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Brief paper

Novel Gramians for linear semistable systems ☆

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ABSTRACT

In this paper, the notions of pseudo Gramians are introduced for linear time-invariant semistable systems, which allow multiple semisimple poles at the origin. The proposed Gramians are the generalizations of standard Gramian matrices defined for asymptotically stable systems, and they can be computed by a set of Lyapunov equations. Furthermore, it is shown that the controllability and observability of a semistable system are indicated by the ranks of the pseudo Gramians, and the controllability and observability energy functions are also characterized using the pseudo Gramians. Additionally, the $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm of a semistable system are analyzed, and then the results are used for the model reduction of semistable systems. Finally, the effectiveness of the methods is illustrated by an example of a gene regulation network.

1. Introduction

Controllability and observability of a system are central concepts in system and control theory. Closely related is the definition of Gramians (Moore, 1981). For an asymptotically stable system with system matrices A, B, C, the controllability and observability Gramians are defined as

$$P_s = \int_0^\infty e^{A\tau} BB^\top e^{A^\top \tau} \mathrm{d}\tau, \qquad Q_s = \int_0^\infty e^{A^\top \tau} C^\top C e^{A\tau} \mathrm{d}\tau, \tag{1}$$

respectively. The Gramians have many important applications. For instance, since the ranks of Gramians determine the controllability and observability of a linear system, the Gramians are used for selecting optimal locations of actuators and sensors (Georges, 1995; Marx, Koenig, & Georges, 2004; Summers, Cortesi, & Lygeros, 2016; Taylor, Luangsomboon, & Fooladivanda, 2017). Furthermore, the well-known model order reduction method, balanced truncation, is based on the singular value decomposition of the two Gramians, and the projection subspaces are constructed by discarding the less observable and controllable states.

In this paper, we consider linear semistable systems, which allow multiple semisimple poles at the origin. In contrast to asymptotically stable systems, semistable ones contain nonisolated equilibrium points, and each of them is Lyapunov stable (Bhat & Bernstein, 1999; Hui & Berg, 2013; Hui, Haddad, & Bhat, 2009). Semistable systems are an important class of linear systems appearing in many applications. In recent decades, a great deal of attention has been paid to the problem of network consensus, which describes a phenomenon in dynamical networks where the states of different agents in a network converge to a common value (Cheng & Scherpen, 2019; Ren, Beard, & Atkins, 2005; Wu, 2007). The analysis of semistability plays a central role here, as it ensures that the agents will converge to synchronized states even when the network is influenced by external control signals and disturbances. For instance, in the control of microgrids (Cucuzzella et al., 2018; Simpson-Porco, Dörfler, & Bullo, 2013), a distributed control law can be applied such that the closed-loop system is semistable, achieving a proportional current sharing as load demands are changing.

Note that the Gramian matrices in (1) are ill-defined in the case of semistable systems, since A is not necessarily Hurwitz. Thus, we generalize the definitions in (1) and propose *pseudo controllability and observability Gramians*. Relevant to this concept, recent advances have been seen in the context of network systems. In Hui (2011) an $\mathcal{H}_2$ optimal semistable control problem is studied for linear coupled systems, where a semistable Lyapunov equation admitting multiple solutions is used. For model reduction of semistable positive networks, Ishizaki et al. (2015)

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define a projected controllability Gramian, which is presented as the unique solution of a Lyapunov equation associated with the asymptotically stable part of a semistable network. In our previous works in Cheng, Kawano, and Scherpen (2017) and Cheng and Scherpen (2016, 2019), the notion of network Gramians is proposed, which facilitates the computation of vertex dissimilarities in terms of the $\mathcal{H}_2$ norm.

In this paper, the proposed pseudo Gramians extend the relevant definitions from the setting of network systems to the more general case of semistable systems, and a systematic analysis of the main features of pseudo Gramians is provided. First, explicit definitions of the Gramians are presented, which are shown to be the unique solutions of semistable Lyapunov equation and a kernel constraint. Compared to Cheng, Kawano and Scherpen (2019) and Ishizaki et al. (2015), where the Gramians are defined by a projected Lyapunov equation using symmetric left and right kernel spaces of the system matrix A, what we defined are augmented Gramians based on only the system matrices. Second, two rank conditions of pseudo Gramians are given to characterize the controllability and observability of a semistable system, and then the controllability and observability functions are established, interpreting the pseudo Gramians in terms of minimal input and output energy. Third, we derive the expressions of the $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm of a semistable system, and then we show how to utilize the pseudo Gramians for the model reduction of semistable systems. The proposed pseudo Gramians can be potentially used in applications, e.g., chemical reaction networks (Ahsendorf, Wong, Eils, & Gunawardena, 2014; Gunawardena, 2012; Ishizaki et al., 2015) and power systems (Cheng & Scherpen, 2018; Monshizadeh, De Persis, van der Schaft, & Scherpen, 2017), where semistable dynamics arise. In this paper, we apply the pseudo Gramians to the model reduction of semistable systems and propose two different reduction approaches, namely, balanced truncation and state aggregation. The effectiveness of the methods is then demonstrated by an example of a gene regulation network.

This paper is organized as follows. Section 2 provides the definitions of pseudo Gramians of semistable systems, and some important features of the new Gramians are also discussed. Section 3 gives the applications of the pseudo Gramians to the model reduction of semistable systems. Section 4 provides an illustrative example of model reduction of a gene regulation network using the proposed Gramians, and finally, concluding remarks are made in Section 5.

Notation: Denote $\mathbb{R}$ as the set of real numbers, and $\mathbb{R}^n$ is a vector space of n dimension. Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$, then $\mathcal{W}^\perp$ denotes the orthogonal complement of $\mathcal{W}$ in $\mathbb{R}^n$. $\mathtt{dim}(\mathcal{W})$ represents the dimension of space $\mathbb{W}$ The trace, rank, span and nullspace of A are denoted by $\mathtt{tr}(A)$, $\mathtt{rank}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively.

2. Gramians of semistable systems

This section extends the definition of *controllability and observability Gramians* from asymptotically stable systems to semistable ones. In our preliminary results in Cheng et al. (2017) and Cheng and Scherpen (2016), new Gramians are introduced for first-order and second-order network systems. Here, we present a generalization of the results to general semistable systems.

Consider the state-space model of a linear time-invariant system

$$\mathbf{\Sigma}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{2}$$

with states $x \in \mathbb{R}^n$, inputs $u \in \mathbb{R}^p$ and outputs $y \in \mathbb{R}^q$.

Definition 1 (*Bernstein & Bhat, 1995*)**.** The system $\mathbf{\Sigma}$ in (2) is **semistable** if $\lim_{t\to\infty} x(t)$ exists for all initial states $x(0)$ and $u(t) = 0$.

A necessary and sufficient condition for the semistability of $\mathbf{\Sigma}$ is provided in Bernstein and Bhat (1995).

Lemma 2. *The system $\mathbf{\Sigma}$ is semistable if and only if the zero eigenvalues of A in (2) are **semisimple**, i.e., the geometric multiplicity of the zero eigenvalue coincides with the algebraic multiplicity,[1] and all the other eigenvalues have negative real parts.*

Semistability is a more general concept than asymptotic stability as it allows for multiple zero poles in a system, whose trajectories thus may converge to a nonzero stable equilibrium. Generally, semistable systems are not in the $\mathcal{H}_2$ space, implying that the standard controllability and observability Gramians in (1) are not well-defined for the semistable case. Thus, we propose new definitions of Gramians for semistable systems.

Definition 3. Consider the semistable system $\mathbf{\Sigma}$ as in (2). The **pseudo controllability and observability Gramians** are defined as

$$\mathcal{P} = \int_0^\infty (e^{A\tau} - \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J}^\top)\mathrm{d}\tau \in \mathbb{R}^{n\times n}, \tag{3a}$$

$$\mathcal{Q} = \int_0^\infty (e^{A^\top\tau} - \mathcal{J}^\top)C^\top C(e^{A\tau} - \mathcal{J})\mathrm{d}\tau \in \mathbb{R}^{n\times n}, \tag{3b}$$

where $\mathcal{J} := \lim_{\tau\to\infty} e^{A\tau}$ is a constant matrix.

Note that the pseudo Gramians in (3a) and (3b) are well-defined, since the integrands in both integrals converge to zero when $\tau \to \infty$. Furthermore, using the matrix $\mathcal{J}$, the Lyapunov characteristics of $\mathcal{P}$ and $\mathcal{Q}$ in Definition 3 are provided.

Theorem 4. *Consider the semistable system $\mathbf{\Sigma}$ in (2). The pseudo controllability and observability Gramians of $\mathbf{\Sigma_s}$, $\mathcal{P}$ and $\mathcal{Q}$ defined in (3), are the unique symmetric solutions of the following sets of linear matrix equations*

$$\begin{cases} 0 = A\mathcal{P} + \mathcal{P}A^\top + (I - \mathcal{J})BB^\top(I - \mathcal{J}^\top), & \text{(a)} \\ 0 = \mathcal{J}\mathcal{P}\mathcal{J}^\top. & \text{(b)} \end{cases} \tag{4}$$

$$\begin{cases} 0 = A^\top\mathcal{Q} + \mathcal{Q}A + (I - \mathcal{J}^\top)C^\top C(I - \mathcal{J}), & \text{(a)} \\ 0 = \mathcal{J}^\top\mathcal{Q}\mathcal{J}. & \text{(b)} \end{cases} \tag{5}$$

Proof. Assume that A in (2) has zero eigenvalues with the geometric (or algebraic) multiplicity m, which means that the eigenspace of the zero eigenvalues has dimension m. Therefore, there exists a similarity transformation

$$A = \mathcal{U}D\mathcal{U}^{-1} = \begin{bmatrix} U & \bar{U} \end{bmatrix} \begin{bmatrix} 0_{m\times m} & \\ & \bar{A} \end{bmatrix} \begin{bmatrix} V^\top \\ \bar{V}^\top \end{bmatrix}, \tag{6}$$

such that $\bar{A} \in \mathbb{R}^{(n-m)\times(n-m)}$ is Hurwitz, and the matrices $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{n\times m}$ fulfill

$$\mathcal{R}(U) = \mathcal{N}(A),\ \ \mathcal{R}(V) = \mathcal{N}(A^\top),\ \text{ and } V^\top U = I_m. \tag{7}$$

Note that the product $UV^\top$ is invariant to the choices for U and V, and it holds that

$$\mathcal{J} = \lim_{\tau\to\infty} e^{A\tau} = UV^\top. \tag{8}$$

[1] Let λ be an eigenvalue of A. Then, the algebraic multiplicity of λ is the number of times λ appears as a root of the characteristic polynomial $|A - \lambda I|$, and the geometric multiplicity of λ is the dimension of the eigenspace of λ (Horn & Johnson, 2012).

Therefore, the following equations are obtained:

$$\mathcal{J}^2 = \mathcal{J}, \; A\mathcal{J} = 0, \; \text{and } \mathcal{J}A = 0. \tag{9}$$

Furthermore, for any $\tau \in \mathbb{R}$, we have

$$\mathcal{J}e^{A\tau} = \mathcal{J}\left(I + \sum_{k=1}^{\infty} \frac{A^k \tau^k}{k!}\right) = \mathcal{J}, \; \text{and } e^{A\tau}\mathcal{J} = \mathcal{J}. \tag{10}$$

Notice that

$$\begin{aligned}&\frac{d}{d\tau}\left[(e^{A\tau} - \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J}^\top)\right] \\ =&Ae^{A\tau}BB^\top(e^{A^\top\tau} - \mathcal{J}^\top) + (e^{A\tau} - \mathcal{J})BB^\top e^{A^\top\tau}A^\top.\end{aligned} \tag{11}$$

Taking the integral of each term leads to

$$\begin{aligned}&\int_0^\infty Ae^{A\tau}BB^\top(e^{A^\top\tau} - \mathcal{J})d\tau \\ =&\int_0^\infty A(e^{A\tau} - \mathcal{J} + \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J})d\tau \\ =&A\mathcal{P} + A\mathcal{J}BB^\top\int_0^\infty(e^{A^\top\tau} - \mathcal{J})d\tau = A\mathcal{P},\end{aligned} \tag{12}$$

and similarly, $\int_0^\infty(e^{A\tau} - \mathcal{J})BB^\top e^{A^\top\tau}A^\top d\tau = \mathcal{P}A^\top$ is obtained. As a result, we have

$$\begin{aligned}&A\mathcal{P} + \mathcal{P}A \\ =&\int_0^\infty \frac{d}{d\tau}\left[(e^{A\tau} - \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J}^\top)\right]d\tau \\ =&\left.(e^{A\tau} - \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J}^\top)\right|_0^\infty \\ =&(I - \mathcal{J})BB^\top(I - \mathcal{J}^\top).\end{aligned} \tag{13}$$

The second equation in (4)(b) can be seen from the fact that $\mathcal{J}\left(e^{A\tau} - \mathcal{J}\right) = \mathcal{J} - \mathcal{J} = 0$.

Next, we prove the uniqueness of the solution of (4) by contradiction. Assume that two symmetric matrices $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy (4) and $\mathcal{P}_1 \neq \mathcal{P}_2$. From (4)(a), we have

$$A(\mathcal{P}_1 - \mathcal{P}_2) + (\mathcal{P}_1 - \mathcal{P}_2)A^\top = 0, \tag{14}$$

which leads to

$$\begin{aligned}&e^{A\tau}\left[A(\mathcal{P}_1 - \mathcal{P}_2) + (\mathcal{P}_1 - \mathcal{P}_2)A^\top\right]e^{A^\top\tau} \\ =&\frac{d}{d\tau}\left[e^{A\tau}(\mathcal{P}_1 - \mathcal{P}_2)e^{A^\top t}\right] = 0.\end{aligned} \tag{15}$$

Therefore, $\int_0^\infty \frac{d}{d\tau}\left[e^{A\tau}(\mathcal{P}_1 - \mathcal{P}_2)e^{A^\top\tau}\right]d\tau = 0$, which implies that

$$\mathcal{P}_1 - \mathcal{P}_2 = \mathcal{J}(\mathcal{P}_1 - \mathcal{P}_2)\mathcal{J}^\top. \tag{16}$$

As both $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy (4)(b), Eq. (16) becomes zero, which, however, contradicts the assumption that $\mathcal{P}_1 \neq \mathcal{P}_2$. Therefore, the common solution of (4)(a) and (b) is unique. The proof of the pseudo observability Gramian in (5) is similar to the controllability Gramian part, and thus the details are omitted here. □

Remark 5. It is implied by (4)(b) and (5)(b) that the pseudo Gramians $\mathcal{P}$ and $\mathcal{Q}$ are positive semidefinite. Particularly, when A is Hurwitz, i.e., $\boldsymbol{\Sigma}$ is asymptotically stable, it follows that $\mathcal{J} = 0$, implying that $\mathcal{P}$ and $\mathcal{Q}$ in (3) become the standard Gramians. Thus, the pseudo Gramians are generalizations of the standard ones.

Due to the singularity of the A matrix, there may exist multiple solutions of the Lyapunov equations in (4)(a) and (5)(a). For instance, suppose a symmetric matrix $\mathcal{P}$ is a solution of (4)(a), then any matrix $\mathcal{P} + \Delta_\mathcal{P}$, with $\Delta_\mathcal{P} = \Delta_\mathcal{P}^\top$ and $A\Delta_\mathcal{P} = 0$, is also a solution of (4)(a). However, combining the Lyapunov equations in (4)(a) and (5)(a) with the constraints in (4)(b) and (5)(b), we can determine the pseudo Gramians $\mathcal{P}$ and $\mathcal{Q}$ uniquely.

Corollary 1. *Let $\mathcal{P}_a$ and $\mathcal{Q}_a$ be arbitrary solutions of the Lyapunov equations in (4)(a) and (5)(a), respectively. Then, the pseudo controllability and observability Gramians, $\mathcal{P}$ and $\mathcal{Q}$ are computed as*

$$\mathcal{P} = \mathcal{P}_a - \mathcal{J}\mathcal{P}_a\mathcal{J}^\top. \tag{17a}$$

$$\mathcal{Q} = \mathcal{Q}_a - \mathcal{J}^\top\mathcal{Q}_a\mathcal{J}. \tag{17b}$$

with $\mathcal{J}$ a constant matrix defined in (3).

Proof. Since both $\mathcal{P}_a$ and $\mathcal{P}$ are solutions of (4)(a), it follows from (16) that

$$\mathcal{P}_a - \mathcal{P} = \mathcal{J}(\mathcal{P}_a - \mathcal{P})\mathcal{J}^\top = \mathcal{J}\mathcal{P}_a\mathcal{J}^\top, \tag{18}$$

where the second equality holds due to (4)(b). Thus, (17a) is verified, and (17b) can be proven analogously. □

Hereafter, we discuss the relation between the controllability and the observability of the semistable system $\boldsymbol{\Sigma}$ and the pseudo Gramians.

Theorem 6. *Consider a semistable system $\boldsymbol{\Sigma}$ with pseudo controllability and observability Gramians $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let m be the algebraic (or geometric) multiplicity of the zero eigenvalues of A. Then,*

(1) $\boldsymbol{\Sigma}$ is controllable if and only if $\mathtt{rank}(\mathcal{P}) = n - m$ and $\xi^\top B \neq 0$, for any nonzero vector $\xi \in \mathcal{N}(A^\top)$;
(2) $\boldsymbol{\Sigma}$ is observable if and only if $\mathtt{rank}(\mathcal{Q}) = n - m$ and $C\xi \neq 0$, for any nonzero vector $\xi \in \mathcal{N}(A)$.

Proof. We prove the first statement. Consider the *finite-time controllability Gramian* of $\boldsymbol{\Sigma}$:

$$P_s(0, t_f) = \int_0^{t_f} e^{A\tau}BB^\top e^{A^\top\tau}d\tau, \tag{19}$$

which is bounded and positive semidefinite as t_f is finite, and from Antoulas (2005), we have $\boldsymbol{\Sigma}$ is controllable on $[0, t_f]$ if and only if $P_s(0, t_f)$ in (19) is full rank. Analogously, *finite-time pseudo controllability Gramian* of $\boldsymbol{\Sigma}$ is defined as

$$\mathcal{P}(0, t_f) = \int_0^{t_f}(e^{A\tau} - \mathcal{J})BB^\top(e^{A^\top\tau} - \mathcal{J}^\top)d\tau \succcurlyeq 0. \tag{20}$$

We then prove the necessary and sufficient condition of the controllability of $\boldsymbol{\Sigma}$ using the rank of $\mathcal{P}(0, t_f)$.

Necessity: Let $\boldsymbol{\Sigma}$ be controllable. Thus, $P_s(0, t_f) \succ 0$, i.e., for all nonzero vector ξ,

$$\xi^\top P_s(0, t_f)\xi = \int_0^{t_f}\xi^\top e^{A\tau}BB^\top e^{A^\top\tau}\xi d\tau > 0. \tag{21}$$

Equivalently, there is no vector $\xi \neq 0$ such that $\xi^\top e^{A\tau}B = 0$, $\forall\, \tau \in [0, t_f]$.

To determine the rank of $\mathcal{P}(0, t_f)$, we first show that $\mathtt{dim}(\mathcal{N}(\mathcal{P}(0, t_f))) = m$. Consider the decomposition of A in (6), where $\mathcal{R}(U) \cup \mathcal{R}(\bar{U}) = \mathcal{R}(U) \cup \mathcal{R}(V)^\perp = \mathbb{R}^n$, such that an arbitrary nonzero vector $\xi \in \mathbb{R}^n$ can be written as

$$\xi = \alpha\xi_1 + \beta\xi_2, \tag{22}$$

where α, β are scalars, and $\xi_1 \in \mathcal{R}(V)$, $\xi_2 \in \mathcal{R}(U)^\perp$, which satisfy

$$\xi_1^\top(e^{A\tau} - \mathcal{J})B = 0, \; \text{and } \xi_2^\top\mathcal{J} = \xi_2^\top UV^\top = 0. \tag{23}$$

The first equation in (23) holds due to

$$\begin{aligned} & V^\top\left(e^{A\tau}-\mathcal{J}\right)B \\ =& V^\top\left(I+\sum_{k=1}^{\infty}\frac{A^k\tau^k}{k!}-\mathcal{J}\right)B \\ =&(V^\top - V^\top U V^\top)B = 0, \end{aligned} \tag{24}$$

where the equations $V^\top A = 0$ and $V^\top U = I_m$ are used.

Any nonzero vector $\tilde{\xi} \in \mathcal{N}(\mathcal{P}(0,t_f))$ is characterized by

$$\tilde{\xi}^\top(e^{A\tau}-\mathcal{J})B = 0,\ \forall\ \tau \in \left[0, t_f\right]. \tag{25}$$

With the decomposition of the vector $\tilde{\xi}$ as in (22), we rewrite (25) as

$$\begin{aligned} & \tilde{\xi}^\top(e^{A\tau}-\mathcal{J})B \\ =& \alpha\tilde{\xi}_1^\top(e^{A\tau}-\mathcal{J})B + \beta\tilde{\xi}_2^\top(e^{A\tau}-\mathcal{J})B \\ =& \beta\tilde{\xi}_2^\top e^{A\tau}B. \end{aligned} \tag{26}$$

Therefore, $\tilde{\xi} \in \mathcal{N}(\mathcal{P}(0,t_f))$ if and only if $\beta = 0$ and $\alpha \neq 0$ in (26), namely, $\mathcal{N}(\mathcal{P}(0,t_f)) = \mathcal{R}(V)$, which yields

$$\mathtt{rank}(\mathcal{P}(0,t_f)) = n - \mathtt{dim}(\mathcal{R}(V)) = n - m. \tag{27}$$

Furthermore, when $\mathbf{\Sigma}$ is controllable, we also obtain $\xi^\top B \neq 0$, for all nonzero vector $\xi \in \mathcal{N}(A^\top)$. Otherwise, there will exist a nonzero vector $\xi \in \mathcal{R}(V)$ such that $\xi^\top \mathcal{J} = 0$, which implies that $\xi^\top e^{A\tau}B = \xi^\top(e^{A\tau}-\mathcal{J})B = 0$. This contradicts (21).

Sufficiency: Note that any nonzero vector $\xi \in \mathbb{R}^n$ can be decomposed as a linear combination of $\xi_1 \in \mathcal{R}(V)$ and $\xi_2 \in \mathcal{R}(U)^\perp$ as in (22). Since $\mathcal{R}(V)$ is in the nullspace of $\mathcal{P}(0,t_f)$, and $\mathtt{dim}(\mathcal{R}(V)) = m$, the rank of $\mathcal{P}(0,t_f)$ then implies that $\xi_2^\top(e^{A\tau}-\mathcal{J})B \neq 0,\ \forall\ \xi_2 \in \mathcal{R}(U)^\perp$. It follows from (23) that $\xi_2^\top e^{A\tau}B \neq \xi_2^\top\mathcal{J}B = 0$. Moreover,

$$\xi_1^\top e^{A\tau}B = \xi_1^\top(e^{A\tau}-\mathcal{J}+\mathcal{J})B = \xi_1^\top\mathcal{J}B. \tag{28}$$

Observe that $\mathcal{J}B \neq 0$ is sufficient for $V^\top B \neq 0$. Thus, (28) is nonzero for all $\xi_1 \in \mathcal{R}(V)$ since $V^\top\mathcal{J}B = V^\top UV^\top B = V^\top B \neq 0$. Consequently, we obtain $\xi^\top e^{A\tau}B \neq 0$, for any nonzero vector ξ, i.e., $P_s(0,t_f)$ is positive definite. It means that $\mathbf{\Sigma}$ is controllable.

Finally, the first statement in the theorem is obtained as $t_f \to \infty$. The proof of the observability part follows a dual statement, Hence, the details are omitted here. □

Moreover, the proposed pseudo Gramians are also relevant to the minimum input and output energy of a semistable system $\mathbf{\Sigma}$.

Theorem 7. *Consider the semistable system $\mathbf{\Sigma}$ and its pseudo controllability and observability Gramians $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

(1) If (A, B) is controllable, then the least input energy required to steer the system state from 0 to $x_0 \in \mathcal{N}(A)^\perp$ in infinite time is given by

$$L_c(x_0) = \min\left\{\int_{-\infty}^{0}\|u(\tau)\|^2\mathrm{d}\tau\right\} = x_0^\top\mathcal{P}^\dagger x_0, \tag{29}$$

where $\mathcal{P}^\dagger$ is the pseudoinverse of $\mathcal{P}$, and $u(\tau) \in \mathcal{L}_2$, $x(-\infty) = 0$, $x(0) = x_0 \in \mathcal{N}(A)^\perp$.

(2) If (C, A) is observable, then the energy of the outputs produced by a given initial state $x_0 \in \mathcal{N}(A)^\perp$ and zero input is

$$L_o(x_0) = \int_0^\infty \|y(\tau)\|^2\mathrm{d}\tau = x_0^\top\mathcal{Q}x_0, \tag{30}$$

with $x(0) = x_0 \in \mathcal{N}(A)^\perp$, $u(\tau) = 0$, $\forall\ \tau \geq 0$.

Proof. First, the controllability energy function $L_c(x_0)$ in (29) is proven. Consider a coordinate transformation $z(t) := \mathcal{U}^{-1}x(t)$, with $\mathcal{U}^{-1}$ in (6). Then, we obtain

$$\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & \bar{A}\end{bmatrix}\begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix} + \begin{bmatrix}V^\top B\\ \bar{V}^\top B\end{bmatrix}u(t), \tag{31}$$

with $z_1(t) = Vx(t) \in \mathbb{R}^m$, and $z_2(t) = \bar{V}x(t) \in \mathbb{R}^{n-m}$. Note that the subsystem with the state $z_2(t)$ is asymptotically stable due to the Hurwitz matrix $\bar{A}$, and its controllability Gramian is given as

$$\bar{\mathcal{P}} := \int_0^\infty e^{\bar{A}\tau}\bar{V}^\top BB^\top\bar{V}e^{\bar{A}^\top\tau}\mathrm{d}\tau = \bar{V}^\top\mathcal{P}\bar{V}, \tag{32}$$

where the latter equation is obtained by multiplying $\bar{V}^\top$ and $\bar{V}$ to the left and right sides of (3a), respectively.

For any $x_0 \in \mathcal{N}(A)^\perp$, we have $z_1(0) = 0$ and $z_2(0) = \bar{V}^\top x_0$. Thus, the input energy $L_c(x_0)$ that required to steer $x(t)$ from 0 to $x_0 \in \mathcal{N}(A)^\perp$ is equivalent to the energy needed to steer $z_2(t)$ the state from $z_2(-\infty) = 0$ to $z_2(0)$. Therefore, it follows from Antoulas (2005) that

$$L_c(x_0) = z_2(0)^\top\bar{\mathcal{P}}^{-1}z_2(0) = x_0^\top\bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top x_0. \tag{33}$$

We then show that $\mathcal{P}^\dagger := \bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top$ is the pseudoinverse of $\mathcal{P}$. Consider the similarity transformation in (6), where $UV^\top + \bar{U}\bar{V}^\top = \mathcal{U}\mathcal{U}^{-1} = I$, and $V^\top\mathcal{P} = 0$. The following Moore–Penrose conditions are verified.

$$\begin{aligned} \mathcal{P}\mathcal{P}^\dagger\mathcal{P} &= (UV^\top + \bar{U}\bar{V}^\top)\mathcal{P}\bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top\mathcal{P} \\ &= \bar{U}\bar{V}^\top\mathcal{P} = (I - UV^\top)\mathcal{P} = \mathcal{P}, \\ \mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger &= \bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top\mathcal{P}\bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top = \mathcal{P}^\dagger \\ (\mathcal{P}^\dagger\mathcal{P})^\top &= \mathcal{P}\bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top = \mathcal{P}\mathcal{P}^\dagger \\ (\mathcal{P}\mathcal{P}^\dagger)^\top &= \bar{V}(\bar{V}^\top\mathcal{P}\bar{V})^{-1}\bar{V}^\top\mathcal{P} = \mathcal{P}^\dagger\mathcal{P} \end{aligned}$$

Thus, $\mathcal{P}^\dagger$ the Moore–Penrose inverse of $\mathcal{P}$, which leads to (29) from (33).

Next, we derive the observability energy function $L_o(x_0)$ as follows. With $y(\tau) = Ce^{A\tau}x_0$, we obtain

$$L_o(x_0) = \int_0^\infty x_0^\top e^{A^\top\tau}C^\top Ce^{A\tau}x_0\mathrm{d}\tau, \tag{34}$$

which is equal to $x_0^\top\mathcal{Q}x_0$, since $\mathcal{J}x_0 = UV^\top x_0 = 0$, for all $x_0 \in \mathcal{N}(A)^\perp = \mathcal{R}(U)^\perp$. □

Next, we provide a sufficient and necessary condition for the semistable $\mathbf{\Sigma}$ being in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ space.

Theorem 8. *Consider the semistable system $\mathbf{\Sigma}$ in (2), and denote $\eta(s) = C\,(sI_n - A)^{-1}B$. We have $\mathbf{\Sigma} \in \mathcal{H}_2$ and $\mathbf{\Sigma} \in \mathcal{H}_\infty$ if and only if*

$$C\mathcal{J}B = 0 \tag{35}$$

Furthermore, let (35) hold, then

$$\|\eta(s)\|_{\mathcal{H}_2}^2 = \mathtt{tr}(C\mathcal{P}C^\top) = \mathtt{tr}(B^\top\mathcal{Q}B), \tag{36}$$

and $\|\eta(s)\|_{\mathcal{H}_\infty} \leq \gamma$ if there exist $\gamma > 0$ and $\mathcal{K} \succcurlyeq 0$ satisfying

$$\begin{bmatrix} A^\top\mathcal{K} + \mathcal{K}A & \star & \star \\ B^\top\mathcal{K} & -\gamma I & \star \\ C(I-\mathcal{J}) & 0 & -\gamma I \end{bmatrix} \preccurlyeq 0, \tag{37}$$

$$\mathcal{J}^\top\mathcal{K}\mathcal{J} = 0. \tag{38}$$

Proof. Consider a coordinate transformation $z(t) := \mathcal{U}^{-1}x(t)$, where $\mathcal{U}^{-1}$ is defined in (6). We then obtain (31) and $y(t) = \begin{bmatrix}CU & C\bar{U}\end{bmatrix}z(t)$. Thereby, the transfer function of $\mathbf{\Sigma}$ is written as

$$\eta(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \frac{1}{s}C\mathcal{J}B, \tag{39}$$

with $\bar{C} = C\bar{U}$ and $\bar{B} = \bar{V}^\top B$. Note that $C\bar{U}(sI - \bar{A})^{-1}\bar{V}^\top B$ is asymptotically stable. Thus, $\eta(s) \in \mathcal{H}_2$ (or $\mathcal{H}_\infty$) if and only if (35) holds.

Let $g(\tau) := Ce^{A\tau}B$ be the impulse response of Σ. It follows from Antoulas (2005) that

$$\|\eta(s)\|_{\mathcal{H}_2}^2 = \mathtt{tr}\left(\int_0^\infty g(\tau)^\top g(\tau)\mathrm{d}\tau\right), \tag{40}$$

which is well-defined if and only if $g(\tau)$ is absolutely integrable, namely, in this case,

$$\lim_{\tau\to\infty} g(\tau) = C\left(\lim_{\tau\to\infty} e^{A\tau}\right)B = C\mathcal{J}B = 0, \tag{41}$$

which then immediately yields

$$\mathtt{tr}(C\mathcal{P}C^\top) = \mathtt{tr}(B^\top \mathcal{Q}B) = \mathtt{tr}\left(\int_0^\infty g(\tau)^\top g(\tau)\mathrm{d}\tau\right).$$

Next, we derive the $\mathcal{H}_\infty$ norm of Σ. If (35) is satisfied, $\|\eta(s)\|_{\mathcal{H}_\infty} = \|\bar{C}(sI - \bar{A})^{-1}\bar{B}\|_{\mathcal{H}_\infty}$. By the well-known bounded real lemma (Scherer & Weiland, 2005), $\|\eta(s)\|_{\mathcal{H}_\infty} \leq \gamma$, if there exists $K \succ 0$ satisfies

$$\bar{A}^\top K + K\bar{A} + \bar{C}^\top\bar{C} + \gamma^2 K\bar{B}\bar{B}^\top K \preccurlyeq 0. \tag{42}$$

Note that due to (38),

$$\mathcal{U}^\top \mathcal{K}\mathcal{U} = \begin{bmatrix} U^\top\mathcal{K}U & U^\top\mathcal{K}\bar{U} \\ \bar{U}^\top\mathcal{K}U & \bar{U}^\top\mathcal{K}\bar{U}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{K}\end{bmatrix}, \tag{43}$$

with $\bar{K} = \bar{U}^\top\mathcal{K}\bar{U}$. Moreover, we have

$$\mathcal{U}^{-1}(I - \mathcal{J})\mathcal{U} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-m}\end{bmatrix}. \tag{44}$$

Thus, the following equations hold.

$$\mathcal{U}^\top A^\top \mathcal{K}\mathcal{U} = (\mathcal{U}^{-1}A\mathcal{U})^\top \mathcal{U}^\top\mathcal{K}\mathcal{U} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}^\top\bar{K}\end{bmatrix}, \tag{45}$$

$$\mathcal{U}^\top\mathcal{K}B = \mathcal{U}^\top\mathcal{K}\mathcal{U}\mathcal{U}^{-1}B = \begin{bmatrix} 0 \\ \bar{K}\bar{B}\end{bmatrix}, \tag{46}$$

$$C(I - \mathcal{J})\mathcal{U} = C\mathcal{U}^{-\top}\mathcal{U}^\top(I - \mathcal{J})\mathcal{U} = \begin{bmatrix} 0 & \bar{C}\end{bmatrix}. \tag{47}$$

It leads to (42) if we multiply (37) by $\mathtt{blkdiag}(\mathcal{U}^\top, I, I)$ and its transpose from the left and right simultaneously. Thus, $\|\eta(s)\|_{\mathcal{H}_\infty} \leq \gamma$. □

A specific case is discussed.

Corollary 2. *Consider the semistable transfer function $g(s) = C(sI - A)^{-1}B + D$. If A is dissipative, i.e., $A + A^\top \preccurlyeq 0$, and $(I - \bar{V}\bar{V}^\top)B = 0$ with $\bar{V}$ defined in (6), then $\|g(s)\|_{\mathcal{H}_\infty} \leq \kappa$ with κ satisfying*

$$\begin{bmatrix} A^\top + A & \star & \star \\ B^\top & -\kappa I & \star \\ C(I - \mathcal{J}) & D & -\kappa I\end{bmatrix} \preccurlyeq 0, \tag{48}$$

Proof. To characterize the $\mathcal{H}_\infty$ norm of $g(s)$ with the feedthrough term D, (37) is modified, where the second entry on the bottom is replaced by D. Now, let $\mathcal{K} := \bar{V}\bar{V}^\top$, which satisfies (38) due to $\bar{V}^\top U = 0$. Furthermore, we have

$$A\mathcal{K} = \bar{U}\bar{A}\bar{V}^\top\bar{V}\bar{V}^\top = A, \ \mathcal{K}B = \bar{V}\bar{V}^\top B = B, \tag{49}$$

which then leads to (48). □

Note that if A is Hurwitz in (2), i.e., $\mathcal{J} = 0$, the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm are well-defined and can be characterized by the standard Gramians and a Riccati inequality, respectively (Antoulas, 2005). However, when A contains semistable eigenvalues, both characterizations are not feasible any more. In contrast, Theorem 8 can be used.

Remark 9. The definitions and analysis in this section can be also extended to the discrete-time semistable systems, which allow for multiple semisimple poles on the unit circle. Instead of (4) and (37), discrete-time algebraic Lyapunov equation and Riccati inequality are used. Due to the limited space, we leave this discussion open for the future work.

3. Model reduction of semistable systems

On the basis of the proposed pseudo Gramians, we provide two approaches in this section, namely balanced truncation and state aggregation, to reduce the dimension of a given semistable system. Both approaches aim to preserve the semistability. More precisely, we aim to find a reduced-order model

$$\hat{\Sigma}: \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \end{cases} \tag{50}$$

with smaller matrices $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times p}$, and $\hat{C} \in \mathbb{R}^{q\times r}$, which is still semistable and the reduction error indicated by $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_\infty}$ or $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is small, where

$$\eta(s) = C\,(sI_n - A)^{-1}\,B, \hat{\eta}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}, \tag{51}$$

are the transfer matrices of Σ and $\hat{\Sigma}$, respectively.

3.1. Balanced truncation

Since the standard Gramians in (1) are not well-defined in the semistable case, we cannot apply the Lyapunov balanced truncation approach directly for the model reduction of semistable systems. A feasible approach in Cheng, Scherpen and Besselink (2019) suggests decomposing a given semistable system into two subsystems, which correspond to the zero poles and asymptotically stable poles, respectively. Then, the standard balanced truncation method can be applied to reduce the stable subsystem only, while the other subsystem is retained. However, in this paper, we would like to avoid the extra decomposition of the original system, as it cost additional computation effort. Thus, the balanced truncation method is extended to the semistable case, where the pseudo Gramians proposed in Section 2 are directly used.

Consider the semistable system Σ in (2), which is assumed to be controllable and observable. Thus, the rank conditions of the pseudo controllability and observability Gramians, $\mathcal{P}$ and $\mathcal{Q}$, in Theorem 6 are satisfied. Similar to the asymptotically stable case (Antoulas, 2005), balancing the semistable system in (2) amounts to find a nonsingular matrix that diagonalizes the two Gramians $\mathcal{P}$ and $\mathcal{Q}$ in a covariant and contravariant manner, respectively.

Lemma 10. *Let Σ in (2) be a minimal system, i.e., Σ is controllable and observable. Then, there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$T\mathcal{P}T^\top = T^{-\top}\mathcal{Q}T^{-1} = \mathcal{S} = \begin{bmatrix} 0_{m\times m} & 0 \\ 0 & S\end{bmatrix}, \tag{52}$$

where $S := \mathtt{diag}\,(s_1, s_2, \ldots, s_n)$ with $s_1 \geq s_2 \geq \cdots \geq s_{n-m} > 0$.

Proof. The computation of T basically follows a similar procedure as in the asymptotically stable case (Antoulas, 2005) but with important modifications in constructing T due to the semistability of Σ.

Consider the singular value decomposition (SVD) of the pseudo controllability Gramian:

$$\mathcal{P} = \mathcal{U}_c\mathcal{S}_c\mathcal{U}_c^\top = \begin{bmatrix} U_c & \bar{U}_c\end{bmatrix}\begin{bmatrix} 0_{m\times m} & 0 \\ 0 & S_c\end{bmatrix}\begin{bmatrix} U_c^\top \\ \bar{U}_c^\top\end{bmatrix}, \tag{53}$$

where the partition of $\mathcal{U}_c$ holds due to the controllability of $\boldsymbol{\Sigma}$ and Theorem 6. Let $\mathcal{Q} = \mathcal{U}_o \mathcal{S}_o \mathcal{U}_o^\top$ be the SVD of pseudo observability Gramian, and denote

$$\mathcal{P}_h = \mathcal{S}_c^{\frac{1}{2}} \mathcal{U}_c^\top, \text{ and } \mathcal{Q}_h = \mathcal{S}_o^{\frac{1}{2}} \mathcal{U}_o^\top. \tag{54}$$

Then, the SVD of $\mathcal{P}_h \mathcal{Q}_h$ can be written as

$$\mathcal{P}_h \mathcal{Q}_h = \mathcal{U}_b \mathcal{S}_b \mathcal{V}_b^\top = \begin{bmatrix} U_b & \bar{U}_b \end{bmatrix} \begin{bmatrix} 0_{m\times m} & 0 \\ 0 & S_b \end{bmatrix} \begin{bmatrix} V_b^\top \\ \bar{V}_b^\top \end{bmatrix}. \tag{55}$$

Note that $\mathcal{R}(U_c) = \mathcal{N}(A^\top)$. There exists a matrix $V_o \in \mathbb{R}^{n\times m}$ such that $\mathcal{R}(V_c) = \mathcal{N}(A)$ and $U_c^\top V_o = I_m$. Then, we construct the nonsingular matrix as

$$T := \begin{bmatrix} U_c^\top \\ S_b^{-\frac{1}{2}} \bar{V}_b^\top \mathcal{Q}_h \end{bmatrix}, \text{ with } T^{-1} := \begin{bmatrix} V_o & \mathcal{P}_h \bar{U}_b S_b^{-\frac{1}{2}} \end{bmatrix}. \tag{56}$$

Using the equations $\mathcal{P} U_c = 0$ and $\mathcal{Q} V_o = 0$, it is not hard to verify that T satisfies (52) with $\mathcal{S} = \mathcal{S}_b$. □

Using T as a coordinate transformation, we obtain the Lyapunov balanced realization of $\boldsymbol{\Sigma}$, in which the states corresponding to the zero singular values in (52) are related to the zero poles of $\boldsymbol{\Sigma}$ and should not be truncated in order to preserve the semistability. On the other hand, the states corresponding to S are in $\mathcal{N}(A)^\perp$, and from Theorem 7, the one related to a smaller singular value in Σ is relatively difficult to control and observe. Thus, a reduced-order model $\hat{\boldsymbol{\Sigma}}$ with dimension r ($m \le r \le n$) is acquired by truncating the states with smallest singular values in S. The following *a priori* error bound holds.

Proposition 1. *Consider the semistable system $\boldsymbol{\Sigma}$ in (2) and its reduced-order model $\hat{\boldsymbol{\Sigma}}$ obtained by the Lyapunov balanced truncation. Then, the upper bound of the model reduction error can be measured by*

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_\infty} \le 2 \sum_{i=r+1}^{n} s_i, \tag{57}$$

where s_i are neglected nonzero singular values in (52).

Proof. Consider the balancing transformation matrix T in (56). Let (A_b, B_b, C_b) be the system matrices of the balanced system, which are given as

$$A_b = TAT^{-1} = \begin{bmatrix} 0_{m\times m} & 0 \\ 0 & \bar{A}_b \end{bmatrix},$$

$$B_b = TB = \begin{bmatrix} U_c^\top B \\ \bar{B}_b \end{bmatrix}, C_b = CT^{-1} = \begin{bmatrix} CV_o & \bar{C}_b \end{bmatrix},$$

with $\bar{A}_b := S^{-\frac{1}{2}} \bar{V}_b^\top \mathcal{Q}_h A \mathcal{P}_h \bar{U}_b S^{-\frac{1}{2}}$, $\bar{B}_b := S^{-\frac{1}{2}} \bar{V}_b^\top \mathcal{Q}_h B$, and $\bar{C}_b := C \mathcal{P}_h \bar{U}_b S^{-\frac{1}{2}}$. Clearly, $\bar{A}_b$ is Hurwitz, and it is verified that S is the controllability and observability Gramian of the asymptotically stable system $(\bar{A}_b, \bar{B}_b, \bar{C}_b)$. Thus, the truncation of (A_b, B_b, C_b) is actually performed on the stable part $(\bar{A}_b, \bar{B}_b, \bar{C}_b)$, while the part corresponding to the zero singular values is untouched. Consequently, we have the truncated model $\hat{\boldsymbol{\Sigma}} := (\hat{A}, \hat{B}, \hat{C})$ with

$$\hat{A} = \begin{bmatrix} 0_{m\times m} & 0 \\ 0 & \bar{A}_r \end{bmatrix}, \hat{B} = \begin{bmatrix} U_c^\top B \\ \bar{B}_r \end{bmatrix}, \hat{C} = \begin{bmatrix} CV_o & \bar{C}_r \end{bmatrix},$$

where $(\bar{A}_r, \bar{B}_r, \bar{C}_r)$ is obtained by truncating the model $(\bar{A}_b, \bar{B}_b, \bar{C}_b)$. Furthermore, the error between the semistable systems $\boldsymbol{\Sigma}$ and $\hat{\boldsymbol{\Sigma}}$ can be evaluated by the error on their asymptotically stable components, i.e.,

$$\eta(s) - \hat{\eta}(s) = \bar{C}_b (sI_n - \bar{A}_b)^{-1} \bar{B}_b - \bar{C}_r (sI_r - \bar{A}_r)^{-1} \bar{B}_r, \tag{58}$$

which satisfies $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$, since both $\bar{A}_b$ and $\bar{A}_r$ are Hurwitz. Following Antoulas (2005), the error bound (57) is obtained using the singular values in S, which corresponds to the asymptotically stable part $(\bar{A}_b, \bar{B}_b, \bar{C}_b)$. □

It is worth noting that we can also implement a bounded real balanced truncation using the maximal and minimal solutions of (37), $\mathcal{K}_M$ and $\mathcal{K}_m$, subject to (38). In that case, if the $\mathcal{H}_\infty$ norm of original system $\boldsymbol{\Sigma}$ is less or equal to γ, the reduced-order system $\hat{\boldsymbol{\Sigma}}$ will preserve the $\mathcal{H}_\infty$ bound. Meanwhile the upper bound of the reduction error follows the same expression as in (57), where s_i is replaced by the nonzero singular values of $\mathcal{K}_M \mathcal{K}_m$.

3.2. State aggregation

In the section, we provide the other model reduction technique, called state aggregation for semistable systems. Unlike the balanced truncation approach, the state aggregation is to partition system variables into several nonempty and disjoint subsets and merging all the variables in each subset as a single state. Thus, this method does not mix up state variables so that the aggregated model can retain some key qualitative or structural properties of the original system, such as semistability or network structure (Cheng et al., 2017; Ishizaki et al., 2015).

Define a set $\mathcal{X} := \{1, 2, \ldots, n\}$, with n the dimension of the original system $\boldsymbol{\Sigma}$. To implement the state aggregation, we partition $\mathcal{X}$ into r nonempty and disjoint subsets, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$, such that $\bigcup_{k=1}^r \mathcal{C}_k = \mathcal{X}$ and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset, \forall\, i, j = 1, 2, \ldots, r$. Define $\Pi \in \mathbb{R}^{n\times r}$ as a binary matrix such that $\Pi_{ij} = 1$ if vertex $i \in \mathcal{C}_j$ and $\Pi_{ij} = 0$ otherwise, where Π_{ij} denotes the entry of Π that lies in the ith row and the jth column.

Then, the projection matrix is given as $\Gamma = WW^\dagger$ with

$$W = M\Pi, \; W^\dagger = (\Pi^\top M N \Pi)^{-1} \Pi^\top N \tag{59}$$

where M and N are weighting matrices to be determined. The aggregate model is thereby established via the Petrov–Galerkin framework (Antoulas, 2005) with

$$\hat{A} := W^\dagger A W, \; \hat{B} = W^\dagger F, \text{ and } \hat{C} = CW. \tag{60}$$

The following result indicates how to select M, N and Π such that the reduction error $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is bounded.

Theorem 11. *Consider the semistable system $\boldsymbol{\Sigma}$ in (2) and its reduced-order model $\hat{\boldsymbol{\Sigma}}$ with system matrices in (60). If there exist diagonal and positive definite matrices M and N such that for each pair $i, j \in \mathcal{C}_k$, and $k \in \{1, 2, \ldots, r\}$,*

$$(\mathbf{e}_i - \mathbf{e}_j)^\top M^{-1} \mathcal{J} = 0, \text{ and } \mathcal{J} N^{-1} (\mathbf{e}_i - \mathbf{e}_j) = 0, \tag{61}$$

with $\mathbf{e}_i$ the ith column of I_n, then $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$.

Proof. From Theorem 8, $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$ holds if and only if $C(\mathcal{J} - W\hat{\mathcal{J}}W^\dagger)B = 0$, with $\mathcal{J} := \lim_{\tau\to\infty} e^{A\tau}$ and $\hat{\mathcal{J}} := \lim_{\tau\to\infty} e^{\hat{A}\tau}$, respectively. Thus, it sufficient to show that

$$\mathcal{J} = W \hat{\mathcal{J}} W^\dagger, \tag{62}$$

when (61) holds. To this end, we assume that $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r$ are nonempty and disjoint subsets of the original state set $\mathcal{X}$, and without loss of generality, let

$$\Pi = \texttt{blkdiag}\left(\mathbb{1}_{|\mathcal{C}_1|}, \mathbb{1}_{|\mathcal{C}_2|}, \ldots, \mathbb{1}_{|\mathcal{C}_r|}\right). \tag{63}$$

Consider the matrices U and V in (7). Accordingly, we obtain the following matrix partitions:

$$U = \begin{bmatrix} U_1 \\ \vdots \\ U_r \end{bmatrix}, \text{ and } V = \begin{bmatrix} V_1 \\ \vdots \\ V_r \end{bmatrix}, \tag{64}$$

where $U_k, V_k \in \mathbb{R}^{|\mathcal{C}_k|\times m}$ with $k = 1, 2, \ldots, r$. Meanwhile, the projection matrix $\Gamma = WW^{\dagger}$ is written in a block diagonal form with the kth diagonal entry as

$$\Gamma_k = M_k \mathbb{1}_{|\mathcal{C}_k|}(\mathbb{1}_{|\mathcal{C}_k|}^{\top} M_k N_k \mathbb{1}_{|\mathcal{C}_k|})^{-1}\mathbb{1}_{|\mathcal{C}_k|}^{\top} N_k, \tag{65}$$

where $M_k, N_k \in \mathbb{R}^{|\mathcal{C}_k|\times|\mathcal{C}_k|}$ are the corresponding principal submatrices of M and N, respectively.

It follows from (61), where $\mathcal{J} = UV^{\top}$, that $\mathbf{e}_{ij}^{\top} M^{-1} U = 0$, and $VN^{-1}\mathbf{e}_{ij} = 0$, which imply that $\mathcal{R}(M_k^{-1}U_k) = \mathcal{R}(N_k^{-1}V_k) = \mathcal{R}(\mathbb{1}_{|\mathcal{C}_k|})$, namely, there exist two row vectors $\mu_k, \nu_k \in \mathbb{R}^{1\times m}$ such that $U_k = M_k \mathbb{1}_{|\mathcal{C}_k|}\mu_k$, and $V_k = N_k \mathbb{1}_{|\mathcal{C}_k|}\nu_k$. Therefore, it is not hard to verify that $U_k = \Gamma_k U_k$, and $V_k^{\top} = V_k^{\top}\Gamma_k$, which yield

$$U = WW^{\dagger}U, \text{ and } V^{\top} = V^{\top}WW^{\dagger}. \tag{66}$$

Now we compute $\hat{\mathcal{J}}$ in (62) for the reduced-order system $\hat{\Sigma}$. Let $\hat{U} := W^{\dagger}U$ and $\hat{V}^{\top} := V^{\top}W$, which satisfies

$$\hat{V}^{\top}\hat{U} = V^{\top}WW^{\dagger}U = V^{\top}U = I_m,$$
$$\hat{A}\hat{U} = W^{\dagger}AWW^{\dagger}U = W^{\dagger}AU = 0,$$
$$\hat{V}^{\top}\hat{A} = V^{\top}WW^{\dagger}AW = V^{\top}AW = 0.$$

Thus, we obtain that

$$\hat{\mathcal{J}} := \hat{U}\hat{V}^{\top} = W^{\dagger}UV^{\top}W, \tag{67}$$

leading to $W\hat{\mathcal{J}}W^{\dagger} = WW^{\dagger}UV^{\top}WW^{\dagger} = UV^{\top} = \mathcal{J}$, i.e., (62) holds. As a result, the error $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is bounded. □

The conditions in (61) indicate that the ith and jth rows of $\mathcal{J}$ should span the same one-dimensional row space, and the ith and jth columns of $\mathcal{J}$ also belong to the same one-dimensional column space. Different from Ishizaki et al. (2015), the matrices M and N are selected based on $\mathcal{J}$, rather than the Frobenius vector, having all positive entries. Note that the matrix A in this paper may contain multiple zero eigenvalues and the associated eigenvectors are allowed to have zero or negative entries.

Example 1. Consider the following semistable matrix

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 12 & -18 & 0 \\ 8 & -6 & -2 \end{bmatrix}, \tag{68}$$

which gives

$$\mathcal{J} = \lim_{\tau\to\infty} e^{A\tau} = \begin{bmatrix} 0.9 & 0.15 & 0 \\ 0.6 & 0.1 & 0 \\ 1.8 & 0.3 & 0 \end{bmatrix}. \tag{69}$$

From Theorem 11, we can select $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{3\}$ with $M^{-1} = \mathtt{diag}(2, 3, 1)$ and $N^{-1} = \mathtt{diag}(1, 6, 1)$. △

Proposition 2. *If $\eta(s) - \hat{\eta}(s) \in \mathcal{H}_2$, and*

$$MA^{\top}N + NAM \preccurlyeq 0, \tag{70}$$

then the approximation error is upper bounded by

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2} \leq \kappa_a \cdot \sqrt{\mathtt{tr}\left[(I-\Gamma)\mathcal{P}(I-\Gamma^{\top})\right]}, \tag{71}$$

where Γ and $\mathcal{P}$ are the projection matrix and pseudo controllability Gramian, respectively, and $\kappa_a > 0$ is a scalar satisfying

$$\begin{bmatrix} MA^{\top}N + NAM & \star & \star \\ A^{\top}N & -\kappa_a I & \star \\ C(I-\mathcal{J}) & C & -\kappa_a I \end{bmatrix} \preccurlyeq 0. \tag{72}$$

Proof. Using a similar argument as Theorem 4 in Ishizaki, Kashima, Imura, and Aihara (2014), we can show that the reduction error $\eta_e(s) := \eta(s) - \hat{\eta}(s)$ can be written in the cascade form of

$$\eta_e(s) = \underbrace{\left[C + CW(sI - \hat{A})W^{\dagger}A\right]}_{\eta_a(s)} \underbrace{(I-\Gamma)(sI - A)^{-1}B}_{\eta_b(s)}.$$

Thus, $\|\eta_e(s)\|_{\mathcal{H}_2} \leq \|\eta_a(s)\|_{\mathcal{H}_\infty} \cdot \|\eta_b(s)\|_{\mathcal{H}_2}$.

Since $CW\hat{\mathcal{J}}W^{\dagger}A = C\mathcal{J}A = 0$, we obtain from Theorem 8 that $\eta_a(s) \in \mathcal{H}_\infty$. Moreover, due to $(I-\Gamma)\mathcal{J} = 0$, it holds that $\|\eta_b(s)\|_{\mathcal{H}_2} = \sqrt{\mathtt{tr}\left[(I-\Gamma)\mathcal{P}(I-\Gamma^{\top})\right]}$.

Next, we show that $\|\eta_a(s)\|_{\mathcal{H}_\infty} \leq \kappa_a$, which fulfills (72). Let $D := N^{-\frac{1}{2}}M^{\frac{1}{2}}$, and $\tilde{A} := D^{-1}AD$. It then follows from (70) that $\tilde{A}^{\top} + \tilde{A} \preccurlyeq 0$, and

$$\eta_a(s) = C + C\tilde{W}(sI - \tilde{W}^{\top}\tilde{A}\tilde{W})^{-1}\tilde{W}^{\top}\tilde{A}, \tag{73}$$

where $\tilde{W} := (\Pi^{\top}MN\Pi)^{-\frac{1}{2}}\Pi^{\top}M^{\frac{1}{2}}N^{\frac{1}{2}}$. Using the result in Corollary 2, we obtain $\|\eta_a(s)\|_{\mathcal{H}_\infty} \leq \hat{\kappa}_a$, which satisfies

$$\begin{bmatrix} \tilde{W}^{\top}\tilde{A}^{\top}\tilde{W} + \tilde{W}^{\top}\tilde{A}\tilde{W} & \star & \star \\ \tilde{A}^{\top}\tilde{W}^{\top} & -\hat{\kappa}_a I & \star \\ C\tilde{W}(I-\hat{J}) & C & -\hat{\kappa}_a I \end{bmatrix} \preccurlyeq 0, \tag{74}$$

where $\hat{J} := \lim_{\tau\to\infty} e^{\tilde{W}^{\top}\tilde{A}\tilde{W}\tau}$. Denote $J = \lim_{\tau\to\infty} e^{\tilde{A}\tau} = D^{-1}\mathcal{J}D$. Following the similar reasoning to prove (62), it is not hard to verify that $\tilde{W}(I-\hat{J}) = \tilde{W} - \tilde{W}\tilde{W}^{\top}J\tilde{W} = (I-J)\tilde{W}$. Notice that (74) is obtained by multiplying $\mathtt{blkdiag}(\tilde{W}^{\top}M^{-\frac{1}{2}}N^{-\frac{1}{2}}, I, I)$ and its transpose from the left and right sides of (72), which holds due to the inequality (70). Thus, we obtain $\|\eta_a(s)\|_{\mathcal{H}_\infty} \leq \hat{\kappa}_a \leq \kappa_a$. □

4. Illustrative example

In this section, we consider a model of gene regulation from Ahsendorf et al. (2014) and Mirzaev and Gunawardena (2013) as an example to illustrate the pseudo Gramians and their application in model order reduction. A gene regulation network describing unimolecular reactions among 7 species is presented in Fig. 1, in which the nodes and directed edges represent the chemical species and chemical reactions, respectively. Under mass–action kinetics with the edge weights as rate constants, a linear differential equation is obtained in form of (2), in which each state is the mass of a chemical specie, and the input $u(t)$ represents the synthesis or degradation (see Gunawardena, 2012 for the definitions), and the output $y(t)$ indicates the production of chemical species of interest. The system matrices are given as

$$A = \begin{bmatrix} -a & 0 & b & 0 & 0 & 0 & 0 \\ a & -(c+e+f) & d & 0 & 0 & 0 & 0 \\ 0 & c & -(b+d+g) & 0 & 0 & 0 & 0 \\ 0 & e & 0 & -h & i & 0 & 0 \\ 0 & 0 & 0 & h & -i & 0 & 0 \\ 0 & f & g & 0 & 0 & -j & k \\ 0 & 0 & 0 & 0 & 0 & j & -k \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}, \text{ and}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this example, we specify the rate constants as $a = 0.5$, $b = 0.5$, $c = 0.5$, $d = 0.5$, $e = 0.4$, $f = 0.1$, $g = 0.2$, $h = 0.1$, $i = 0.8$, $j = 0.1$, $k = 0.8$. The matrix A is the *Laplacian matrix* of the graph in Fig. 1. Due to the eigenvalues of A with $m = 2$, the system is semistable. Thus, the classical model reduction methods, e.g., balanced truncation and moment matching, cannot be applied to such a system to handle the semistability and preserve the network structure.

Following Corollary 1, we compute the pseudo Gramians of the system as $\mathcal{P}$ and $\mathcal{Q}$ given in Box I, which satisfy $\mathtt{rank}(\mathcal{P}) = \mathtt{rank}(\mathcal{Q}) = 5$. According to Theorem 6, the system is controllable and observable.

$$\mathcal{P} = \begin{bmatrix} 1.3627 & 0.6823 & 0.3627 & -1.4047 & -0.2048 & -0.6944 & -0.1037 \\ 0.6823 & 0.5020 & 0.3217 & -0.8635 & -0.1324 & -0.4399 & -0.0701 \\ 0.3627 & 0.3217 & 0.5507 & -0.6848 & -0.0991 & -0.3902 & -0.0609 \\ -1.4047 & -0.8635 & -0.6848 & 1.6964 & 0.2504 & 0.8724 & 0.1340 \\ -0.2048 & -0.1324 & -0.0991 & 0.2504 & 0.0374 & 0.1285 & 0.0200 \\ -0.6944 & -0.4399 & -0.3902 & 0.8724 & 0.1285 & 0.4539 & 0.0698 \\ -0.1037 & -0.0701 & -0.0609 & 0.1340 & 0.0200 & 0.0698 & 0.0109 \end{bmatrix},$$

$$\mathcal{Q} = \begin{bmatrix} 0.0276 & 0.0205 & 0.0237 & 0.0088 & -0.0707 & 0.0041 & -0.0328 \\ 0.0205 & 0.0160 & 0.0179 & 0.0078 & -0.0626 & 0.0037 & -0.0298 \\ 0.0237 & 0.0179 & 0.0208 & 0.0073 & -0.0586 & 0.0050 & -0.0403 \\ 0.0088 & 0.0078 & 0.0073 & 0.0069 & -0.0549 & 0.0000 & -0.0000 \\ -0.0707 & -0.0626 & -0.0586 & -0.0549 & 0.4390 & 0.0000 & -0.0000 \\ 0.0041 & 0.0037 & 0.0050 & 0.0000 & -0.0000 & 0.0069 & -0.0549 \\ -0.0328 & -0.0298 & -0.0403 & -0.0000 & -0.0000 & -0.0549 & 0.4390 \end{bmatrix},$$

Box I.

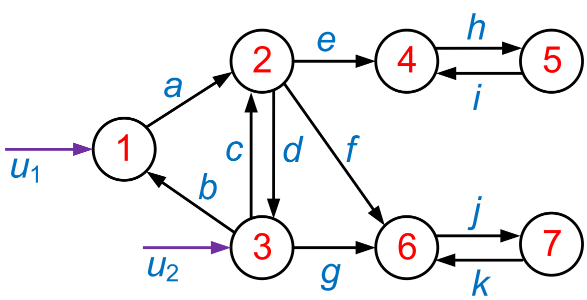

Fig. 1. A gene regulation network with seven species. The edge weights are rate constants for the corresponding reactions.

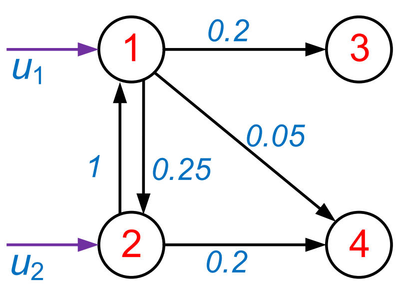

Fig. 2. A reduced reaction network obtained by the state aggregation method.

First, we use the results in Section 3.1 and reduce the semistable system by balanced truncation. The resulting reduced-order model with dimension 4 is obtained as

$$\hat{A}_b = \begin{bmatrix} 0_{2\times 2} & \\ & \begin{matrix} -0.1898 & -0.1524 \\ 0.1374 & -0.6331 \end{matrix} \end{bmatrix}, \hat{B}_b = \begin{bmatrix} -0.3862 & -0.3293 \\ -0.0895 & -0.1857 \\ 0.2770 & 0.2407 \\ -0.1422 & -0.0283 \end{bmatrix},$$

$$\hat{C}_b = \begin{bmatrix} -0.2004 & 0.0134 & -0.3235 & -0.1085 \\ -0.0526 & -0.1632 & -0.1733 & -0.0962 \end{bmatrix},$$

which gives the approximation error $\|\eta(s) - \hat{\eta}_b(s)\|_{\mathcal{H}_2} = 0.0148$.

Next, we apply the state aggregation method in Section 3.2 and partition the states into 4 subsets $\mathcal{C}_1 = \{1, 2\}, \mathcal{C}_2 = \{3\}, \mathcal{C}_3 = \{4, 5\}, \mathcal{C}_4 = \{6, 7\}$. Choose

$$M = \mathtt{diag}(1, 1, 1, 8, 1, 8, 1), \text{ and } N = I_7,$$

which satisfy the conditions in Theorem 11. Thus, the approximation error between the original and aggregated models is bounded and is computed as $\|\eta(s) - \hat{\eta}_a(s)\|_{\mathcal{H}_2} = 0.0596$. The system matrices of the aggregated model are given as

$$\hat{A}_a = \begin{bmatrix} -0.5 & 1 & 0 & 0 \\ 0.25 & -1.2 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.05 & 0.2 & 0 & 0 \end{bmatrix}, \hat{B}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \hat{C}_a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}.$$

This numerical example shows that both balanced truncation and state aggregation methods are effective for the model reduction of the semistable system. The balanced truncation method provides a more accurate approximation than the aggregation method. However the latter one preserves the interconnection structure of the reaction network. Note that $\hat{A}_a$ is still a Laplacian matrix, thus the obtained reduced-order model can be interpreted as a simplified reaction network, see Fig. 2.

5. Conclusions

In this paper, we have extended the notions controllability and observability Gramians from linear asymptotically stable systems to semistable systems. These extended Gramians are characterized by the unique solutions of a set of Lyapunov equations and kernel constraints. It is shown that the degrees of controllability and observability of a semistable system can be related to the ranks of these Gramians, which also can be used to interpret the controllability and observability functions in terms of minimal input and output energy. Furthermore, the $\mathcal{H}_2$ norm and $\mathcal{H}_\infty$ norm of a semistable system are studied, and then the results are applied to the model reduction of semistable systems. For future works, the extension to the discrete-time case is relevant, and the analysis of pseudo Gramians for nonlinear semistable systems remains an open question.

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