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J.H.J. Einmahl
M. Geilen

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John H.J. Einmahl[*]
Eindhoven University of Technology

Mario Geilen
Brabantia Nederland

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Abstract

A shortt of a one dimensional probability distribution is defined to be an interval which has at least probability t and minimal length. The length of a shortt, $U(t)$, and its obvious estimator, $U_n(t)$, are significant measures of scale of a probability distribution and the corresponding random sample, respectively. The shortt process is defined to be $\sqrt{n}(U_n(t) - U(t))/U'(t)$, similarly to the definition of the quantile process. It is known that this process converges weakly, under natural regularity conditions, to a Brownian bridge. In this note a strong approximation of the shortt process by a Kiefer process is established, which yields the weak convergence as a corollary. Applications of the result to the global and local strong limiting behaviour of the shortt process are also presented.

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1 Introduction and main result

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common distribution function F. A famous robust scale estimator based on the first n of these random variables is the length of a shortest closed interval containing at least fraction t (shortt) of these n observations. So the length of a shortt is defined to be

$$U_n(t) = \inf\{b - a : F_n(b) - F_n(a-) \geq t\},\ 0 < t < 1,$$

where F_n denotes the right-continuous empirical distribtution function (df). Similarly we define the the theoretical counterpart of the length of a shortt (i.e. our parameter of interest) by

$$U(t) = \inf\{b - a : F(b) - F(a-) \geq t\},\ 0 < t < 1.$$

The estimator $U_n(\frac{1}{2})$ can be shown to be a least median of squares estimator (cf. Croux and Rousseeuw (1992)) with asymptotic breakdown point 50% (see Rousseeuw and Leroy (1988))

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and it is approximately min-max bias robust (see Martin and Zamar (1993)). A functional central limit theorem (CLT) for the shortt process

$$\delta_n = \sqrt{n}\frac{U_n - U}{U'}$$

has been established in Grübel (1988). In Einmahl and Mason (1992) a functional CLT for a generalized quantile process is derived, which contains Grübel's result as well as the functional CLT for the classic quantile process as special cases.

It is the purpose of this note to refine the functional CLT for δ_n by establishing a strong approximation with a Kiefer process of the sequence of shortt processes $\{\delta_n\}$. We also briefly present applications to the strong limiting behaviour of the shortt process. In order to present our results we introduce more notation and some assumptions. Throughout let $f = F'$ be continuous, strictly increasing on $(\alpha, x_0]$, strictly decreasing on $[x_0, \beta)$ and 0 elsewhere, for certain $-\infty \leq \alpha < x_0 < \beta \leq \infty$. In addition let f' be continous on $\mathbb{R}$ with $|f'(x)| > \varepsilon$ on $x \in [x_1, x_2] \cup [x_3, x_4]$, with $\alpha \leq x_1 < x_2 < x_0 < x_3 < x_4 \leq \beta$ for some $\varepsilon > 0$. Also let $0 < \eta < 1/2$ be such that the endpoints of the interval determining $U(\eta)$ and $U(1-\eta)$ are contained in $[x_1, x_2] \cup [x_3, x_4]$. Set, as above,

$$\delta_n(t) = \sqrt{n}(U_n(t) - U(t))/U'(t) = \sqrt{n}h(U(t))(U_n(t) - U(t)), 0 < t < 1,$$

where $h = H'$ with $H = U^{-1}$; cf. the definition of the classical quantile process or see Einmahl and Mason (1992). Let K be a Kiefer process, i.e.

$$K(n,t) = W(n,t) - tW(n,1),\ n \in \mathbb{N},\ 0 \leq t \leq 1,$$

where W is a two-parameter Wiener process, see, e.g., Csörgő and Révész (1981), section 1.15. Observe that $n^{-1/2}K(n,t)$ is a Brownian bridge for any $n \in \mathbb{N}$.

Theorem 1.1

Under the above assumptions there exists a Kiefer process K such that

$$\sup_{\eta \leq t \leq 1-\eta} |\delta_n(t) - n^{-1/2}K(n,t)| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad a.s.$$

Remark 1. We only present the approximation with a Kiefer process since our method gives the same results for the approximation with a sequence of Brownian bridges. Note that the functional CLT for δ_n in Grübel (1988) immediately follows from Theorem 1.1.

Remark 2. It is not clear if the rate in Theorem 1.1 is optimal. However, in the recent paper Deheuvels (1997) it is shown that in the Kiefer process approximation of the classical quantile process the power of $1/n$ in the approximation cannot be higher than $\frac{1}{4}$. Therefore we expect that the power of $1/n$ in Theorem 1.1 cannot be improved beyond $\frac{1}{4}$, since the shortt process is a generalized quantile process.

In section 2 we present two applications of Theorem 1.1. The proof of Theorem 1.1 is given in section 3.

2 Applications

We first present the analogue of Chung's (1949) upper and lower class result for $D_n = \sup_{\eta \le t \le 1-\eta} |\delta_n(t)|$.

Theorem 2.1

Let $\lambda_n \uparrow \infty$. Then

$$P(D_n > \lambda_n \ i.o.) = \begin{cases} 0 \\ 1 \end{cases}$$

according as

$$\sum_{i=1}^{n} \frac{\lambda_n^2}{n} e^{-2\lambda_n^2} \begin{Bmatrix} < \\ = \end{Bmatrix} \infty.$$

Proof We present a brief sketch of the proof. It is well known that Theorem 2.1 holds with D_n replaced by $\sup_{0<t<1} |n^{-1/2}K(n,t)|$. Now by the law of the iterated logarithm (LIL) for $\sup_{t\in(0,\eta)\cup(1-\eta,1)} |n^{-1/2}K(n,t)|$ it can be easily shown, using $\lambda_n > \sqrt{\frac{1}{2}\text{loglog } n}$ eventually, that Theorem 2.1 holds with D_n replaced by $\sup_{\eta\le t\le 1-\eta} |n^{-1/2}K(n,t)|$. Finally, it rather easily follows from Theorem 1.1 that we can replace the latter expression by D_n itself, i.e. we have obtained Theorem 2.1. □

Note that Theorem 2.1 immediately yields the LIL for D_n. In fact the LIL for D_n and even the functional LIL for $\{\delta_n(t), \eta \le t \le 1-\eta\}$ (cf. Finkelstein (1971)) follow direct from Theorem 1.1 in conjunction with the corresponding results for the Kiefer process.

Theorem 1.1 also easily yields a result on the strong local behaviour of δ_n, provided the bandwidth is tending to 0 slowly enough. Define the oscillation modulus of δ_n by

$$\omega_{\delta_n}(a) = \sup_{\eta\le s\le t\le 1-\eta, t-s\le a} |\delta_n(t) - \delta_n(s-)|, \ 0 < a \le 1-2\eta.$$

Theorem 2.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of numbers satisfying $a_n \downarrow 0, na_n \uparrow, \log(\frac{1}{a_n})/\text{loglog } n \to c \in [0,\infty]$ and $a_n(n/\log n)^{1/3} \to \infty$, then

$$\limsup_{n\to\infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n(\log\frac{1}{a_n} + \text{loglog } n))^{1/2}} = 1 \ a.s.,$$
$$\liminf_{n\to\infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n(\log\frac{1}{a_n} + \text{loglog } n))^{1/2}} = \left(\frac{c}{c+1}\right)^{1/2} \ a.s.$$

When $c = \infty$, $\frac{c}{c+1}$ should be read as 1 and hence we then have

$$\lim_{n\to\infty} \frac{\omega_{\delta_n}(a_n)}{(2a_n \log\frac{1}{a_n})^{1/2}} = 1 \ a.s.$$

Proof These results are well-known for the Kiefer process (or for the uniform empirical process); see, e.g., Chapter 14 of Shorack and Wellner (1986). Hence, using $a_n(n/\log n)^{1/3} \to \infty$, Theorem 2.2 immediately follows from Theorem 1.1. □

3 Proof of Theorem 1.1

In order to prove Theorem 1.1 we first need a few auxiliary results. One of these results, Proposition 3.1, may be of independent interest. We begin with introducing more notation. We write P and P_n for the probability measures corresponding to F and F_n respectively. Also for a function $z: \mathbb{R} \to \mathbb{R}$ and a closed interval $I = [a, b] \subset \mathbb{R}$ we set $z(I) = z(b) - z(a-)$, provided $z(a-)$ exists. The class of intervals $I \subset \mathbb{R}$ is denoted with $\mathbb{I}$ and the length of such an I is written as $|I|$. The empirical process of $X_1, \ldots, X_n$ is denoted by

$$\alpha_n = \sqrt{n}(P_n - P),$$

and we will briefly write $\alpha_n(x)$ for $\alpha_n((-\infty, x])$, $x \in \mathbb{R}$.

Recall that the Komlós-Major-Tusnády (1975) Kiefer process approximation of the empirical process yields the existence of a Kiefer process $\tilde{K}$ such that

$$\sup_{x\in\mathbb{R}} |\alpha_n(x) - \frac{\tilde{\mathcal{K}}(n, F(x))}{\sqrt{n}}| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \ a.s. \tag{1}$$

Write $B_{P,n}(x) = \frac{\tilde{\mathcal{K}}(n,F(x))}{\sqrt{n}}$. Then (1) immediately yields

$$\sup_{I\in\mathbb{I}} |\alpha_n(I) - B_{P,n}(I)| = O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \ a.s. \tag{2}$$

Let I_x be the unique interval with $|I_x| = x$, $U(\eta) \le x \le U(1-\eta)$, and maximal probability P. Denote with $I_{n,x}$ a sample analogue of I_x, i.e. an interval with length x and maximal empirical measure P_n. Denote the left endpoints of I_x and $I_{n,x}$ by ℓ_x and $L_{n,x}$, respectively. The next lemma on the distance between ℓ_x and $L_{n,x}$ is a key ingredient for the proof of Theorem 1.1.

Lemma 3.1

There exists a $C \in (0, \infty)$ such that with probability one

$$\sup_{U(\eta)\le x\le U(1-\eta)} |\ell_x - L_{n,x}| \le C\left(\frac{\log n}{n}\right)^{1/3}$$

eventually.

Proof Define $I_{1,n} = I_{n,x} \setminus I_x$ and $I_{2,n} = I_x \setminus I_{n,x}$. Then $|I_{1,n}| = |I_{2,n}| = |\ell_x - L_{n,x}| =: B_n$. Observe that

$$\begin{aligned} P(I_{n,x}) - P(I_x) &= P(I_{1,n}) - P(I_{2,n}), \\ P_n(I_{n,x}) - P_n(I_x) &= P_n(I_{1,n}) - P_n(I_{2,n}). \end{aligned}$$

Write $b = f'(\ell_x)$ and $d = -f'(\ell_x + x)$. Then by a Taylor series expansion and the Glivenko-Cantelli theorem

$$P(I_x) - P(I_{n,x}) = \frac{1}{2}B_n^2(b + d) + o(B_n^2) \ a.s.$$

Hence with probability one

$$\begin{aligned}\frac{1}{2}B_n^2(b+d)+o(B_n^2) &= P(I_x)-P(I_{n,x})\\ &= P_n(I_{n,x})-P_n(I_x)-(P(I_{n,x})-P(I_x))\\ &= P_n(I_{1,n})-P_n(I_{2,n})-(P(I_{1,n})-P(I_{2,n}))\\ &= \frac{1}{n^{1/2}}\{\alpha_n(I_{1,n})-\alpha_n(I_{2,n})\}.\end{aligned} \tag{3}$$

We have

$$\begin{aligned}|\alpha_n(I_{j,n})| &\le |\alpha_n(I_{j,n})|1_{\{|I_{j,n}|<n^{-1/2}\}} \vee \frac{|\alpha_n(I_{j,n})|}{|I_{j,n}|^{1/2}}1_{\{|I_{j,n}|\ge n^{-1/2}\}}|I_{j,n}|^{1/2}\\ &\le \omega_n(\frac{1}{n^{1/2}}) \vee \tilde{\omega}_n(\frac{1}{n^{1/2}})B_n^{1/2},\ j=1,2,\end{aligned} \tag{4}$$

where ω_n and $\tilde{\omega}_n$ are the oscillation modulus and the Lipschitz-$\frac{1}{2}$ modulus of α_n respectively, see Mason, Shorack and Wellner (1983). Using the results in that paper it follows that the right-hand-side of (4) is almost surely bounded by $\tilde{C}\left(\frac{1}{n^{1/2}} \vee B_n\right)(\log n)^{1/2}$ eventually for some $\tilde{C} \in (0,\infty)$. Combining this with (3), (4) and the fact that b and d are bounded away from 0, Lemma 3.1 follows. □

Note that

$$H(x) = \sup_{I\in\mathbb{I},|I|\le x} P(I).$$

Set

$$H_n(x) = \sup_{I\in\mathbb{I},|I|\le x} P_n(I)$$

and

$$\gamma_n(x) = \sqrt{n}(H_n(x)-H(x)),\ x\ge 0.$$

Proposition 3.1

There exists a Kiefer process K such that

$$\sup_{U(\eta)\le x\le U(1-\eta)} |\gamma_n(x)+n^{-1/2}K(n,H(x))| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right)\ a.s.$$

Proof We will take K as follows:

$$K(n,H(x)) = -\left\{\tilde{K}(n,F(\ell_x+x))-\tilde{K}(n,F(\ell_x))\right\} = -n^{1/2}B_{P,n}(I_x).$$

Note that $K(n,t),\ n\in\mathbb{N},\ \eta\le t\le 1-\eta$, is a Kiefer process. Now we have

$$\begin{aligned}&\sup_{U(\eta)\le x\le U(1-\eta)} |\gamma_n(x)+n^{-1/2}K(n,H(x))|\\ &= \sup_{U(\eta)\le x\le U(1-\eta)} \left|\sqrt{n}\left(\sup_{|I|\le x}P_n(I)-H(x)\right)-B_{P,n}(I_x)\right|\\ &= \sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sqrt{n}\left(\sup_{|I|=x}P_n(I)-H(x)\right)-B_{P,n}(I_x)\right\}\\ &\vee \sup_{U(\eta)\le x\le U(1-\eta)} \left\{B_{P,n}(I_x)-\sqrt{n}\left(\sup_{|I|=x}P_n(I)-H(x)\right)\right\}.\end{aligned} \tag{5}$$

The second term in the right-hand-side of (5) is bounded from above by

$$
\begin{aligned}
&\sup_{U(\eta)\le x\le U(1-\eta)} \left\{B_{P,n}(I_x) - \sqrt{n}\,(P_n(I_x) - P(I_x))\right\} \\
&= \sup_{U(\eta)\le x\le U(1-\eta)} (B_{P,n}(I_x) - \alpha_n(I_x)) \\
&\le \sup_{I\in\mathbb{I}} |B_{P,n}(I) - \alpha_n(I)| \\
&= O\left(\frac{(\log n)^2}{\sqrt{n}}\right) \ a.s.,
\end{aligned}
$$

where the last 'equality' follows from (2).

Using Lemma 3.1, we see that with probability one the first term on the right in (5) can be bounded from above for large n by

$$
\begin{aligned}
&\sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sqrt{n}\left(\sup_{|I|=x,|\ell_x-L_{n,x}|\le C\left(\frac{\log n}{n}\right)^{1/3}} P_n(I) - H(x)\right) - B_{P,n}(I_x)\right\} \\
&\le \sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sqrt{n} \sup_{|I|=x,|\ell_x-L_{n,x}|\le C\left(\frac{\log n}{n}\right)^{1/3}} (P_n(I) - P(I)) - B_{P,n}(I_x)\right\} \\
&\le \sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sup_{|I|=x,|\ell_x-L_{n,x}|\le C\left(\frac{\log n}{n}\right)^{1/3}} |\alpha_n(I) - B_{P,n}(I)|\right\} \\
&+ \sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sup_{|I|=x,|\ell_x-L_{n,x}|\le C\left(\frac{\log n}{n}\right)^{1/3}} B_{P,n}(I) - B_{P,n}(I_x)\right\} \\
&\le \sup_{I\in\mathbb{I}} |\alpha_n(I) - B_{P,n}(I)| \\
&+ \sup_{U(\eta)\le x\le U(1-\eta)} \left\{\sup_{|I|=x,|\ell_x-L_{n,x}|\le C\left(\frac{\log n}{n}\right)^{1/3}} B_{P,n}(I) - B_{P,n}(I_x)\right\} \\
&= O\left(\frac{(\log n)^2}{\sqrt{n}}\right) + O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right),
\end{aligned}
$$

where again (2) is applied to bound the first term; the bound on the second term follows from the well-known oscillation behaviour of the Kiefer process (see, e.g., Theorem 1.15.2 in Csörgő and Révész (1981)). □

The shortt process δ_n can be seen as a quantile process version of the empirical-type process γ_n. Therefore we will obtain Theorem 1.1 by 'inverting' Proposition 3.1. We will begin with proving a version of Theorem 1.1 for the 'uniformized' version $\bar{\delta}_n$ of δ_n. Set

$$\bar{P}_n(t) = \sup_{|I|\le U(t)} P_n(I),\ 0 < t < 1,\ \bar{P}_n(0) = 0,\ \bar{P}_n(1) = 1,$$

and

$$\bar{P}_n^{-1}(s) = \inf\{t: \ \bar{P}_n(t) \geq s, \ 0 \leq t \leq 1\}, \ 0 \leq s \leq 1.$$

Then

$$\gamma_n(x) = \sqrt{n}(\bar{P}_n(H(x)) - H(x));$$

set

$$\bar{\delta}_n(t) = \sqrt{n}(\bar{P}_n^{-1}(t) - t), \ 0 \leq t \leq 1.$$

Proposition 3.2

For the Kiefer process of Proposition 3.1 we have

$$\sup_{\eta \leq t \leq 1-\eta} |\bar{\delta}_n(t) - n^{-1/2}K(n,t)| = O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \ a.s.$$

Proof Since H_n and hence $\bar{P}_n$ make only jumps of size $\frac{1}{n}$ almost surely, we obtain that

$$(6) \qquad \sup_{\eta \leq t \leq 1-\eta} |\bar{\delta}_n(t) + \sqrt{n}\left(\bar{P}_n(\bar{P}_n^{-1}(t)) - \bar{P}_n^{-1}(t)\right)| = \frac{1}{\sqrt{n}} \ a.s.$$

So by Proposition 3.1 and the LIL for the Kiefer process it follows that

$$(7) \qquad \sup_{\eta \leq t \leq 1-\eta} |\bar{P}^{-1}(t) - t| = O\left(\sqrt{\frac{\text{loglog } n}{n}}\right) \ a.s.$$

Using (7) in conjunction with the oscillation modulus results for a Kiefer process, again Proposition 3.1, and (6), we have

$$\begin{aligned} &\sup_{\eta \leq t \leq 1-\eta} |\bar{\delta}_n(t) - n^{-1/2}K(n,t)| \\ \leq &\sup_{\eta \leq t \leq 1-\eta} |\sqrt{n}\left(\bar{P}_n(\bar{P}_n^{-1}(t)) - \bar{P}_n^{-1}(t)\right) + n^{-1/2}K(n, \bar{P}_n^{-1}(t))| \\ &+ \sup_{\eta \leq t \leq 1-\eta} n^{-1/2}|K(n, \bar{P}_n^{-1}(t)) - K(n,t)| + \tfrac{1}{\sqrt{n}} \\ = &\, O\left(\tfrac{(\log n)^{2/3}}{n^{1/6}}\right) + O\left(\tfrac{(\text{loglog } n)^{1/4}(\log n)^{1/2}}{n^{1/4}}\right) + \tfrac{1}{\sqrt{n}} = O\left(\tfrac{(\log n)^{2/3}}{n^{1/6}}\right) \ a.s. \end{aligned}$$

Note that actually Proposition 3.1 has to be used on an interval slightly larger than $[U(\eta), U(1-\eta)]$. This causes no problems however, since f' is continuous. □

Now we have in hands the necessary ingredients to give a short proof of our main result.

Proof of Theorem 1.1 It is shown in Einmahl and Mason (1992) that almost surely

$$U_n(t) = U(P_n^{-1}(t)), \ \eta \leq t \leq 1 - \eta.$$

So by the mean value theorem, we have for some t_n between t and $\bar{P}_n^{-1}(t)$, that

$$(8) \qquad \begin{aligned} &\delta_n(t) - n^{-1/2}K(n,t) \\ &= \frac{h(U(t))}{h(U(t_n))}\left(\bar{\delta}_n(t) - n^{-1/2}K(n,t)\right) + \left(\frac{h(U(t))}{h(U(t_n))} - 1\right) n^{-1/2}K(n,t) \ a.s. \end{aligned}$$

The following facts, which can be shown by elementary analysis, will be needed: for $U(\eta) \leq x \leq U(1-\eta)$ we have

$$h(x) = f(\ell_x) = f(\ell_x + x),$$
$$h'(x) = \frac{f'(\ell_x)f'(\ell_x + x)}{f'(\ell_x) - f'(\ell_x + x)} < 0.$$

Hence we have that h and $|h'|$ are bounded away from 0 and ∞ on $[U(\eta), U(1-\eta)]$; in fact this even holds on a slightly larger interval. Observe that by the mean value theorem $h(U(t) - h(U(t_n)) = (t - t_n)h'(U(\tilde{t}_n))/h(U(\tilde{t}_n))$, with $\tilde{t}_n$ between t and t_n. So Proposition 3.2, (7), (8) and the LIL for the Kiefer process yield

$$\begin{aligned}
&\sup_{\eta \leq t \leq 1-\eta} |\delta_n(t) - n^{-1/2}K(n,t)| \\
&= O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) + O\left(\sqrt{\frac{\text{loglog } n}{n}}\right) O\left(\text{loglog } n)^{1/2}\right) \\
&= O\left(\frac{(\log n)^{2/3}}{n^{1/6}}\right) \quad a.s.
\end{aligned}$$

□

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Department of Mathematics
and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Brabantia Nederland
P.O. Box 25
5580 AA Waalre
The Netherlands