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Abstract

In the terminology of Martin Gardner, a pentomino cube is defined as a cube covered with a set of the twelve (folded) pentominoes. The very first pentomino cube was published in 1948 by H.D.Benjamin, but virtually ignored in the literature, which instead deals with the second example published in 1954 by W.Stead, also in The Fairy Chess Review. Until 1994 only three other unpublished and unidentified pentomino cubes were known to the author. In spite of George Martin's referring to a "fiendishly difficult problem", the author constructed more than twenty pentomino cubes by hand. In the mean time he realized that pentominoes folded around the cube are very different from the usual ones in the plane, in that two non-neighbouring squares of a pentomino in the plane can become neighbours when folded around a corner of the cube. They then get, what the author calls, an internal edge and they are not easy to recognize. For example, pentominoes P, U, and V all appear the same when each of them is folded around its own corner. By the way, the pentomino cubes of Benjamin and Stead have no such internal edges. In 1995 the author started constructing pentomino cubes by computer. The total number of distinct solutions modulo rotation and reflection appears to be astronomical: 26,358,584. The majority of them have at least one internal edge. The number of so-called nice solutions, without internal edge, is 284,402.

1. Introduction

Fifty years ago the following problem was posed and solved by Herbert Daniel Benjamin (1899-1950) in The Fairy Chess Review published in England but now extinct [1]:
Cover a root-10-edge cube exactly with the 12-five-pieces properly fitting.

Only the proposer's own solution was published, reading as follows:
Put the pieces on b10-14, c11-13de13, c10d10-12e11, d89e910f10, g10-11h10i10-11, f89ghi9, cd7e678, f67g567, a6b456c4, c5d456e5, f45g234, e1234f3. Draw the root 10 lines, lower left corner of f1-b13, same of i2-e14, complete the 4 root-10 squares between these with one to left and one to right over obvious pieces, and fold up completely to cover the root-10 cube, a beautiful result.

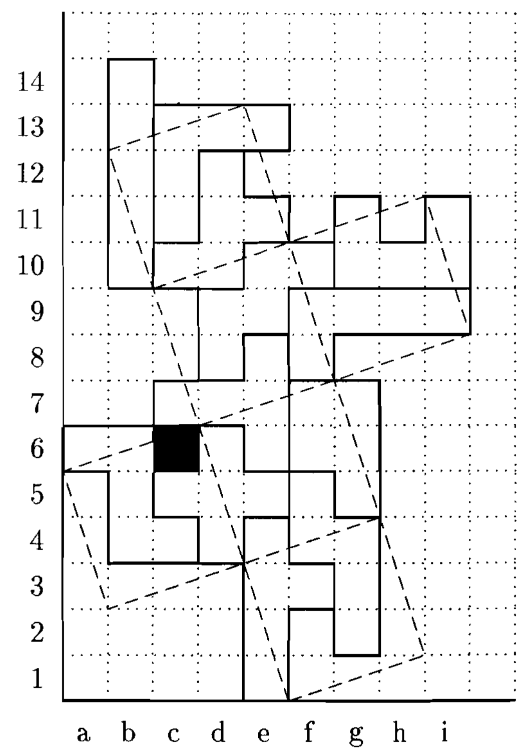


Fig.1. Benjamin's solution decoded.

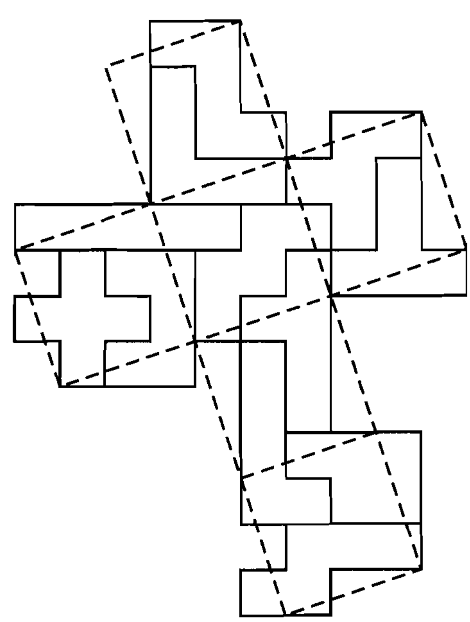


Fig.2. Stead's solution.

Some explanation is in order. First, five-piece is the old name for pentomino. Secondly, the solution although being a bit cryptic is in conformity with the coding system used in The Fairy Chess Review. A square of the chess board is identified by a letter and a number. Assume the origin of the board at bottom-left. The columns are marked a, b, c, ... from left to right, the rows are labeled 1, 2, 3 ... from bottom to top. Then all twelve pentominoes I, V, F, W, U, L, T, P, Z, X, N, and Y are placed in succession. The root-10 lines are drawn similarly and the folding of the pentominoes can be completed. It should be noticed that square c6 (black) is left uncovered.
With the diagram shown in Fig.1, it is easy to actually realize the root-10 cube in three dimensions with its surface tiled by the twelve pentominoes.
Mathematically speaking, a pentomino is defined in the two-dimensional plane as a combination of five (unit) squares connected edge to edge. Folded pentominoes can behave differently from the plane ones, in that two unconnected squares of a pentomino may (appear to) become connected after folding that pentomino. Folded pentominoes can have so-called internal edges [2, 3]. One can be sure that Benjamin was familiar with this phenomenon. In fact, in his solution internal edges are absent and all pentominoes are undeformed and easy to recognize. This may explain why Benjamin called his solution beautiful.
Henceforth a pentomino cube will be called nice if internal edges are absent.

2. References in the literature

Six years after the publication of Benjamin's solution, a different one was printed, also in The Fairy Chess Review. In his column DISSECTION, W. Stead begins with "This gentle relaxation is worthy of a wider following. For too long it has struggled for existence without visual aid". Consequently, the corresponding Fig.2 is given without more ado [4]. Unfortunately, the originator of this second solution is unknown. In what follows it will be referred to as if due to Stead.
Three other authors refer to the problem in their respective books, to wit Martin Gardner [5], Solomon W. Golomb [6], and George E. Martin [7].
Let me quote Gardner: "Another interesting pentomino problem, proposed in The Fairy Chess Review by H.D.Benjamin, is shown in Figure 78. The twelve pentominoes will exactly cover a cube that is the square root of ten units on the side. The cube is formed by folding the pattern along the dotted lines". Of the three authors, Gardner is the only one giving credit to Benjamin, but his very nice Figure 78 is based on Stead's solution, not on that of Benjamin of 1948. That is quite understandable because the diagram was ready for the take, unlike that of Benjamin. A minor defect in the drawing for pentomino T is easy to correct. The other two authors completely ignore the work of Benjamin and join Gardner in dealing with Stead's solution only.
Golomb, mostly dealing with plane polyominoes, invents as it were a new puzzle which, in my opinion, warrants critical approach. He writes: "An unidentified reader of the Fairy Chess Review designed and solved this unusual problem. The 12 pentominoes will cover the irregular shape shown below, which can then be

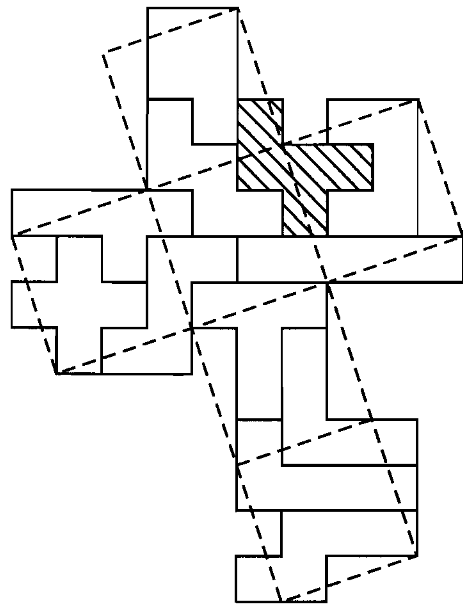


Fig.3. Alternative solution.

folded on the dashed lines to cover the surface of a cube as indicated." This is somewhat misleading. In fact the unidentified reader designed a cube covered with pentominoes, showed the exact positions of them inside an irregular space of 60 unit squares, and gave directions for actual constructing the cube with its tiled surface.
Diagrams as Fig.1 enable us to recover in a unique way the pentomino cube under consideration. The converse is not true. Given one such cube, we can construct very many different diagrams, and the irregular space may even become multiply connected. Perhaps overlooked by Golomb, the irregular space may have more than one covering by the 12 pentominoes. This is obvious from Fig.1. The block formed by pentominoes F and W can be reflected in the horizontal. Curiously, this type of symmetry was never reported. As a matter of fact, such symmetry is easily overlooked on the cube itself. Anyhow, there are at least 2 solutions in the sense of Golomb.
The third author, Martin, follows Golomb closely, but he adds the corresponding configuration of pentominoes in an extra figure. His reference 5.23 to it should have been 5.27. Martin ends the discussion by calling the problem "fiendishly difficult".

3. Golomb's new puzzle

The new puzzle of Golomb as mentioned in Section 2 does have the merit of being helpful in finding new solutions from old ones. We already found two solutions for the irregular space in Fig.1. By computer it is easy to verify that there are no more. Surprisingly may be, Benjamin's cube has another symmetry not directly evident from Fig.1. If we change the irregular space near the pentominoes L and P, by turning L over 90 degrees towards P, we get another symmetric block. The new irregular space has just 4 different pentomino covers, as the computer proves. However, of the 4 cubes so obtained only Benjamin's is nice. In the other 3 either W or P has an internal edge. Very likely, Benjamin was aware of these symmetries not leading to further nice cubes.
We now turn to Fig.2. The corresponding irregular space has, by computer, just two solutions, one of which (Fig.2) is that of Stead. The other, see Fig.3, is completely different, not directly derivable from Stead's and not nice because after folding F has an internal edge. By the way, Stead's solution leads to two other solutions by interchange of two congruent blocks each made of two pentominoes, F+Z and P+T, and after interchange of T and V pentominoes. They can be found from Fig.2 by appropriate turns and shifts of a few pentominoes. They are more easily derived from the pentomino cube itself. The new two solutions are not nice either.

4. How to solve a difficult problem easily

More than thirty years ago, I found a pentomino cube on my desk in my office

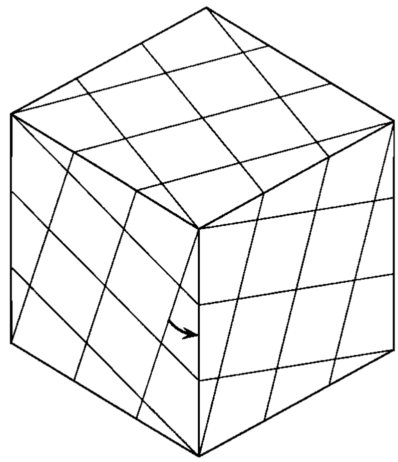


Fig.4. The angle indicated equals arctan(1/3).

at Philips Research Laboratories, Eindhoven. It was clearly a present but I never learned where it came from. I still have it, made of green paper with pentominoes drawn in red lines, not nice because F is deformed. About three years ago, when I learned of the existence of yet another pentomino cube, also of unknown origin, I had already started making my own pentomino cubes. By consulting recreational and other journals, I was highly surprised that only very few cubes had been published. Moreover, why did nobody mention the symmetries in the published solutions? Sure, I was acquainted with Gardner [5] and Golomb [6]. Much later I found the compilation of dissection problems in The Fairy Chess Review by G.P.Jelliss [8].
Up to now I do not understand how Benjamin discovered his cube. The presentations by Golomb [6] and Martin [7] are of no help. In my opinion they do not encourage the study of pentomino cubes. Martin's "fiendishly difficult" is not very inviting at all, or is it?
We need two things, a cube and a set of pentominoes made of wood and thin paper respectively. An appropriate grid of 60 squares is drawn on the cube surface. It can be done in two ways. We choose that shown in perspective in Fig.4, not its reflection in 3-space. For many people this fair geometric object will be a striking disclosure indeed. As to the set of pentominoes, it is obvious that those for sale are useless because they cannot be folded.
Having not enough fingers to set and keep more than a few pentominoes folded on the cube, I drilled a tiny hole at the center of most squares. With tiny nails as applied in fasten asphalt paper on wood, I can stick the pentominoes to the cube. Moreover, I am not afraid at all to renew my set of pentominoes now and again!
By trial and error, or more systematically by backtrack, I succeeded in constructing more than twenty solutions by hand. One of them was found in eight hours of puzzling, others required less than one hour or so. Anyhow, it was never a question of a few minutes. What I did learn from this experiment was the idiosyncrasies of pentominoes upon folding. For example, pentomino P and its reflection can cover the same five squares if they are folded around a corner of the cube. The same holds for pentomino U. Again, pentominoes P, U and V can cover the same squares on the cube. Also pentominoes L, N, Y can cover each other at certain positions around a corner. The pentominoes involved get an internal edge, they are deformed and not uniquely identifiable. By hand, I constructed a cube for which pentominoes P, U, V can be permuted, leading to six different solutions modulo rotation (and reflection, of course).
Possibly, Benjamin used a similar gadget as mine with holes and nails. Perhaps Stead's cube was indeed Benjamin's second solution, who knows?

5. Invoking the computer

About June 1994 I started attacking Benjamin's problem with the computer, and it became soon clear to me that the job could not be done in a couple of days, weeks or even months, in view of the astronomically large number of solutions to be expected. I wrote various programs for a Philips P3230/12.5 MHz computer

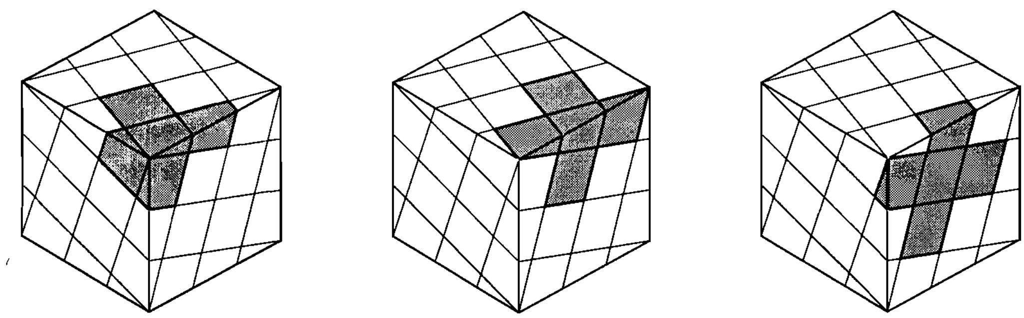


Fig.5. Definition of the three classes of pentomino cubes.

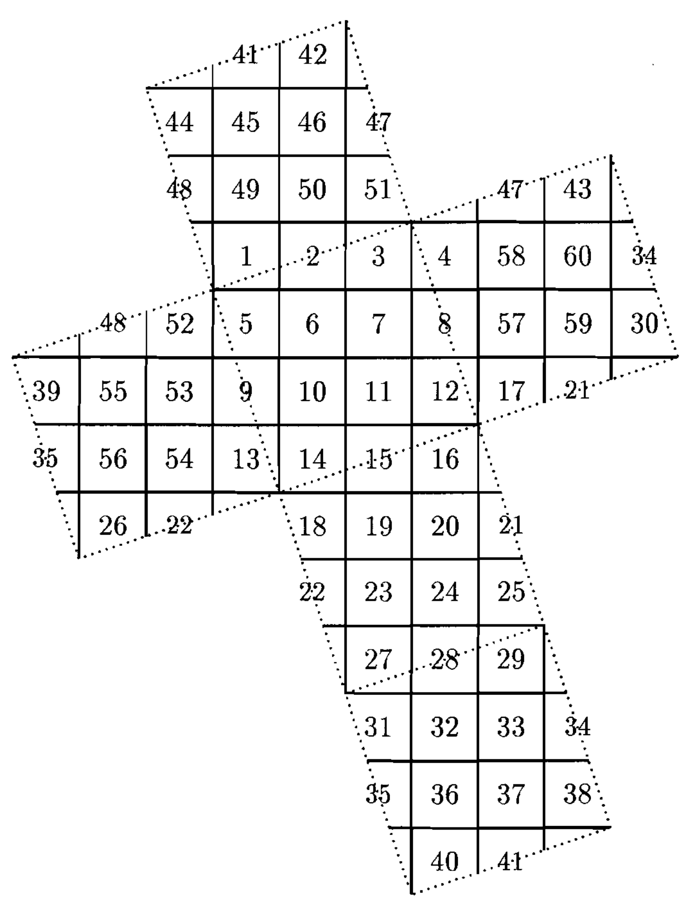


Fig.6. Cell numbers on the cube's layout.

in BASIC with IBM PC compiler version 1.00, but only as late as September 1994 did I really start to produce many solutions with a 80486-25MHz computer at the Technical University Eindhoven, with GWBASIC and IBM's compiler versions 1.00 and/or 2.00. The first 100000 solutions (of Class1, see below) were obtained in one run of almost 59 hours. On December 31, 1994 over 3000000 solutions had been found. Then, in January 1995 I bought a 80486 DX2-66MHz, twice as fast as the earlier machine, with GWBASIC and the 2.00 compiler. Sometimes the two computers were running day and night. It soon became obvious that breaking up the whole computation in short runs was preferable in view of possible malfunction of the systems. In January 1997 I bought a Pentium 200, but the job was done already. I used it for timing and checking purposes only.

We distinguish three cases, according to the location of pentomino X on the cube. This pentomino can be placed in sixty different ways, but only three of them are different modulo rotation of the cube. Therefore three classes are introduced: In Class 1, pentomino X is fixed at a specific corner, where three squares meet, which are covered by X. In Class 2, X is fixed symmetrically with respect to a specific edge, and Class 3 is the last alternative as shown in Fig.5.

In Class 1, X has an internal edge so that no nice solutions occur here at all. In Class 2, pairs of solutions identical modulo rotation of the cube are to be expected. Each class is split up in subclasses by fixing a second pentomino. That leads to 133, 91, 66 subclasses for the three classes 1, 2, 3, as we shall indicate below.

Clearly, further standardization of representing solutions is a must. First we unfold the cube surface onto the plane in the form of a cross, so that all faces become visible, although some of them are turned about. If the cube stands before you on the table, the upper square of the cross represents the back, the next three squares represent left, top, and right, while the remaining faces show front and bottom of the cube.

There are two reasons why our cross is somewhat tilted: (1) the drawings are similar to those of Benjamin and Stead, and (2) they are of better quality if only a matrix printer is available, as the boundaries of the pentominoes remain straight lines.

In the sequel the unit squares on the cube surface will be called cells, and they will be numbered as indicated in Fig.6. So, in Class 1, X covers cells 1, 2, 5, 49, 52. For Class 2 these numbers are 1, 2, 3, 6, 50 and for Class 3 they are 2, 5, 6, 7, 10. If only X is placed on the cube, the so-called first free cell is cell 3, 4, and 1, respectively. The second (to be fixed) pentomino shall cover the first free cell, so long as it does not interfere with X. For an illustration see Fig.7, where pentomino T was chosen as to be fixed. The T can be placed in eight different ways, leading to eight subclasses of cubes. Two other allowable positions of T need not be considered since X+T blocks one cell, number 6 or 50. Together the 11 pentominoes give rise to 133 subclasses of Class 1. In a similar way Class 2 has 91 and Class 3 has 66 subclasses.

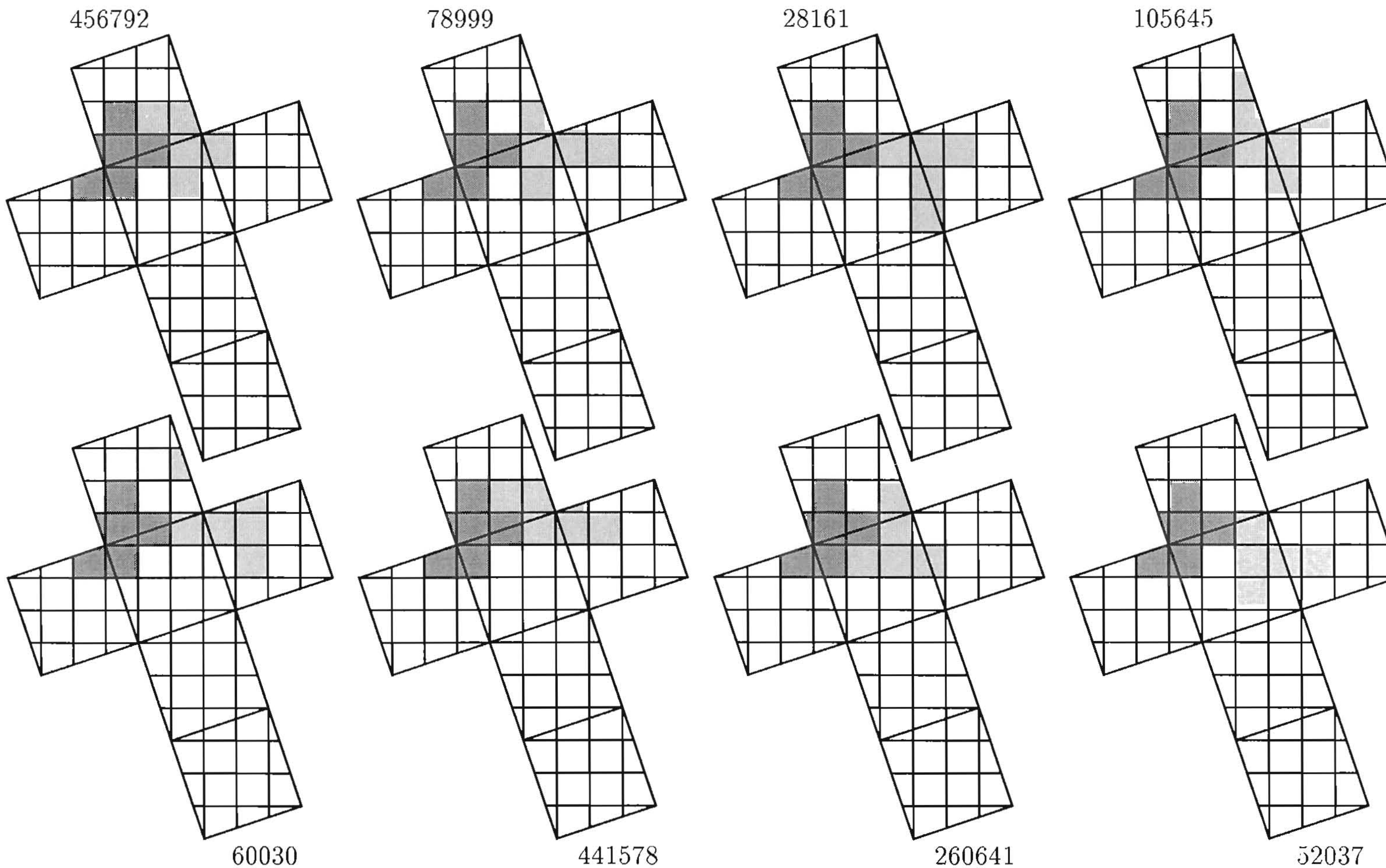


Fig.7. Eight subclasses (T second fixed pentomino) with their respective numbers of solutions.

For the computer, a pentomino placed on the cube is an integer array of dimension five, the elements of which are the five numbers (ordered to increasing value) of cells covered by the pentomino. All eleven pentominoes beyond X together produce 3576 such arrays, which are ordered pentomino-wise in the order U, T, Z, V, W, I, F, N, Y, L, P. The arrays for each pentomino are ordered to increasing lexicographical value. Pentomino U has 216 arrays, T, Z, V, W have 240, I has 120, F 456, N 480, Y 456, L 480, and P has 408. The corresponding matrix of 5 columns and 3576 rows is stored on disk. It is quite a job to construct this matrix, but the computer can help us. For example, if we know one array, we can construct 23 others by rotation of the cube. With some coding by hand I got the matrix in two different ways, and lateron my friend Jaap Zonneveld, inspired by my holes and nails gadget, confirmed my results by a completely different and ingenious program.

Solutions are obtained by the process of backtrack, and they are output in the form of an ordered (according to the respective first free cells) array of eleven numbers in the range 1 to 3576. Storing all solutions is out of the question. Instead, I printed one solution on my matrix printer for every thousand found by the computer. The missing 999 solutions are easy to retrace if need be. Many different options were installed. For example, I can place five pentominoes as fixed (anywhere on the cube) and construct all solutions (if any) different in the other six. Further, the computer can be instructed to compute nice cubes only. It is also possible to cope with the pairs in Class 2, skip the solutions with the larger of the two codes, and store the remaining on disk, what I did. Table II does not include the corresponding numbers, exept for the totals.

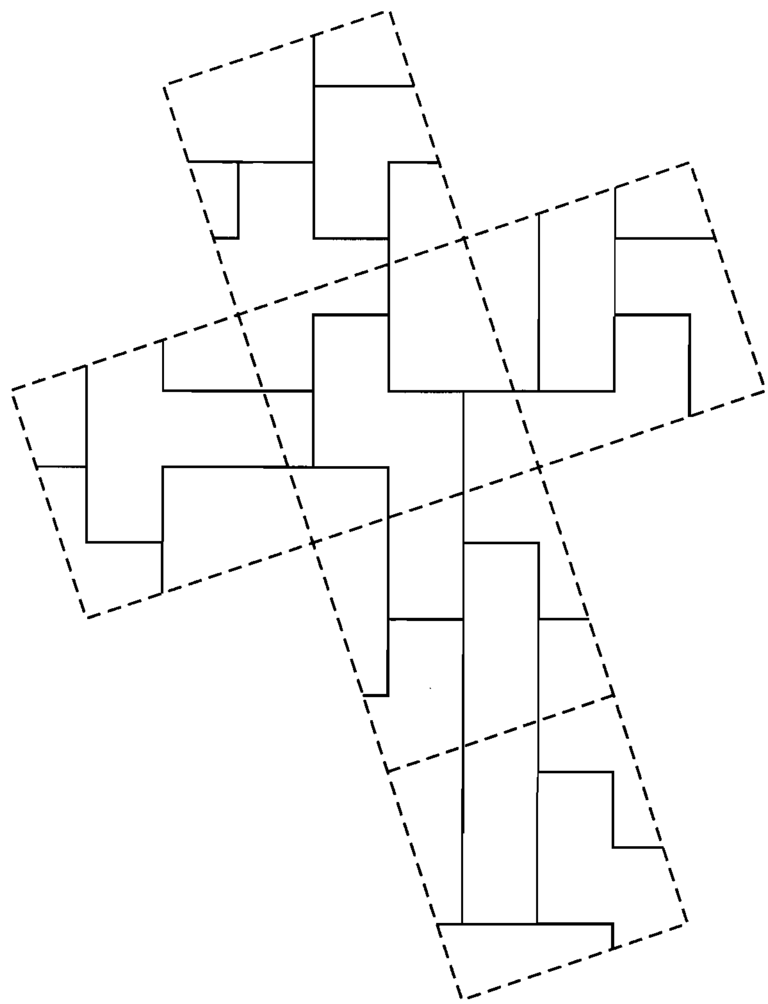
Also, I determined all solutions with a deformed pentomino around each of the eight corners, and in this case too all solutions were stored on disk. There are 22972 of them, one of which is shown in Fig.8. It really presents 36 solutions, since F, W, Z can be permuted and so can P, U, V. It belongs to Class 1 and is of type FLT, meaning that F, L, T (and I) are not deformed. This type FLT has 648 different solutions. The coded form of any solution in eleven numbers is sufficient to produce a drawing on screen or printer, as shown in Fig.8. See also my former paper [2], of which the results were obtained after those reported here.

Movies and still pictures, all in colour, were programed on screen for fun. But Jaap Zonneveld's movie of rotating and waggling coloured cubes cannot be beaten.

At the end of this section, the standard figures of the cubes of Benjamin and Stead are given in Fig.9 and Fig.10 respectively. It should be recalled that Class-2 solutions came in pairs, and thus we take the representation with the smaller (lexicographical) code. The larger code is

```
2323  275  524 1235 1523  826 1101  150 3091 3531 2218
   Y    T    Z    I    F    V    W    U    L    P    N
```

but the corresponding drawing is omitted.



28	1880	299	552	798	1248	2517	1093	1659	3521	3162
U	N	T	Z	V	I	Y	W	F	P	L

The cube is of Class 1 The type is LNT

Fig.8. Eight pentominoes folded around the corners of the cube.

6. Tables

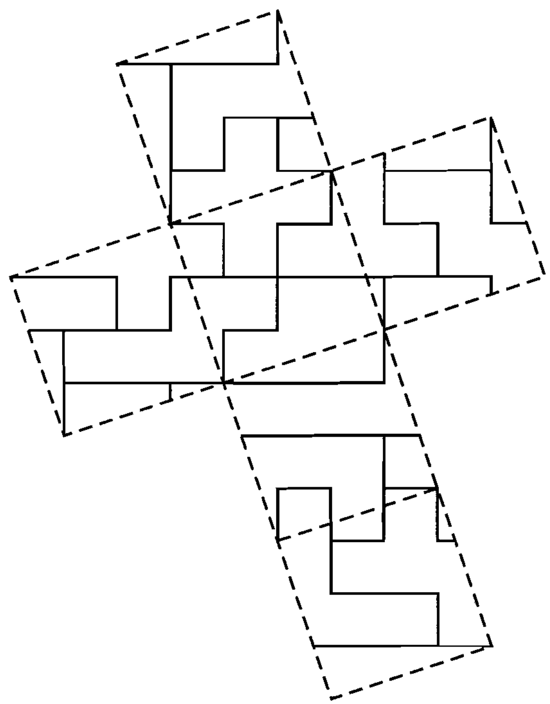
Solutions of Class 1 are specified in Table I. First, we take pentominoes in alphabetical order F, I, L, N, P, T, U, V, W, Y, Z. For example, F can be the second fixed pentomino in 18 different ways as shown in the first column. The next five columns indicate the five cells covered by F in the corresponding subclasses. The last column shows the number of solutions. Thus, if F is placed on cells 3, 4, 6, 7, 11, the number of solutions is 86489. Remarkably, the following two subclasses have the same number of solutions, namely 230212. This could have been predicted, because of the symmetries in the blocks X+F. The number of solutions with F is 1892436. The six subclasses for pentomino I give rise to 810579 solutions, ... , and the total number of pentomino cubes in Class 1 is 14755166.

Tables II and III have a different structure, in that they also show the number of solutions that are nice. For example, subclass F-1 of Class 2 has 35436 solutions of which 1041 are nice.

All in all, there are 26358584 different Benjamin cubes but only 284402 of them are nice.

Acknowledgement

My sincere thanks are due to Herman Willemsen for his enthousiastic support in editing this report and especially for his magnificent graphics.

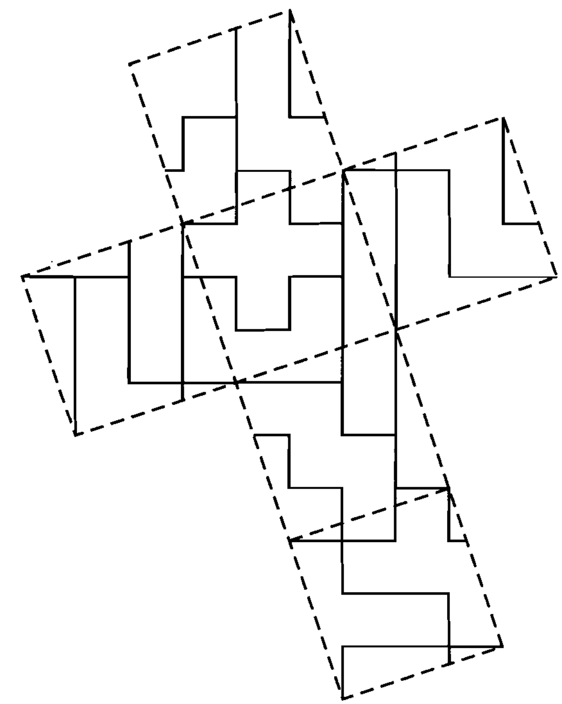


262 2347 1906 3308 2929 120 1095 867 1623 1288 689

T Y N P L U W V F I Z

The cube is of Class 2

Fig.9. Standard representation of Benjamin's cube.



2727 1839 1199 69 819 1058 2508 1623 398 636 3534

L N I U V W Y F T Z P

The cube is of Class 3

Fig.10. Standard representation of Stead's cube.

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Eindhoven, October 1997.

TABLE I

Class 1. Pentomino X covers cells 1 2 5 49 52.

F-1	3	4	6	7	11	86489
F-2	3	4	6	7	51	230212
F-3	3	4	7	50	51	230212
F-4	3	4	7	51	58	30383
F-5	3	4	8	12	57	16988
F-6	3	4	8	47	51	50240
F-7	3	4	8	47	58	20819
F-8	3	4	8	51	57	48262
F-9	3	4	8	51	58	31003
F-10	3	4	46	50	51	219350
F-11	3	4	47	50	51	236174
F-12	3	4	50	51	58	229628
F-13	3	4	51	57	58	40821
F-14	3	6	7	8	10	101105
F-15	3	6	7	8	12	100659
F-16	3	6	7	11	12	93997
F-17	3	7	8	12	57	17295
F-18	3	47	50	51	58	108799
						1892436

I-1	3	4	34	58	60	117429
I-2	3	7	11	15	19	184319
I-3	3	7	11	15	51	140747
I-4	3	7	11	47	51	142431
I-5	3	7	43	47	51	110114
I-6	3	38	43	47	51	115539
						810579

L-1	3	4	7	11	15	24377
L-2	3	4	7	11	51	52997
L-3	3	4	7	58	60	17140
L-4	3	4	8	12	16	10880
L-5	3	4	8	12	51	37119
L-6	3	4	43	47	51	60203
L-7	3	4	43	58	60	8779
L-8	3	4	51	58	60	62440
L-9	3	4	58	59	60	16181
L-10	3	6	7	47	51	128506
L-11	3	7	8	47	51	32577
L-12	3	7	8	57	59	27225
L-13	3	7	11	12	51	21269
L-14	3	7	11	14	15	31710
L-15	3	7	11	15	16	28894
L-16	3	7	11	50	51	98013
L-17	3	7	47	51	58	26713
L-18	3	42	43	47	51	27081
L-19	3	43	47	51	60	41148
						753252

N-1	3	4	6	7	58	92139
N-2	3	4	7	11	51	51088
N-3	3	4	7	50	51	229850
N-4	3	4	7	51	58	38340
N-5	3	4	8	12	51	40727
N-6	3	4	8	47	51	41080
N-7	3	4	8	57	59	20828
N-8	3	4	43	47	51	59022
N-9	3	4	43	47	58	27798
N-10	3	4	50	51	58	200623
N-11	3	4	51	58	60	56834
N-12	3	4	57	58	59	23140
N-13	3	6	7	10	14	102946
N-14	3	6	7	10	51	90424
N-15	3	7	8	12	16	12896
N-16	3	7	8	12	51	19546
N-17	3	7	11	12	16	19354
N-18	3	7	46	50	51	75170
N-19	3	42	46	50	51	76494
N-20	3	47	51	58	60	32243
						1310542

P-1	3	4	6	7	8	139380
P-2	3	4	7	8	11	28743
P-3	3	4	7	8	12	24130
P-4	3	4	7	8	51	65612
P-5	3	4	7	8	57	23726

T-1	3	4	7	50	51	456792
T-2	3	4	7	51	58	78999
T-3	3	4	8	12	58	28161
T-4	3	4	8	47	51	105645
T-5	3	4	47	57	58	60030

P-6	3	4	7	8	58	21416
P-7	3	4	8	57	58	20175
P-8	3	4	47	51	58	89682
P-9	3	6	7	10	11	174649
P-10	3	7	8	11	12	23204
P-11	3	46	47	50	51	137784
						748501

T-6	3	4	50	51	58	441578
T-7	3	6	7	8	51	260641
T-8	3	7	8	11	57	52037
						1483883

U-1	3	4	7	8	51	193208
U-2	3	4	7	11	12	35046
U-3	3	4	7	57	58	27914
U-4	3	4	8	11	12	37785
U-5	3	4	47	51	58	211812
U-6	3	6	7	50	51	1030396
U-7	3	7	8	57	58	45906
						1582067

V-1	3	4	7	8	51	199251
V-2	3	4	7	11	58	38524
V-3	3	4	17	57	58	24485
V-4	3	4	47	51	58	234736
V-5	3	7	8	51	57	60811
V-6	3	7	11	12	17	30277
V-7	3	47	51	57	58	81816
						669900

W-1	3	4	6	7	10	128754
W-2	3	4	6	7	51	518935
W-3	3	4	7	47	51	97457
W-4	3	4	8	17	57	19301
W-5	3	4	8	50	51	501040
W-6	3	4	8	51	57	100153
W-7	3	4	46	50	51	396368
W-8	3	4	51	57	58	99042
W-9	3	6	7	9	10	430607
W-10	3	7	8	12	17	26921
W-11	3	45	46	50	51	496180
						2814758

Y-1	3	4	7	11	51	49218
Y-2	3	4	7	47	51	48738
Y-3	3	4	8	12	51	40035
Y-4	3	4	8	50	51	266091
Y-5	3	4	8	58	60	22682
Y-6	3	4	43	47	51	60513
Y-7	3	4	47	58	60	41945
Y-8	3	4	51	58	60	60072
Y-9	3	4	57	58	60	23398
Y-10	3	6	7	8	57	152030
Y-11	3	6	7	11	15	171108
Y-12	3	6	7	11	51	86724
Y-13	3	7	8	11	15	27116
Y-14	3	7	8	11	51	24220
Y-15	3	7	11	12	15	18760
Y-16	3	7	47	50	51	90970
Y-17	3	43	47	50	51	94053
Y-18	3	43	47	51	58	26082
						1303755

Z-1	3	4	6	7	51	467021
Z-2	3	4	7	47	58	56021
Z-3	3	4	8	12	17	26118
Z-4	3	4	8	51	57	89488
Z-5	3	4	46	50	51	434783
Z-6	3	4	51	57	58	95117
Z-7	3	7	8	17	57	35927
Z-8	3	7	8	50	51	181018
						1385493

Total number of solutions of Class 1 is 14755166

TABLE II

Class 2. Pentomino X covers cells 1 2 3 6 50.

F-1	4	7	8	11	57	35436	1041
F-2	4	7	8	12	17	42076	2924
F-3	4	7	8	12	58	54957	4395
F-4	4	7	8	17	57	41493	2393
F-5	4	7	8	51	58	248134	9593
F-6	4	8	17	57	59	4575	278
F-7	4	17	57	58	59	10190	653
F-8	4	43	46	47	51	105812	6915
F-9	4	43	57	58	60	14474	997
F-10	4	51	57	58	60	59315	1758
						616462	30947

I-1	4	8	12	16	20	60750	3246
I-2	4	8	12	16	51	225569	11597
I-3	4	33	34	58	60	96275	777
						382594	15620

L-1	4	8	12	15	16	11440	1189
L-2	4	8	12	16	17	29351	0
L-3	4	8	12	16	58	11458	929
L-4	4	8	12	17	51	57432	3460
L-5	4	8	12	47	51	65740	3739
L-6	4	8	30	57	59	16533	640
L-7	4	8	34	58	60	18809	490
L-8	4	16	17	57	58	5886	261
L-9	4	30	34	58	60	13477	725
L-10	4	34	38	58	60	19135	649
L-11	4	34	51	58	60	109536	1041
L-12	4	38	43	47	51	37060	2249
						395857	15372

N-1	4	7	8	11	15	65367	1188
N-2	4	7	8	11	51	235866	7946
N-3	4	7	8	58	60	64114	2195
N-4	4	8	12	16	17	29232	0
N-5	4	8	16	17	57	4355	350
N-6	4	8	17	51	57	45138	2275
N-7	4	8	46	47	51	110330	3049
N-8	4	17	51	57	58	41541	2795
N-9	4	30	57	58	59	16165	654
N-10	4	30	58	59	60	9950	726
N-11	4	38	43	58	60	15178	1153
N-12	4	45	46	47	51	138047	4304
						775283	26635

P-1	4	7	8	11	12	51098	3731
P-2	4	7	8	57	58	96598	2515
P-3	4	8	12	17	57	13860	686
P-4	4	8	12	57	58	18624	1220
P-5	4	8	17	57	58	14155	862
P-6	4	8	47	51	58	61640	1411
P-7	4	8	51	57	58	79527	1775
P-8	4	8	57	58	59	20531	501
P-9	4	8	57	58	60	19822	437
P-10	4	43	47	51	58	53220	2367
P-11	4	46	47	51	58	161062	2692
P-12	4	47	51	57	58	77924	804
P-13	4	47	51	58	60	86624	1284
P-14	4	57	58	59	60	24100	313
						778785	20598

T-1	4	7	8	51	57	555901	14287
T-2	4	8	12	57	59	28049	1532
T-3	4	8	51	58	60	115408	3947
T-4	4	17	57	58	60	22357	1457
T-5	4	43	58	59	60	29801	2419
						751516	23642

U-1	4	8	12	17	58	13194	1271
U-2	4	8	47	51	57	84181	2068
U-3	4	8	57	59	60	12391	492
U-4	4	8	58	59	60	12393	452

V-1	4	8	12	17	21	34010	2150
V-2	4	8	12	58	60	25937	2257
V-3	4	8	43	47	51	80779	5233
V-4	4	8	51	57	59	136840	3280

U-5	4	12	17	57	58	15784	1920
U-6	4	43	47	51	60	96592	7078
U-7	4	43	51	58	60	132819	7933
						367354	21214

V-5	4	21	58	59	60	28052	557
V-6	4	42	43	58	60	37457	2369
						343075	15846

W-1	4	7	8	10	11	283876	11036
W-2	4	7	8	11	58	91494	5755
W-3	4	8	17	21	57	13426	1446
W-4	4	21	57	58	59	31110	470
W-5	4	42	46	47	51	203046	15425
W-6	4	51	57	58	59	111919	3587
						734871	37719

Y-1	4	7	8	12	16	41693	2384
Y-2	4	7	8	12	51	290961	14071
Y-3	4	7	8	57	59	65258	1858
Y-4	4	8	12	16	17	25062	0
Y-5	4	8	12	16	57	7399	515
Y-6	4	8	12	51	57	60061	3319
Y-7	4	8	12	51	58	52862	4512
Y-8	4	34	43	58	60	26553	1238
Y-9	4	34	57	58	60	21169	314
Y-10	4	34	58	59	60	15293	405
						606311	28616

Z-1	4	7	8	47	51	473516	16496
Z-2	4	8	21	57	59	28047	673
Z-3	4	8	43	58	60	25763	1941
Z-4	4	17	21	57	58	27475	1995
Z-5	4	42	43	47	51	73928	6224
Z-6	4	51	58	59	60	118807	2958
						747536	30287

Totals : 3249822 133248
different modulo rotation and reflection

TABLE III

Class 3. Pentomino X covers cells 2 5 6 7 10.

F-1	1 9 48 52 53	318421	5302
F-2	1 44 45 49 50	49017	2114
F-3	1 44 48 49 50	52082	2004
F-4	1 44 48 49 55	20956	728
F-5	1 44 48 52 55	59266	1635
F-6	1 45 46 48 49	16632	381
F-7	1 45 49 50 52	148962	2752
F-8	1 46 48 49 50	51815	741
F-9	1 46 49 50 51	53137	2665
		770288	18322

I-1	1 22 52 53 54	407698	2143
I-2	1 37 41 45 49	278969	877
		686667	3020

L-1	1 40 41 45 49	35168	541
L-2	1 40 44 48 52	57088	1369
L-3	1 41 42 45 49	46795	190
L-4	1 41 45 49 52	105316	504
L-5	1 48 49 55 56	40269	490
L-6	1 49 52 53 54	91667	898
L-7	1 52 53 54 56	91698	1145
		468001	5137

N-1	1 9 13 52 53	235866	7946
N-2	1 40 44 45 49	21459	766
N-3	1 40 44 48 49	32891	779
N-4	1 42 45 46 49	29626	861
N-5	1 42 46 49 50	70646	2136
N-6	1 48 52 55 56	78834	695
N-7	1 49 50 51 52	144606	6517
N-8	1 49 50 52 53	110362	2635
N-9	1 52 53 55 56	49507	1297
		773797	23632

P-1	1 44 45 48 49	39126	721
P-2	1 44 48 49 52	108065	1496
P-3	1 45 46 49 50	104441	1121
P-4	1 45 48 49 52	97584	656
P-5	1 48 49 50 52	209762	1828
P-6	1 48 49 52 53	77703	866
P-7	1 48 49 52 55	100965	498
P-8	1 48 52 53 55	88874	1044
		826520	8230

T-1	1 9 52 53 55	704230	7791
T-2	1 44 45 46 49	50775	2328
T-3	1 44 48 52 53	98957	3398
T-4	1 45 48 49 55	83604	804
T-5	1 45 49 50 51	141115	6174
		1078681	20495

U-1	1 3 49 50 51	408127	18329
U-2	1 44 45 48 52	147115	3979
U-3	1 44 45 49 52	139228	2432
U-4	1 49 52 53 55	83132	1058
		777602	25798

V-1	1 39 44 45 49	46483	1353
V-2	1 39 52 53 55	98958	3072
V-3	1 45 46 47 49	101186	0
V-4	1 45 49 52 53	130330	1309
		376957	5734

W-1	1 39 44 48 49	15145	471
W-2	1 39 48 52 55	103639	3270
W-3	1 46 47 49 50	95913	0
W-4	1 46 49 50 52	281497	5641
		496194	9382

Y-1	1 9 52 53 54	433235	3641
Y-2	1 41 44 45 49	27625	594
Y-3	1 41 45 46 49	28716	440
Y-4	1 41 45 48 49	27912	219
Y-5	1 41 45 49 50	97932	791
Y-6	1 48 49 50 51	54158	2491
Y-7	1 48 49 50 55	79863	1084

Y-8	1	48	52	53	54	90426	1047
Y-9	1	52	53	54	55	78435	834
						918302	11141

Z-1	1	9	49	52	53	694178	8421
Z-2	1	39	44	48	52	101216	2223
Z-3	1	39	48	49	55	70280	1375
Z-4	1	45	46	49	52	160883	2028
Z-5	1	47	49	50	51	154030	6216
						1180587	20263

Totals : 8353596 151154
different modulo rotation and
reflection