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A Queuing Model for Due Date Control in a Multi Server Repair Shop with Subcontracting

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Abstract: This paper deals with a repair shop with multiple parallel servers, which has to carry out planned overhauls, consisting of a lot of maintenance jobs. These overhauls are disturbed by randomly arriving emergency jobs. To control the delivery performance of the overhauls, knowledge about the overhaul makespan distribution should be available. However, past approaches do not have an answer to this problem. Using a 2-dimensional Markov model, we derive theoretical solutions for the first and second moment of the overhaul makespan for the case that the repair times of all overhaul jobs are identically and exponentially distributed. In case of non identical repair time distributions, an approximation for the moments is presented. These moments are used to approximate the makespan distribution by a mixture of two Erlang distributions. We show by simulation that this approximation gives very good results.

1. INTRODUCTION

Maintenance departments of large production facilities are concerned with the objective to provide an operational availability of the technical systems in accordance with the production schemes of the production departments. The activities for achieving this

availability are divided into planned preventive overhauls and unplanned corrective repair jobs.

Three types of resources are available for these activities: manpower, tools and materials. Particularly manpower is an important resource, because it is needed during the entire processing time of a maintenance job. Consequently, the planning and control of these resources and the resource requirements are the key decisions in achieving a satisfactory due date reliability. In general, the complexity of the technical systems to be maintained makes that highly skilled resources are needed and consequently, the volume flexibility of the work force is, especially on the short term, rather limited. At the medium term, subcontracting part of a planned overhaul gives an opportunity to vary the capacity level.

Unlike the resource capacity levels, the demand for the capacity resources is highly stochastic for two reasons. Firstly, the random (and thus unplanned) breakdowns of operational technical systems create emergency jobs for the maintenance department. Since capacity has to be assigned to these jobs immediately, they disturb the progress of outstanding overhauls. Secondly, at the start of processing a job the contents of the maintenance activities - and therefore the required amount of resource capacity - are not precisely known. As a result, the completion date of a planned overhaul becomes stochastic.

The uncertainty about the demands for resources complicates the due date control for planned overhauls. If the delivery performance is uncontrolled, the operational availability of the technical systems is also uncontrolled. In this paper we present a model to calculate the probability of completing an overhaul in time. It can be used dynamically in the sense that during the progress of an overhaul this probability can be evaluated constantly. Based upon

this probability it can be decided whether subcontracting is necessary to ensure a timely completion of the overhaul. Of course, subcontracting is only done at the latest possible time. Furthermore, a quantitative indication for the amount of work load to be subcontracted is given.

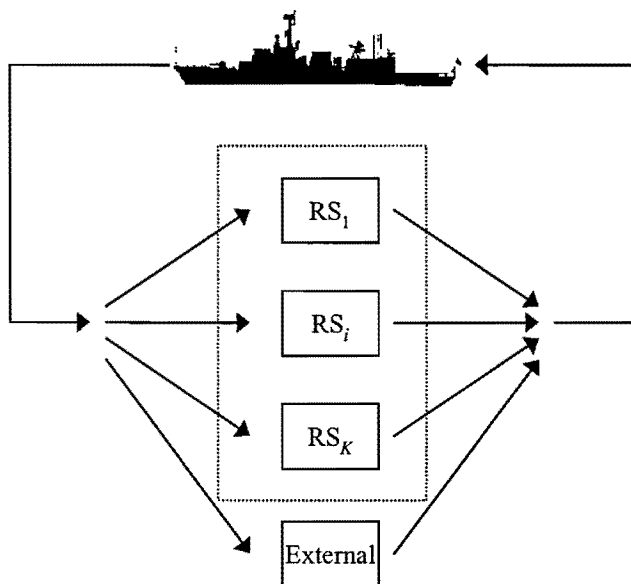


Fig. 1. The flow of the technical systems through the different repair shops (RS) and subcontractors.

2. MOTIVATION

This research is based upon the maintenance department of the Royal Netherlands Navy. The Navy has to provide fighting strength on sea. The maintenance department has to ensure that the operational availability of the technical systems is in accordance with the mission schemes of the Navy. These mission schemes can be compared to production schemes in the civil industry: these indicate the time periods during which the technical systems are required to be in a non-failed condition. The maintenance department consists of several parallel multi-server repair shops. Each repair shop is responsible for the maintenance

of a certain number of technical systems, whereas each technical system can be maintained by only one repair shop. Every repair shop consists of a number of equally skilled repair men. In practice, the number of repair men per repair shop varies between 1 and 13. For some (sub)systems it is also possible to subcontract them to an external (civil or military) organisation (see Fig. 1).

Demands for the resources are classified into preventive overhauls of entire warships and corrective emergency jobs of failed technical systems. Firstly, the planned overhauls consume a large part of the available resource capacity. An overhaul results into packages of jobs for each repair shop. Upon arrival of the ship at the department, the actual condition of the technical systems is not exactly known and consequently the needed resource capacity to process these jobs is not known. All jobs in a package have the same external due date, which is the start date of the next mission of the ship being maintained. In general, the start date of an overhaul is before the end date of the previous overhaul: there is a certain amount of overlap. Fig. 2 shows a typical scheme of planned overhauls with for each ship the earliest possible start date and the due date of its overhaul. The lead time requested by the Navy for an overhaul varies between five and ten months. On arrival an overhaul explodes into many *planned* jobs, which are divided among the different repair shops. The number of planned jobs per repair shop varies from 0 to 150 jobs, based upon the type of ship. Secondly, the unplanned breakdowns of operational systems on sea create *emergency* jobs for the maintenance department. About a quarter of the resource capacity is used to process these emergency jobs. The processing times of the jobs vary between 1 and 1000 hours, whereas 80 percent of the jobs have a processing time which is smaller than 40 hours. Each repair shop has a different mixture of planned and emergency jobs and different processing time characteristics.

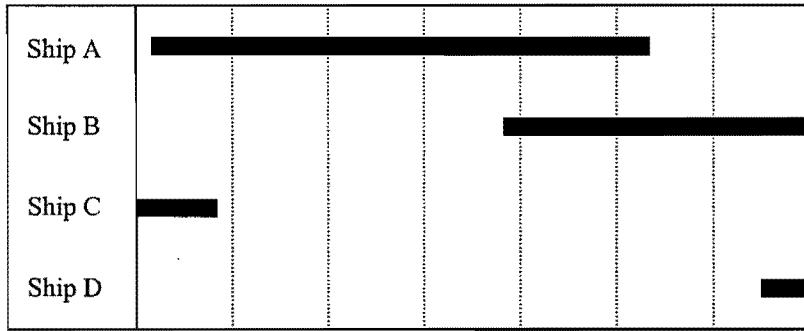


Fig. 2. A typical time scheme with planned overhauls.

The possibility of subcontracting gives the opportunity to temporarily increase the capacity level. Because the subcontracting lead time (i.e. the elapsed time between subcontracting and getting back a certain job) is usually larger than the internal throughput time, timely subcontracting is of extreme importance.

Surprisingly, little is known about the probability distribution of throughput times of bunches of jobs interrupted by higher priority jobs in a shop with multiple parallel servers and random repair times. Gaver [2], Kim and Alden [4] and Keizers and Bertrand [3] present some results for the one-server shop. However, for the multi-server case, no literature is available and therefore this research is about the derivation of the distribution function in a repair shop with multiple parallel servers.

3. PROBLEM FORMULATION

In this section, we formulate the mathematical problem. In this paper we restrict ourselves to the due date control problem at repair shop level. Once this problem has been

solved, we can easily extend it to the case of several parallel repair shops, each receiving packages of jobs from the same overhaul.

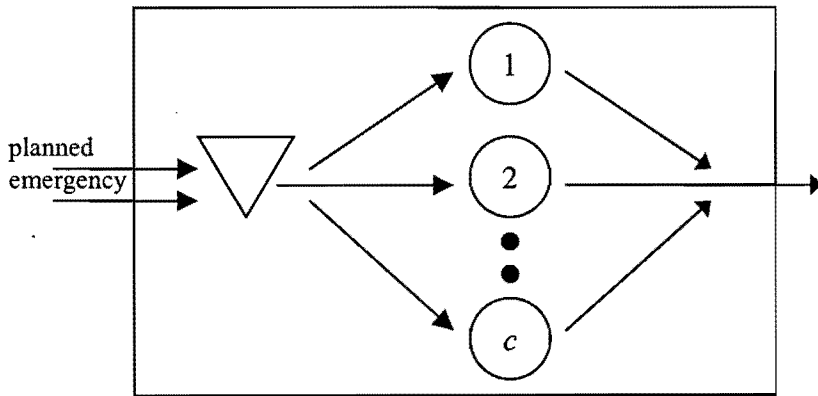


Fig. 3. A c -server repair shop

Consider a repair shop with c parallel servers, as depicted in Fig. 3. At time $t = 0$ an overhaul arrives containing n planned jobs. These jobs have identical due dates, say $t = T_D$. The processing time of each planned job is exponentially distributed with mean $1/\mu_p$. The progress of the overhaul is disturbed by the arrival of emergency jobs. These arrive one at the time according to a Poisson process with intensity λ . Their processing times are exponentially distributed with mean $1/\mu_E$. The interruption process by emergency jobs is assumed to be pre-emptive resume: upon arrival of an emergency job, a planned job in service is put back in the queue and the emergency job is immediately taken into service if the total number of emergency jobs in the system is less than c . Otherwise, the emergency job is put in the queue, waiting for another emergency job to be finished. After a server has finished an emergency job and the queue with emergency jobs is empty, the planned job in the queue is resumed from the point at which interruption took place.

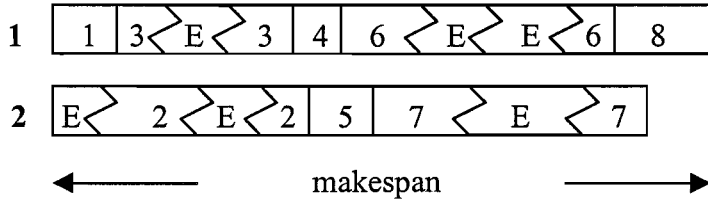


Fig. 4. Realised makespan of an overhaul with 8 planned jobs in a 2-server repair shop with initially 1 emergency job (E).

If C_i is the completion time of planned job i ($i = 1, \dots, n$), the makespan of the entire overhaul is equal to $\max(C_i | i = 1, \dots, n)$ and it is determined by the processing times of the emergency and internally processed planned jobs. Let $T(n, m)$ be the random variable describing the makespan if there are n planned and m emergency jobs in the shop. An example of a realisation of the makespan is shown in Fig. 4. The problem we are facing is to calculate the probability of delivering the entire overhaul in time, that is:

$$P(T(n, m) \leq T_D) \tag{1}$$

Once we are able to calculate this probability, we can use it to control the delivery performance of the repair shop according to the targets set by the organisation.

4. THE DISTRIBUTION OF THE MAKESPAN

The exact determination of the makespan distribution is very complicated. Therefore, in this section we present an approach to approximate the distribution of the makespan. We first

analytically derive the moments of $T(n,m)$. Then we approximate the distribution of the makespan by fitting an appropriate distribution on the first two moments of $T(n,m)$.

Let state (n,m) describe the situation in which there are n planned jobs and m emergency jobs in the system, waiting for or being in service. During processing the overhaul index n decreases to zero and index m fluctuates between zero and infinity. In state (n,m) , one of the three following events may occur. Firstly, an emergency job may arrive and a transition occurs towards state $(n,m+1)$. Secondly if there is at least one emergency job in the system, one of the servers may finish an emergency job and a transition occurs towards state $(n,m-1)$. And finally, one of the servers may finish a planned job and a transition occurs towards state $(n-1,m)$. Because of the pre-emptive resume interruption process by emergency orders, a planned job can only be finished if there are at most $c-1$ emergency jobs in the system. The overhaul is finished as soon as the system moves to one of the states $(0,0)$, $(0,1), \dots, (0,c-1)$. The state transition diagram is depicted in Fig. 5.

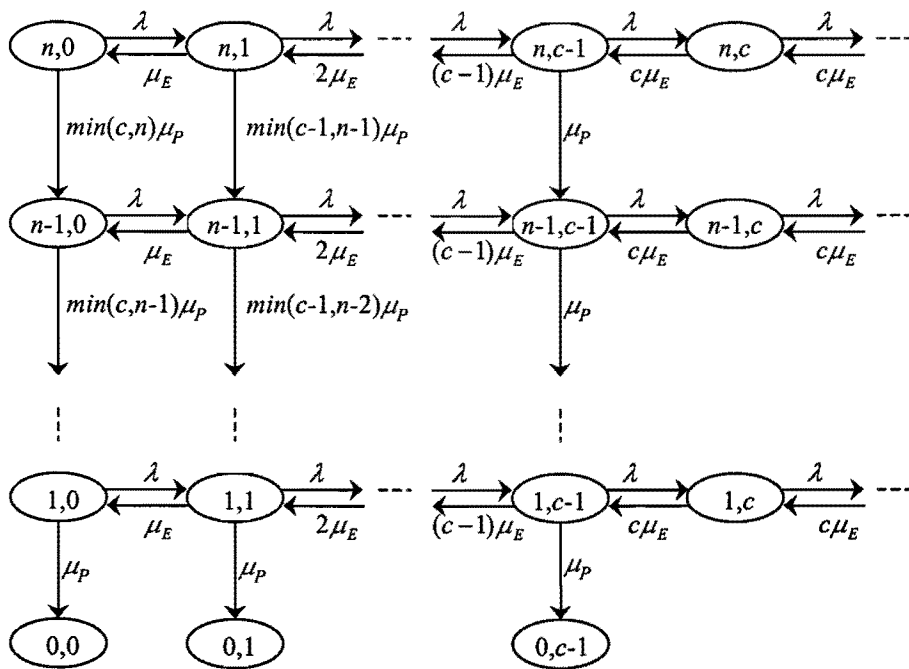


Fig. 5. State-transition rate diagram for the c -server repair shop.

Recall that $T(n, m)$ is the overhaul makespan if there are currently n planned and m emergency jobs in the shop. Knowing the state-transition rates, the mathematical relations between the different $T(n, m)$'s can be formulated. If $Z(\theta)$ denotes an exponentially distributed random variable with expectation $1/\theta$, the following relations hold:

If $m \geq c$ and $n > 0$ then

$$T(n, m) = Z(\lambda + c\mu_E) + \begin{cases} T(n, m-1) & \text{w.p. } \frac{c\mu_E}{\lambda + c\mu_E} \\ T(n, m+1) & \text{w.p. } \frac{\lambda}{\lambda + c\mu_E} \end{cases} \quad (2)$$

If $m < c$ and $n > 0$ then

$$T(n, m) = Z(\lambda + m\mu_E + \min(c-m, n)\mu_P) + \begin{cases} T(n, m-1) & \text{w.p. } \frac{m\mu_E}{\lambda + m\mu_E + \min(c-m, n)\mu_P} \\ T(n, m+1) & \text{w.p. } \frac{\lambda}{\lambda + m\mu_E + \min(c-m, n)\mu_P} \\ T(n-1, m) & \text{w.p. } \frac{\min(c-m, n)\mu_P}{\lambda + m\mu_E + \min(c-m, n)\mu_P} \end{cases} \quad (3)$$

And finally, if $n = 0$ then the overhaul is finished, thus $T(0, m) = 0$ for $m = 0, \dots, c-1$.

The moments of $T(n, m)$ can be derived from its Laplace Stieltjes Transform (LST). If $\varphi(n, m)(s)$ is the LST of $T(n, m)$, that is $\varphi(n, m)(s) = E(e^{-sT(n, m)})$ for $s \geq 0$, then

$$E(T(n, m)^k) = (-1)^k \left. \frac{d^k \varphi(n, m)(s)}{d^k s} \right|_{s=0} \quad \text{for } k = 1, 2, 3, \dots \quad (4)$$

Below we show how $\varphi(n, m)(s)$ can be determined. Based upon the state transition formula (2), the following relations for $\varphi(n, m)(s)$ in case $m \geq c$ and $n > 0$ can be derived

$$\begin{aligned} \varphi(n, m)(s) &= E\left(e^{-sZ(\lambda+c\mu_E)}\right) \left(\frac{c\mu_E}{\lambda+c\mu_E} E\left(e^{-sT(n, m-1)}\right) + \frac{\lambda}{\lambda+c\mu_E} E\left(e^{-sT(n, m+1)}\right) \right) \\ &= \frac{\lambda+c\mu_E}{\lambda+c\mu_E+s} \left(\frac{c\mu_E}{\lambda+c\mu_E} \varphi(n, m-1)(s) + \frac{\lambda}{\lambda+c\mu_E} \varphi(n, m+1)(s) \right) \\ &= \frac{c\mu_E}{\lambda+c\mu_E+s} \varphi(n, m-1)(s) + \frac{\lambda}{\lambda+c\mu_E+s} \varphi(n, m+1)(s) \end{aligned} \quad (5)$$

Rewriting gives

$$(\lambda+c\mu_E+s)\varphi(n, m)(s) = c\mu_E\varphi(n, m-1)(s) + \lambda\varphi(n, m+1)(s) \quad (6)$$

For fixed n and s this is a second order recurrence relation in m . The general solution of (6) is given by

$$\varphi(n, m)(s) = A_1\alpha_1(s)^m + A_2\alpha_2(s)^m \quad (7)$$

where

$$\alpha_1(s) = (-q - \sqrt{q^2 - 4pr})/2p, \quad \alpha_2(s) = (-q + \sqrt{q^2 - 4pr})/2p$$

with $p = \lambda$, $q = -\lambda - c\mu_E - s$, $r = c\mu_E$ and A_1, A_2 are constants. Note that the stability condition $\lambda/c\mu_E < 1$ implies that $0 < \alpha_1(s) \leq 1 < \alpha_2(s)$ for $s \geq 0$. Since $|\varphi(n, m)(s)| \leq 1$ for all n, m and $s \geq 0$, it follows that we have to set $A_2 = 0$. Hence, from (7) we can conclude that for $m \geq c$ the following holds:

$$\begin{aligned} \varphi(n, m)(s) &= \varphi(n, c-1)\alpha_1(s)^{m-c+1} \\ &= \varphi(n, c-1) \left(\frac{\lambda + c\mu_E + s - \sqrt{(-\lambda - c\mu_E - s)^2 - 4\lambda c\mu_E}}{2\lambda} \right)^{m-c+1} \end{aligned} \quad (8)$$

The value of $\varphi(n, c-1)$ can be calculated from the system of difference equations for the LST in case $m < c$ and $n > 0$. Similarly, relation (3) can be transformed to

$$\begin{aligned} (\lambda + m\mu_E + \min(c-m, n)\mu_p + s)\varphi(n, m)(s) &= m\mu_E\varphi(n, m-1)(s) + \lambda\varphi(n, m+1)(s) \\ &\quad + \min(c-m, n)\mu_p\varphi(n-1, m)(s) \end{aligned} \quad (9)$$

Finally, if $n = 0$ then $\varphi(0, m)(s) = E(e^{-sT(0, m)}) = E(e^0)$ and thus

$$\varphi(0, m)(s) = 1 \quad \text{for } m = 0, \dots, c-1 \quad (10)$$

Now, we are able to calculate $\varphi(n, m)(s)$ for all $n, m \geq 0$. The transform $\varphi(n, c)(s)$ can be eliminated from (9) by substituting (8) for $m = c$. This gives a set of equations for $\varphi(n, m)(s)$ with $m < c$ which in matrix notation can be written as:

$$A_n(s)\varphi_n(s) = D_{n-1}\varphi_{n-1}(s) \quad \text{for } n = 1, 2, 3, \dots \quad (11)$$

$$\varphi_0(s) = (1, \dots, 1)^T \quad (12)$$

where

$$A_n(s) = \begin{pmatrix} \lambda + \min(c, n)\mu_p + s & -\lambda & & & 0 \\ & -\mu_E & & & \\ & & \ddots & & \\ & & & \lambda + (c-2)\mu_E + \min(2, n)\mu_p + s & -\lambda \\ 0 & & & -(c-1)\mu_E & -\lambda\alpha(s) + \lambda + (c-1)\mu_E + \mu_p + s \end{pmatrix}$$

and

$$D_{n-1} = \begin{pmatrix} \min(c, n) & & & 0 \\ & \min(c-1, n) & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \mu_p$$

for $n = 1, 2, 3, \dots$ and $\varphi_n(s) = (\varphi(n, 0)(s) \dots \varphi(n, c-1)(s))^T$ for $n = 0, 1, 2, \dots$. The equation (11) can now easily be solved recursively for $n = 1, 2, 3, \dots$. This concludes the determination of $\varphi(n, m)(s)$.

According to (4), computing the first and second moment of $T(n, m)$ requires the first and second derivative of $\varphi(n, m)(s)$ at $s = 0$. Differentiating the system (11)-(12) once and twice with respect to s results in:

$$\varphi'_n(s) = [A_n(s)]^{-1} [D_{n-1}\varphi'_{n-1}(s) - A'_n(s)\varphi_n(s)] \quad \text{for } n = 1, 2, 3, \dots \quad (13)$$

$$\varphi'_0(s) = (0, \dots, 0)^T \quad (14)$$

and

$$\varphi_n''(s) = [A_n(s)]^{-1} [D_{n-1}\varphi_{n-1}'' - A_n''(s)\varphi_n(s) - 2A_n'(s)\varphi_n'(s)] \quad \text{for } n = 1, 2, 3, \dots \quad (15)$$

$$\varphi_0''(s) = (0, \dots, 0)^T \quad (16)$$

From the structure of $A_n(s)$ it is clear that $-A_n(s)$ is a transient generator for $s \geq 0$ and thus $[A_n(s)]^{-1}$ exists. Furthermore, because $A_n(s)$ is a tridiagonal matrix, the equations (11), (13) and (15) can be solved very efficiently for respectively $\varphi_n(s)$, $\varphi_n'(s)$ and $\varphi_n''(s)$ (see e.g. Atkinson [1]). Of course, in this case only the solution for $s = 0$ is relevant. Based upon the formulas (8), (11)-(16) it is possible to recursively calculate the first two moments of $T(n, m)$ for any given value of n and m . Appendix A presents an algorithm to carry out this task.

Finally, the first two moments are used to fit an appropriate distribution. Let $E(T(n, m))$ and $\sigma(T(n, m))$ denote the mean and standard deviation of $T(n, m)$. And let $c_{T(n, m)}$ denote its coefficient of variation, defined as $c_{T(n, m)} = \sigma(T(n, m)) / E(T(n, m))$. If $c_{T(n, m)}$ is less than 1 it is possible to fit a mixture of two Erlang distributions with the same scale parameter μ and respectively having k and $k-1$ phases: the makespan of an overhaul is now with probability p (resp. probability $1-p$) described by an Erlang distribution with $k-1$ (resp k) phases (see e.g. Tijms [6]). The appropriate values for the parameters μ , k and p can be found in Appendix B. This results in the approximation

$$P(T(n, m) \leq T_D) \cong 1 - p \sum_{j=0}^{k-2} e^{-\mu T_D} \frac{(\mu T_D)^j}{j!} - (1-p) \sum_{j=0}^{k-1} e^{-\mu T_D} \frac{(\mu T_D)^j}{j!} \quad (17)$$

If $c_{T(n,m)}$ is equal to or greater than 1 it is possible to fit another phase type distribution on the first two moments of the makespan. However, this is not relevant in the present situation.

5. DISCUSSION OF THE APPROXIMATION

Though the first and second moment of $T(n, m)$ are theoretically and exactly determined, the fitting procedure may cause a difference between the approximated probability (17) and the exact probability. In this section we investigate the accuracy of this approximation. We generate eight overhauls, each with different repair and arrival rates, starting conditions (n, m) and number of parallel servers. The overhauls are listed in Table 1. Overhaul 1 is used as the basic example: the others are derived from it by varying some characteristics. In that way, the impact of each individual parameter should pop up.

Table 1. Demand and repair shop characteristics.

Overhaul	Emergency jobs (m)	Repair rate (μ_E)	Arrival rate (λ)	Planned jobs (n)	Repair rate (μ_P)	Servers (c)	Due Date (T_D)
1	0	1	1	50	1	4	25
2	0	2	1	50	1	4	20
3	0	1	3	50	1	4	80
4	0	1	1	10	1	4	10
5	0	1	1	50	2	4	10
6	0	1	1	50	1	2	55
7	0	1	1	50	1	15	10
8	20	2	1	40	1	4	25

The difference between the approximated and the exact value is - because of the lack of knowledge about the exact value - investigated by simulation. For each overhaul, the first and second moment of $T(n, m)$, as well as the 50th, 70th, 80th, 90th and the 95th percentiles are both simulated and approximated by our queuing and fitting model. For each overhaul, we conducted 100,000 subruns.

Table 2. Simulation (S) and approximation (A) results.

Overhaul	$E(T(n,m))$	$\alpha T(n,m)$	50 th perc.		70 th perc.		80 th perc.		90 th perc.		95 th perc.	
			(S)	(A)	(S)	(A)	(S)	(A)	(S)	(A)	(S)	(A)
1	17.4	3.2	17.1	17.2	18.8	18.9	19.9	19.9	21.6	21.5	23.0	22.8
2	17.6	2.6	17.4	17.5	18.8	18.9	19.7	19.8	21.0	21.0	22.1	22.1
3	49.6	18.5	46.3	47.6	56.1	57.4	63.1	64.5	74.2	73.9	84.8	83.7
4	4.0	1.6	3.8	3.8	4.6	4.7	5.2	5.3	6.1	6.3	7.0	7.0
5	8.6	1.8	8.4	8.5	9.3	9.4	9.9	10.0	10.9	11.0	11.8	11.8
6	50.1	12.2	48.6	49.3	55.3	55.5	59.6	60.1	66.3	65.8	72.2	71.8
7	5.8	1.3	5.6	5.7	6.3	6.4	6.7	6.8	7.5	7.5	8.2	8.2
8	24.0	3.7	23.8	23.9	25.8	25.8	27.0	27.0	28.9	28.9	30.5	30.6

The results, listed in Table 2, show that the approximation is very accurate for each overhaul and percentile, regardless the parameter setting.

6. SUBCONTRACTING POLICY

In this section we show how the approach can be used to support the subcontracting decisions.

Let α be the required delivery performance (e.g. $\alpha = 0.95$). Define $n^*(\alpha, m)$ as the

maximum number of planned jobs which can internally be finished before the due date T_D with a given probability α given that at time $t = 0$ there are m emergency jobs in the shop. So,

$$n^*(\alpha, m) = \max\{n \mid P(T(n, m) \leq T_D) \geq \alpha, n = 0, 1, 2, \dots\} \quad (18)$$

Then the value of $n^*(\alpha, m)$ can simply be calculated with the use of, e.g., a bi-section method. Of course $n^*(0.95, m)$ should be computed at the latest possible subcontracting time. From Table 3 it follows that - if $t = 0$ is the latest possible subcontracting time - only for the overhauls 3, 5 and 6 subcontracting is necessary to complete these in time with a probability of at least 95%. Furthermore, the expected overhaul makespan is shown.

Table 3. Maximum number of jobs to be processed internal.

Overhaul	n	$n^*(0.95)$	$P(T(n^*(0.95, m), m) \leq T_D)$	$E(T(n^*(0.95, m), m))$	$E(T_D - T(n^*(0.95, m), m))^+$
1	50	55	0.956	19.02	6.24
2	50	51	0.960	15.51	4.67
3	50	47	0.952	46.58	34.34
4	10	17	0.955	6.36	3.77
5	50	41	0.956	7.06	3.03
6	50	36	0.952	36.11	20.32
7	50	75	0.953	7.55	2.51
8	50	58	0.953	20.37	4.88

In general, internally processing at most $n^*(\alpha, m)$ jobs results in a low capacity usage in the interval $[0, T_D)$. If no other activities are carried out in that interval besides the planned overhaul and emergency jobs, Table 3 shows the remaining time that all servers are only used for emergency jobs (if any), i.e. $E(T_D - T(n^*(0.95, m), m))^+$. This illustrates that besides the

control of the delivery performance also the control of the capacity usage is important. The capacity usage can be controlled by overlapping overhauls (see Fig. 2) or the use of low priority jobs, for example the make-to-stock processing of failed repairables. Keizers and Bertand [3] give a framework for control of both the delivery performance and the capacity usage.

7. EXTENSION

The model assumes that all planned jobs have exponentially distributed processing times with the same mean. In practice this may not be a valid restriction: an overhaul may contain both small and large jobs and jobs will not be exponentially distributed. The latter is especially true for preventive maintenance jobs. If these overhaul characteristics show up, *sequencing* and *interruption* rules also effect the makespan distribution. After discussing these rules, we present and validate an extension of the model to be able to cope with more general and non identical processing times distributions.

Sequencing rules are rules to determine the sequence in which the jobs of an overhaul are released to the repair shop. In case all jobs have different expected repair times, usually sequencing rules are used which release the jobs to the repair shop in order of decreasing expected repair time (ELPT rule: Expected Longest Processing Time first). This has a positive effect on the makespan: the smaller jobs keep the (expected) difference between the first and last finished repair man as small as possible. A small makespan positively influences the probability of completing the overhaul in time. For the case without emergency jobs and exponential repair times, Pinedo [5] shows that this sequencing rule gives optimal results with

regard to the delivery performance. However, because in practice some precedence between the jobs relations may exist, this rule is not fully applicable.

Interruption rules are rules to determine the planned job which is interrupted upon arrival of an emergency job. In case jobs have non-identical (residual) repair time distributions, the interruption rule affects the makespan distribution. In practice this choice depends upon the difficulties of interrupting the various planned jobs currently on hand.

Without highly complicating it, the model cannot be extended with sequencing and interruption rules. Therefore, the model is no longer capable to exactly determine the moments of the makespan. However, the queuing model and fitting procedure can be extended to give at least approximations for these moments. Consider an overhaul consisting of n jobs. Let the random variables X_1, \dots, X_n denote their processing times with means $E(X_i)$ and standard deviations $\sigma(X_i)$. Let Y be the total demand for capacity resources induced by the n jobs, that is $Y = X_1 + \dots + X_n$. Assuming that the coefficient of variation of Y is smaller than 1, the fitting procedure of Tijms [6] can be used to fit a mixture of two Erlang distributions. If so, the problem is transformed from processing n non identical jobs into a problem in which with probability p' (resp. $1 - p'$) the overhaul consists of $k' - 1$ (resp. k') exponential phases with identical mean μ' which have to be processed. Notice that no longer a physical relation exists between processing a certain job and a phase. Subsequently, let $E(T(m))$ (resp. $\sigma(T(m))$) be the expectation (resp. standard deviation) of the makespan if the work load Y is represented as a mixture of Erlang distributions and there initially are m emergency jobs in the shop. This gives

$$E(T(m)) = p'E(T(k' - 1, m)) + (1 - p')E(T(k', m)) \quad (19)$$

and

$$E(T(m)^2) = p'E(T(k' - 1, m)^2) + (1 - p')E(T(k', m)^2) \quad (20)$$

and thus

$$\sigma(T(m)) = \sqrt{E(T(m)^2) - E^2(T(m))}. \quad (21)$$

Finally, after again applying the fitting procedure based on $E(T(m))$ and $\sigma(T(m))$, the random variable $T(m)$ approximately has a mixture of two Erlang distributions with parameters k'' , p'' and μ'' .

Table 4. Erlang(θ_i, μ_i) distributed planned job types.

Type	θ_i	μ_i	$E(X_i)$	$\sigma(X_i)$
(i)				
I	2	1	2	$\sqrt{2}$
II	2	0.5	4	$\sqrt{8}$
III	4	1	4	$\sqrt{4}$

Table 5. Overhaul characteristics.

Overhaul	n_I	n_{II}	n_{III}
1	40		10
2	10		40
3	40	10	
4	10	40	
5		40	10
6		10	40

Because we neglect the possible existence of sequencing and interruption rules, we have to investigate the difference between the approximated distribution function and the real distribution function in case sequencing or interruption rules are used. Simulation should show whether this approximation error is acceptable or not. We consider several overhauls, where each is assumed to consists of n_I (resp. n_{II} and n_{III}) type I (resp. II and III) planned jobs. The repair time of job type i is assumed to be Erlang(θ_i, μ_i) distributed: the mean equals θ_i / μ_i and the variance equals θ_i / μ_i^2 . Table 4 displays the characteristics of each job type and Table 5 shows a list of the overhauls. The shop conditions are for each overhauls kept at $m = 0$, $\lambda = 1$, $\mu_E = 1$ and $c = 4$. For each overhaul two simulations are carried out: one with a random sequencing rule and one with the ELPT sequencing rule (Table 6). In both simulations we used a random interruption rule. Each simulation consists of 100,000 subruns. Again, both approximations give good results. Furthermore, the moments for the makespan in case of an ELPT rule indeed turn out to be smaller than in case of a random sequencing rule.

Table 6. Approximations (A) and simulations for non-identical repair time distributions and random (SR) resp. ELPT (SE) sequencing rule.

Overhaul	$E(T(m))$			$\sigma(T(m))$			90 th perc.			95 th perc.		
	(A)	(SR)	(SE)	(A)	(SR)	(SE)	(A)	(SR)	(SE)	(A)	(SR)	(SE)
1	40.7	41.3	41.1	4.8	4.9	4.8	46.8	47.7	47.4	49.2	49.7	49.3
2	60.7	61.7	61.0	5.8	6.0	5.8	68.1	69.5	68.6	71.9	70.2	71.1
3	41.0	41.6	41.0	5.3	5.5	5.3	47.7	48.8	47.9	50.2	51.1	50.1
4	61.5	62.2	61.1	7.4	7.5	7.2	70.9	72.1	70.6	74.1	75.2	73.5
5	68.1	68.9	68.4	7.6	7.8	7.5	77.7	79.0	78.2	80.6	82.2	81.4
6	67.5	68.5	68.4	6.5	6.7	6.6	75.9	77.2	77.1	79.0	80.0	79.9

It is to be expected that in case of more varying distribution functions of the processing times of the planned jobs, the approximation will be less accurate, because now sequencing and interruption rules have more impact on the determination of the makespan. Therefore, Table 7 lists the characteristics of more varying job types. Again we conducted 100,000 subruns for each overhaul again based on the overhaul characteristics in table 5. Table 8 lists the results. Although the results differ somewhat more than the simulation with the jobs in Table 4, the results are still sufficiently accurate.

Table 7. Erlang(θ_i, μ_i) distributed planned job types.

Type	θ_i	μ_i	$E(X_i)$	$\sigma(X_i)$
(i)				
I	20	0.5	40	$\sqrt{80}$
II	2	0.1	20	$\sqrt{200}$
III	10	10	1	$\sqrt{0.1}$

Table 8. Approximations (A) and simulations for non-identical repair time distributions and random (SR) resp. ELPT (SE) sequencing rule.

Overhaul	$E(T(m))$			$\alpha(T(m))$			90 th perc.			95 th perc.		
	(A)	(SR)	(SE)	(A)	(SR)	(SE)	(A)	(SR)	(SE)	(A)	(SR)	(SE)
1	538.3	550.8	547.8	21.9	23.1	23.1	566.0	580.6	577.6	573.8	589.3	586.4
2	148.1	161.6	150.7	11.2	13.6	12.6	162.4	179.4	167.0	167.6	184.9	172.5
3	602.4	614.1	612.6	26.8	27.9	29.5	637.2	650.1	650.8	645.8	661.1	662.8
4	406.5	413.5	412.5	33.5	34.2	35.0	451.2	458.0	458.2	464.0	471.9	472.5
5	278.9	282.5	279.9	32.5	33.2	33.3	322.1	325.8	323.3	336.7	339.4	336.9
6	87.5	92.0	83.6	17.8	20.1	18.3	110.3	118.3	107.8	117.5	128.1	116.7

The subcontracting decision has also become more difficult. If at the latest subcontracting time the probability of delivering in time is smaller than α , it is no longer sufficient to solve the minimising problem (18), because the jobs are not identical. However, the answer to this minimising problem gives the number of phases to be subcontracted and thus gives an indication of the workload to be subcontracted. In an iterative what-if approach a suitable set of jobs to subcontract has to be found. Also job parameters like subcontracting costs and subcontracting lead times play an important role now.

8. CONCLUSION

In this paper we studied the problem of determining the probability of delivering a bunch of planned jobs in time, while the arrivals of emergency jobs disturb the progress of these planned jobs. We developed a queueing model to determine an approximation for the first and second moment of the makespan of the bunch of jobs. These moments are used to fit a mixture of Erlang distributions and approximate the probability of delivering the bunch

before or at its due date. We showed a numerical example for the static case, in which upon arrival of a bunch of jobs the due date reliability is approximated. Moreover, the approach can also be used in a dynamic case, in which during the progress of an overhaul the due date reliability (constantly) is measured.

The queueing model gives exact solutions for the makespan moments in case all planned jobs in a bunch have the same exponential processing time. The fitted distribution function is an approximation, but simulation showed that that the error turned out to be very small for the tested instances. Moreover, the approach can be extended for jobs with non-identically and non-exponentially distributed repair times. Simulation had been used to evaluate the approximation error both for the moments and for some fixed percentiles of the probability density distribution. It turned out that these approximations are also sufficiently accurate.

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Appendix A

ALGORITHM: COMPUTE_OVERHAUL_MOMENTS

Input: $n, m, c, \lambda, \mu_E, \mu$

Output: $E(T(n, m))$ and $E(T(n, m)^2)$

Stability condition: $\lambda / \mu_E < c$

Step 0 $i := 0;$

$$\varphi_0(0) = (1, \dots, 1)^T;$$

$$\varphi'_0(0) = (0, \dots, 0)^T;$$

$$\varphi''_0(0) = (0, \dots, 0)^T;$$

Step 1 **WHILE** $i \leq n$ **DO**

BEGIN

$$i := i+1;$$

$$\varphi_i(0) = [A_i(0)]^{-1} D_{i-1} \varphi_{i-1}(0);$$

$$\varphi'_i(0) = [A_i(0)]^{-1} [D_{i-1} \varphi'_{i-1}(0) - A'_i(0) \varphi_i(0)];$$

$$\varphi''_i(0) = [A_i(0)]^{-1} [D_{i-1} \varphi''_{i-1}(0) - A''_i(0) \varphi_i(0) - 2A'_i(0) \varphi'_i(0)];$$

END

Step 2 **IF** $m < c$ **THEN**

BEGIN

$$E(T(n, m)) = -(\varphi'_i(0))_{m+1}$$

$$E(T(n, m)^2) = (\varphi''_i(0))_{m+1}$$

END ELSE

BEGIN

$$E(T(n, m)) = -(m - c + 1)\alpha'(0)(\varphi_i(0))_c - (\varphi'_i(0))_c$$

$$E(T(n, m)^2) = (m - c + 1)(m - c)\alpha''(0)^2(\varphi_i(0))_c$$

$$+ (m - c + 1)\alpha''(0)(\varphi_i(0))_c$$

$$+ 2(m - c + 1)\alpha'(0)(\varphi'_i(0))_c + (\varphi''_i(0))_c$$

END

Appendix B: Fitting a mixture of Erlang distributions

Consider the random variable $T(n, m)$ with mean $E(T(n, m))$ and standard deviation $\sigma(T(n, m))$. Furthermore, let $c_{T(n, m)}$ be its coefficient of variation, defined as $c_{T(n, m)} = \sigma(T(n, m)) / E(T(n, m))$ and assume that this value is between 0 and 1. If $c_{T(n, m)}$ is less than 1, then according to Tijms [6], a mixture of Erlang(k) and Erlang($k-1$) distributions with density function

$$f(t) = p\mu^{k-1} \frac{t^{k-2}}{(k-2)!} e^{-\mu t} + (1-p)\mu^k \frac{t^{k-1}}{(k-1)!} e^{-\mu t}, \quad t \geq 0$$

fits $E(T(n, m))$ and $\sigma(T(n, m))$ provided that the parameters p and μ are chosen as

$$p = \frac{kc_{T(n, m)}^2 - \sqrt{k(1 + c_{T(n, m)}^2) - k^2 c_{T(n, m)}^2}}{1 + c_{T(n, m)}^2}, \quad \mu = \frac{k - p}{E(T(n, m))}$$

where k is such that

$$\frac{1}{k} \leq c_{T(n, m)}^2 \leq \frac{1}{k-1}.$$