-(5), ϵ is called the anytime exactness of observation of the system (1) if the following holds

$$\|x(t) - \hat{x}(t)\| \leq \epsilon \qquad \forall t \geq 0.$$

The channel provides one-way communication without delay and can transmit messages at times t_k, the choice of which is part of the observer and subject to constraints that will be provided further in this document. The transmission is instantaneous and for every time interval $\Delta t_k = t_{k+1} - t_k$, there is both a minimum $b^-(\Delta t_k) < \infty$ and a maximum number of bits $b^+(\Delta t_k) < \infty$ that can be transmitted. These numbers have the following property

$$\frac{b^{\pm}(\Delta t_k)}{\Delta t_k} \to c, \text{ as } \Delta t_k \to \infty \tag{2}$$

where c is the channel rate. Both the coder and decoder know the map f, the set K and desired anytime exactness ϵ. The times when communication occurs are determined by the quantizer $\mathcal{Q}$

$$t_{k+1} = \mathcal{Q}(t_k, \epsilon), \qquad t_{k+1} > t_k, \qquad t_0 := 0. \tag{3}$$

The coder generates finite-bit messages $e(t_k)$ that are sent over the channel in the following way

$$e(t_k) = \mathcal{C}(x(t_k), \epsilon), \tag{4}$$

and the decoder generates estimates in the following way

$$\hat{x}(t) = \mathcal{D}(e(t_k), \epsilon), \forall t \in \mathbb{R}^+, \tag{5}$$

where $\mathbb{R}^+ := [0, \infty)$.

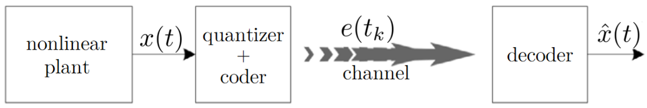

Fig. 1. Observer structure.

We now define the notion of observability through a communication channel. It is this observability property that we want to achieve with our observer.

Definition 2. The system (1) is said to be observable via a communication channel with rate c if for any $\epsilon > 0$, there exists an observer (3)-(5) that observes the system with anytime exactness ϵ and uses a channel that meets (2).

3. THE OBSERVER

In this section, we build an observer and quantify the channel rate required for its implementation. To this end, we need some assumptions.

Assumption 1. The map f is continuously differentiable on $\mathbb{R}^n$. The set K is compact and invariant under the map f.

This assumption yields that any initial state $x_0 \in K$ uniquely determines the solution of (1) and so the system generates a semi-flow in K

$$\varphi^t(x_0) := x(t, x_0), \qquad t \geq 0, x_0 \in K,$$

where $x(t, x_0)$ is the solution of (1) with $x(0) = x_0$. In what follows, the term "system" will sometimes refer to the semi-flow $\{\varphi^t\}_{t\geq 0}$. We also put

$$A(x) := \frac{\partial f}{\partial x}(x).$$

To assess the required channel rate, we need to estimate the rate at which various trajectories diverge from one another as time progresses. To this end, we will use the second Lyapunov method to obtain an upper bound on the largest Lyapunov exponent. The following assumption enables one to apply this method and is a key source of the bound that will be used later in this paper. It should be remarked that for many specific classical chaotic systems, some bounds have been found already.

Assumption 2. There exists a continuously differentiable function $v : \mathbb{R}^n \to \mathbb{R}$, constant Λ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\dot{v}(x) + \lambda_1(x) \leq \Lambda \qquad \forall x \in \mathbb{R}^n \tag{6}$$

where $\dot{v}(x) = \frac{\partial v}{\partial x} f(x)$ and $\lambda_i(x)$ are the solutions of the algebraic equation

$$\det(A^\intercal(x)P + PA(x) - \lambda P) = 0 \tag{7}$$

ordered from largest to smallest ($\lambda_1(x) \geq \cdots \geq \lambda_n(x)$).

We define an inner product $\langle x, y\rangle_P := x^\intercal P y$, and an associated norm $\|\cdot\|_P := \sqrt{\langle \cdot, \cdot\rangle_P}$.

Property 1. Let the matrix P be decomposed as $P = S^\intercal S$. Then the solutions $\lambda_i(x)$ of (7) are the eigenvalues of the symmetric matrix $S^{-\intercal}A^\intercal(x)S^\intercal + SA(x)S^{-1}$.

Proof: Since P is symmetric and positive definite, S is nonsingular so that the solutions of (7) are equal to those of

$$\det(S^{-\intercal}\left[A^\intercal(x)P + PA(x) - \lambda P\right] S^{-1}) = 0.$$

It remains to observe that this equation is equivalent to

$$\det(S^{-\intercal}\left[A^\intercal(x)S^\intercal S + S^\intercal S A(x) - \lambda S^\intercal S\right] S^{-1}) = 0$$
$$\Leftrightarrow \det(S^{-\intercal}A^\intercal(x)S^\intercal + SA(x)S^{-1} - \lambda I) = 0. \qquad \blacksquare$$

Property 1 implies that,

$$x^\intercal(A^\intercal(\xi)P + PA(\xi))x \leq \lambda_1(\xi)\,\|x\|_P^2\,, \tag{8}$$

$\forall x \in \mathbb{R}^n, \xi \in \mathbb{R}^n$.

By Definitions 1 and 2, our basic concern is in fact about the propagation of the estimation accuracy $\|x(t) - \hat{x}(t)\|$ over time. Our first lemma treats this issue in the situation where the estimate $\hat{x}(t)$ is generated as a solution of (1), starting from an appropriate initial state. To formulate the lemma, we first introduce some notations. For any continuously differentiable map $v : \mathbb{R}^n \to \mathbb{R}$, we define

$$\bar{v} := \max_{x\in \mathrm{Conv}(K)} |v(x)|,$$

where $\mathrm{Conv}(K)$ refers to the convex hull of K. For any ϵ, we put

$$\hat{\epsilon} := \frac{\epsilon}{e^{\bar{v}}}. \tag{9}$$

Then, for any δ, such that

$$\hat{\epsilon} > \delta > 0, \tag{10}$$

we define

$$t_{\max} := \frac{2\ln(\hat{\epsilon}) - 2\ln(\delta)}{\Lambda}, \tag{11}$$

where $\ln(\cdot)$ is the natural logarithm and Λ is taken from (6). It will be shown that this $t_{\max}$ is an upper bound on time for which the initial estimation error smaller or equal to δ surely does not grow up above ϵ. Finally, for brevity, we introduce the notations

$$x_0 = x(t), \qquad \hat{x}_0 = \hat{x}(t),$$
$$x_\tau = x(t+\tau), \qquad \hat{x}_\tau = \hat{x}(t+\tau).$$

Lemma 1. Let Assumptions 1 and 2 hold. Then, for any $\epsilon > 0$, there exists $\hat{\delta}$ such that whenever $0 < \delta \leq \hat{\delta}$, $x(t), \hat{x}(t) \in K$ and

$$\|x_0 - \hat{x}_0\|_P \leq \delta,$$

the following inequalities are true

$$t_{\max} > 0,$$
$$\|x_\tau - \hat{x}_\tau\|_P \leq \epsilon, \qquad \forall\ 0 < \tau \leq t_{\max},$$

where $t_{\max}$ is defined in (11).

Proof: For the first inequality to hold, it suffices to choose $\delta^* \leq \hat{\epsilon}$ where $\hat{\epsilon}$ is defined in (9). For the second inequality, we start by noticing that

$$\|x_\tau - \hat{x}_\tau\|_P^2 = [x_\tau - \hat{x}_\tau]^\intercal P\,[x_\tau - \hat{x}_\tau]\,.$$

This implies that

$$\begin{aligned}\frac{d}{d\tau}\,\|x_\tau\text{-}\hat{x}_\tau\|_P^2 &= \\ &= \left[\dot{x}_\tau\text{-}\dot{\hat{x}}_\tau\right]^\intercal P\,[x_\tau\text{-}\hat{x}_\tau] + [x_\tau\text{-}\hat{x}_\tau]^\intercal P\left[\dot{x}_\tau\text{-}\dot{\hat{x}}_\tau\right] \\ &= [f(x_\tau)\text{-}f(\hat{x}_\tau)]^\intercal P\,[x_\tau\text{-}\hat{x}_\tau] + [x_\tau\text{-}\hat{x}_\tau]^\intercal P\,[f(x_\tau)\text{-}f(\hat{x}_\tau)]\,.\end{aligned}$$

By Assumption 1, the map $f(\cdot)$ is C^1. We define the function $\gamma : [0,1] \to \mathbb{R}$,

$$\begin{aligned}\gamma(\alpha) =&f(\hat{x}_\tau + \alpha(x_\tau - \hat{x}_\tau))^\intercal P[x_\tau - \hat{x}_\tau] \\ &+ [x_\tau - \hat{x}_\tau]^\intercal P f(\hat{x}_\tau + \alpha(x_\tau - \hat{x}_\tau)).\end{aligned}$$

This function is clearly continuously differentiable as well. Hence the mean-value theorem guarantees that

$$\frac{\gamma(1) - \gamma(0)}{1-0} = \frac{d\gamma}{d\alpha}(\beta_\tau)$$

for some $\beta_\tau \in (0,1)$. Meanwhile $\frac{d}{d\tau}\,\|x_\tau - \hat{x}_\tau\|_P^2 = \gamma(1) - \gamma(0)$ and so

$$\begin{aligned}\frac{d}{d\tau}\,\|x_\tau - \hat{x}_\tau\|_P^2 =&\gamma(1) - \gamma(0) \\ =&[x_\tau - \hat{x}_\tau]^\intercal A(\xi_\tau)^\intercal P[x_\tau - \hat{x}_\tau] \\ &+ [x_\tau - \hat{x}_\tau]^\intercal PA(\xi_\tau)[x_\tau - \hat{x}_\tau],\end{aligned}$$

where $\xi_\tau = \beta_\tau x_\tau + (1-\beta_\tau)\hat{x}_\tau$. By invoking (8), this becomes

$$\frac{d}{d\tau}\,\|x_\tau - \hat{x}_\tau\|_P^2 \leq \lambda_1(\xi_\tau)\,\|x_\tau - \hat{x}_\tau\|_P^2\,.$$

Hence

$$\|x_\tau - \hat{x}_\tau\|_P^2 \leq e^{\int_0^\tau \lambda_1(\xi_s)ds}\,\|x_0 - \hat{x}_0\|_P^2\,,$$

or equivalently

$$\|x_\tau - \hat{x}_\tau\|_P \leq e^{\frac{1}{2}\int_0^\tau \lambda_1(\xi_s)ds}\,\|x_0 - \hat{x}_0\|_P\,.$$

We now use (6), which gives

$$\begin{aligned}\|x_\tau - \hat{x}_\tau\|_P &\leq e^{\frac{1}{2}\int_0^\tau \lambda_1(\xi_s)ds}\,\|x_0 - \hat{x}_0\|_P \\ &\leq e^{\frac{1}{2}\int_0^\tau \Lambda - \dot{v}(\xi_s)ds}\,\|x_0 - \hat{x}_0\|_P \\ &= e^{\frac{1}{2}[\Lambda\tau + v(\xi_\tau) - v(\xi_0)]}\,\|x_0 - \hat{x}_0\|_P\,.\end{aligned}$$

Since $v(x) \leq \bar{v}, \forall x \in \mathrm{Conv}(K)$, we obtain

$$\|x_\tau - \hat{x}_\tau\|_P \leq e^{\frac{1}{2}\Lambda\tau + \bar{v}}\,\|x_0 - \hat{x}_0\|_P\,.$$

It is now sufficient to pick $\hat{\delta}$ satisfying (10) with $\delta = \hat{\delta}$ to ensure that $t_{\max} > 0$. Then, for any $0 < \tau \leq t_{\max}$,

$$\begin{aligned}\|x_\tau - \hat{x}_\tau\|_P &\leq e^{\frac{1}{2}\Lambda\left(\frac{2\ln(\hat{\epsilon}) - 2\ln(\delta)}{\Lambda}\right) + \bar{v}}\delta \\ &= e^{\ln(\hat{\epsilon}) - \ln(\delta) + \bar{v}}\delta \\ &= e^{\ln(\epsilon) - \ln(\delta) - \bar{v} + \bar{v}}\delta \\ &= \epsilon,\end{aligned}$$

for any δ smaller than $\hat{\delta}$. $\blacksquare$

With this lemma, we are now ready to introduce the communication scheme. Our observer operates in periods of fixed length τ.

Procedure 1. (Observer). The time quantizer equation is

$$t_{k+1} = t_k + \tau, \qquad t_0 = 0.$$

(1) **Initialization Step** ($t = 0$).

(a) Given an anytime exactness ϵ, the coder and the decoder find a common $\hat{\delta}$ that has the properties described in Lemma 1 and meets (10) with $\delta = \hat{\delta}$. They pick a common $\delta \leq \hat{\delta}$ and build an identical finite covering of K with δ-balls whose centers are $\{q_j\} = Q \subset \mathbb{R}^n$, where j is an index.
(b) The coder and decoder choose $\tau = t_{\max}$.
(c) The period index is set to zero ($k = 0$).

(2) **Continuous Step** ($t \in [0, \infty)$).
If $t = t_k$,
(a) The coder computes the index j of an element of the covering that contains $x(t_k)$.
(b) The coder sends the index j over the communication channel.
(c) The decoder assigns $\hat{x}(t) = q_j$.
(d) The period index is increased by one. ($k = k+1$).
If $t \in (t_k, t_{k+1})$,
(e) the decoder computes $\hat{x}(t)$ as the solution of (1) starting at $t = t_k$ with $\hat{x}(t_k)$.

In step 1a, the coder and decoder choose a δ. This δ is only required to be smaller than $\hat{\delta}$. By Lemma 1, this step in fact guarantees the observability over the communication channel. Meanwhile, the choice of δ may affect the resultant channel rate which is why we introduce the following definition.

Definition 3. The observer defined in Procedure 1 is c-feasible for a system $\{\varphi^t\}_{t\geq 0}$ if the observer leads to observability as defined in Definition 2 of the system over a communication channel with channel rate c.

Before we expand on the optimal choice of δ and the resulting channel rate, we take a brief moment to advocate some advantages of the proposed design.

- It tacitly assumes that if the transmission initiated at step 2c fails and so the decoder receives no message to implement step 2d at time t_k, the rule 2e is still applied with the initial data given by the last successful transmission. This may entail violation of the desired accuracy ϵ, which however looks like a somewhat fair price for the message dropout. On the positive side, this violation continues no longer than until the next successful transmission when the desired accuracy is immediately restored. It should be stressed that this advantage is achieved in the absence of feedback communication from the decoder to the coder. This is unlike many other papers in the area, where communication feedback is essentially employed to ensure robustness of the proposed observer against communication errors.
- In terms of the channel rate, the proposed observer outperforms the trivial scheme based on merely sending the quantized state measurement.

In order to give bounds on the channel rate required to implement our observer, we use the upper box dimension.

Definition 4. [Falconer (1997)] The upper box-counting dimension $\bar{d}_B(K)$ of a set $K \subset \mathbb{R}^n$ is given by

$$\bar{d}_B(K) = \limsup_{\delta \to 0} \frac{\log N_\delta(K)}{-\log \delta}, \tag{12}$$

where $N_\delta(K)$ can be defined in any of the following ways (with all of them resulting in a common value (12)):

(i) The smallest number of closed balls of radius δ that cover K.
(ii) The smallest number of cubes of side δ that cover K.
(iii) The number of δ-mesh cubes that intersect K.
(iv) The smallest number of sets of diameter at most δ that cover K.
(v) The largest number of disjoint balls of radius δ with centers in K.

Now we are in a position to state the first main result of the paper, where Λ is taken from Assumption 2 and $\bar{d}_B(K)$ is the upper box dimension introduced in Definition 4.

Theorem 1. Let Assumption 1, and 2 hold, then for any communication channel with rate $c > \frac{\Lambda \bar{d}_\delta(K)}{2\ln(2)}$, the proposed observer is c-feasible for that system.

The proof of this theorem will be provided in the full version of this paper

4. ANALYTICAL BOUNDS ON THE CHANNEL RATE

In the previous section, we have proven that our observer leads to observability over any channel with rate above $\Lambda \bar{d}_B(K)/2\ln 2$. In this section, we offer analytical bounds, depending on the system parameters. The most problematic component in these estimates is the upper box dimension. There are no known methods to compute it in a systematic way for an arbitrary set. In these regards, we will use results from Hunt (1996) and Leonov (2007) to upper bound $\bar{d}_B(K)$ with the Lyapunov dimension of the dynamical system. To this end, we introduce the following notation

$$J^t(x) = \frac{\partial \varphi^t}{\partial x}(x),$$
$$\sigma_i(\varphi^t, x) = \sigma_i\left(\frac{\partial \varphi^t}{\partial x}(x)\right), \qquad x \in K,$$

where $\sigma_i(\cdot)$ are the singular values of the matrix. These singular values are put in descending order, such that $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$, $\forall x \in K$.

Definition 5. The singular value function $\omega_d(J^t(x))$ of order $d \in [0, n]$ at $x \in K$ is defined as

$$\omega_d\left(J^t(x)\right) \coloneqq \begin{cases} 1, & d = 0, \\ \sigma_1(\varphi^t, x) \ldots \sigma_d(\varphi^t, x), & d \in \{1, .., n\}, \\ \sigma_1(\varphi^t, x) \ldots \sigma_{\lfloor d \rfloor + 1}(\varphi^t, x)^{d - \lfloor d \rfloor}, & d \in (0, n) \setminus \{1, .., n\text{-}1\}. \end{cases}$$

With the objective of defining the Lyapunov dimension of a dynamical system, we first introduce the Lyapunov dimension of a map in a point. The next three definitions are taken from Kuznetsov (2016) and use the symbol φ^t to refer to the evolutionary operator $\varphi^t(\cdot)$ for a fixed time t.

Definition 6. The local Lyapunov dimension of a continuously differentiable map $\varphi^1 : \mathbb{R}^n \to \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$ is defined as

$$d_L(\varphi^1, x) \coloneqq \sup\{d \in [0, n] : \omega_d\left(J^1(x)\right) \geq 1\}.$$

Using this definition, we introduce the Lyapunov dimension of a mapping of a set.

Definition 7. The Lyapunov dimension of a continuously differentiable map $\varphi^1 : \mathbb{R}^n \to \mathbb{R}^n$ with respect to a compact set $K \subset \mathbb{R}^n$ is defined as

$$d_L(\varphi^1, K) := \sup_{x\in K} \sup\{d \in [0, n] : \omega_d\left(J^1(x)\right) \geq 1\}.$$

Finally, we introduce the Lyapunov dimension of a dynamical system $\{\varphi^t\}_{t\geq 0}$.

Definition 8. The Lyapunov dimension of a dynamical system $\{\varphi^t\}_{t\geq 0}$ with respect to a compact invariant set K is defined as

$$d_L(\{\varphi^t\}_{t\geq 0}, K) := \inf_{t>0} \sup_{x\in K} \sup\{d \in [0, n] : \omega_d\left(J^t(x)\right) \geq 1\}.$$

Our estimates of the Lyapunov dimension of a dynamical system are based on the final assumption of the paper, which is similar to Assumption 2.

Assumption 3. Let $d = j + s \in [1, n]$ where $j = \lfloor d \rfloor \in \{1, \ldots, n\}$ and $s = d - \lfloor d \rfloor \in [0, 1)$. There exists a continuously differentiable function $w : \mathbb{R}^n \to \mathbb{R}$ and a symmetric positive definite matrix P such that

$$\dot{w}(x) + \sum_{i=1}^{j} \lambda_i(x) + s\lambda_{j+1}(x) < 0, \ \forall x \in \mathbb{R}^n \tag{13}$$

where the λ_i are the solutions of (7) and $\dot{w}(x) = \frac{\partial w}{\partial x} f(x)$.

The following theorem establishes a relationship between the box dimension of a set and the Lyapunov dimension of a map.

Theorem 2. [Hunt (1996)] Let Assumption 1 hold. Then

$$\bar{d}_B(K) \leq d_L(\varphi^1, K).$$

Corollary 1. [Hunt (1996)] Under Assumption 1,

$$\bar{d}_B(K) \leq d_L(\varphi^t, K)$$

for all $t \geq 1$.

The following theorem is a reformulation of Theorem 10.2 p 101 of Leonov (2007).

Theorem 3. if Assumption 3 holds with $d = j + s$, then for sufficiently large $T > 0$

$$d_L(\{\varphi^t\}_{t\geq 0}, K) \leq d_L(\varphi^T, K) \leq j + s.$$

We can now evaluate the channel rate in terms of the Lyapunov dimension. In the following theorem, Λ is taken from Assumption 2, and j and s are taken from Assumption 3.

Theorem 4. Let Assumptions 1, 2, and 3 hold, then for any communication channel with rate $c > \frac{\Lambda(j+s)}{2\ln 2}$ the proposed observation scheme is c-feasible.

Proof: By Assumption 1, the mapping associated with the dynamical system is C^1 and the set K is invariant under the map. We apply Corollary 1, which yields

$$\bar{d}_B(K) \leq d_L(\varphi^t, K).$$

Since this holds for any t, it holds for $t = T$ from Theorem 3. This allows one to write

$$\bar{d}_B(K) \leq j + s. \tag{14}$$

The proof is completed by Theorem 1, along with (14). ■

5. AN APPLICATION: THE LORENZ SYSTEM

This section is devoted to application of the developed theory to a classical chaotic system: the Lorenz system. We compute the theoretical data-rate bound in a closed form for this system. The Lorenz system Lorenz (1963), is a famous example of chaotic continuous-time systems. The system is described by three nonlinear ordinary differential equations

$$\dot{x} = -\sigma x + \sigma y \tag{15}$$

$$\dot{y} = \rho x - xz - y \tag{16}$$

$$\dot{z} = xy - \beta z \tag{17}$$

where σ, ρ, and β are positive parameters. For $\rho \leq 1$, the system has a single equilibrium $(0, 0, 0)$ which is globally asymptotically stable. For $\rho > 1$, the origin becomes a hyperbolically unstable saddle-point and the system has two additional equilibria. We consider the latter case. To guarantee the existence of a non-trivial invariant set, we adopt the following assumption (Leonov et al. (2016))

Assumption 4. Let the following hold

$$\rho > 1,$$

$$\rho \geq \frac{\beta^3 - 2\beta^2 + 6\beta^2\sigma - 3b\sigma^2 - 6b\sigma + b}{3\sigma^2} + 1,$$

$$\sigma\rho > (\beta + 1)(\beta + \sigma),$$

and either

$$\sigma^2(\rho - 1)(\beta - 4) \leq 4\sigma(\sigma\beta + \beta - \beta^2) - \beta(\beta + \sigma - 1)^2,$$

or the following equation has two distinct solutions γ

$$(2\sigma - \beta + \gamma)^2(\beta(\beta + \sigma - 1)^2 - 4\sigma(\sigma\beta + \beta - \beta^2) + \sigma^2(\rho - 1)(\beta - 4)) + 4\beta\gamma(\sigma + 1)(\beta(\beta + \sigma - 1)^2 - 4\sigma(\sigma\beta + \beta - \beta^2) - 3\sigma^2(\rho - 1)) = 0 \tag{18}$$

and

$$\begin{cases} \sigma^2(\rho - 1)(\beta - 4) > 4\sigma(\sigma\beta + \beta - \beta^2) - \beta(\beta + \sigma - 1)^2 \\ \gamma_1 > 0 \end{cases},$$

where γ_1 is the largest root of (18).

With this assumption the following proposition establishes a bound on the channel rate required for the observation of the Lorenz system with our observer.

Proposition 1. Let Assumption 4 hold. Then the proposed observer is c-feasible for the system (15)-(17), where

$$c = \frac{\left[\sqrt{(\sigma\text{-}1)^2 + 4\rho\sigma}\text{-}\sigma\text{-}1\right]\left[3\sqrt{(\sigma\text{-}1)^2 + 4\rho\sigma}\text{-}2\beta + \sigma + 1\right]}{2\ln 2\left[\sqrt{(\sigma\text{-}1)^2 + 4\rho\sigma} + \sigma + 1\right]}.$$

and K is any invariant compact set.

Proof: By using Theorem 9 from Matveev and Pogromsky (2017), it is easy to see that

$$\Lambda = \frac{\sqrt{(\sigma\text{-}1)^2 + 4\rho\sigma}\text{-}\sigma\text{-}1}{\ln 2}.$$

Meanwhile, Theorem 3 from Leonov et al. (2016) yields that

$$d_L(\{\varphi^t\}_{t\geq 0} \leq 3 - \frac{2(\sigma + \beta + 1)}{\sqrt{(\sigma - 1)^2 + 4\sigma\rho} + \sigma + 1},$$

which completes the proof. ■

6. CONCLUSION

In this paper, we presented a solution for the state estimation problem in a non-classic setup: estimates of the state are produced at a remote location to which the system is connected via a data-rate constrained channel. The observer has been proven to be implementable on any channel with a rate above $\Lambda \bar{d}_B(K)/2\ln 2$ where Λ is the largest Lyapunov exponent of the observed system and $\bar{d}_B(K)$ is the upper box dimension of the attractor of that system. Results were also provided that allow one to obtain analytical bounds on the required channel rate in terms of system parameters by using the Lyapunov dimension instead of the upper box dimension. The developed theory was applied on the Lorenz system for which a bound on the channel rate was found in a closed form. The next steps in terms of research on this subject include adding other types of uncertainty (e.g. in the system parameters or in the form of noises in the communication channel), adding a stochastic model for the losses in the communication channel, and using the communication protocol in the context of rate-constrained synchronization problems for large-scale networks.

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