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Uncertainty in Gaussian noise generalized for cross-correlation spectra

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The ensemble average and variance of Gaussian noise measured as cross-correlation spectra are calculated as a function of the number of time averages. The calculations are based on the Carson–Campbell theorem which treats noise spectra as a collection of individual deterministic pulses transformed into the frequency domain. We have compared our calculations with experimental results on metal film resistors. There is an excellent agreement between the predicted and the measured statistical behavior. © 1998 American Institute of Physics. [S0021-8979(98)07820-7]

I. INTRODUCTION

Power density spectra of stationary noise become more precise and hence smoother both through time and frequency averaging. The averaged spectrum will, for instance, become constant as a function of frequency for thermal noise or shot noise, inversely proportional to frequency for the so-called 1/f noise or flat for low frequencies and, beyond a characteristic frequency, inversely proportional to the frequency squared for Lorentzian noise such as generation-recombination noise.1 An observed spectrum often is a combination of these types of subspectra. Owing to the finite average time, an approximation of the spectrum is obtained and the spectrum is always corrupt with an uncertainty. Our study will concentrate on this uncertainty, especially for cross-correlation spectra. The results are independent of the frequency dependence of spectra.

Detailed knowledge of the uncertainty in noise spectra can pinpoint the physical origin of noise sources in a sample. This is possible when statistical analysis shows that the probability density function of the spectral noise can not be classified as Gaussian. A number of groups have published a lot of work concerning non-Gaussian noise in specific situations.2,3 So far, all publications were based on autocorrelation spectra in which the uncertainty in the noise spectra was compared with the expected uncertainty for Gaussian noise. In this article we will calculate and measure the uncertainty of Gaussian noise in cross-correlation spectra. In this situation the measured variance in a Gaussian noise signal can differ significantly from the same measurement performed with an autocorrelation.

The basis of non-Gaussian noise is included in the Carson–Campbell theorem. In 1909 Campbell wrote a series of articles in which he explained the variance in a time signal that consisted of a series of individual deterministic pulses randomly distributed in time.4–6 In 1925 Carson extended this theorem to the frequency domain.7,8 At that time Carson was only interested in the ensemble average of the autocorrelation spectral value.

If the probability density of the magnitude of these individual pulses is Gaussian, the spectrum $S_{FF}(f)$, which is a result of these pulses, will also be Gaussian.9 Saying that $S_{FF}(f)$ is Gaussian actually means that both the real and the imaginary part of the Fourier integral of the signal $F(f)$ have an independent Gaussian probability density distribution. The spectrum $S_{FF}(f)$ is calculated directly from $F(f)$ and because $F(f)$ has a Gaussian distribution $S_{FF}(f)$ will be called Gaussian.

If the magnitude of the individual pulses does not have a Gaussian distribution, one can still get a Gaussian spectrum if the starting times of these pulses is randomly distributed and the number of pulses captured in a time block is large enough. If the noise seems to be non-Gaussian, it is possible to verify whether the number of pulses is causing it by increasing the time window.10 The noise will already appear to be Gaussian when the number of equal pulses, of which the starting times have a Poisson distribution function, are on average ten or more within one time block.

It is also possible to obtain deviations from Gaussianity when the number of pulses generated per second significantly fluctuates in time. This should be called nonstationary noise. In this case the average spectrum will typically increase and decrease over the whole frequency range from one time block to another. The fluctuations of this average power became known as variance noise when the uncertainty in the variance of a band filtered time signal was measured. Stoisiek and Wolf investigated what these variance noise levels should be if a stationary 1/f noise signal is measured.11

Strictly spoken, a real noise signal will never be a true Gaussian signal because the average noise power is determined by physical parameters which are also fluctuating. For instance, resistance and temperature will determine the level of the average thermal noise spectrum. This type of effect was exploited to show that 1/f noise is an equilibrium phenomenon that can be measured in the uncertainty of white noise.12–14 These type of measurements became known as second spectra. Second spectra are used for materials showing a coherent or hierarchical behavior such as ferromagnetic systems15 and spin glasses.16,17 Further more it was used to extract Lorentzian shapes from 1/f noise.18,19
II. THEUNCERTAINTY OF SPECTRAL NOISE WITH A GAUSSIAN DISTRIBUTION

To introduce the main ideas concerning the calculation of the variance of a spectral value, we will first repeat the results given by Restle et al. for an autocorrelation spectral measurement.8

A. The variance of a spectral value obtained from autocorrelation measurements

If both the real and the imaginary part of the original signal \( F(f) \) have a Gaussian probability density function, the probability density \( (pd) \) will be: \( pd(x) = e^{-x^2} \), where \( x \) represents the magnitude of the signal and \( p = \frac{x^2}{2} \) the power. The distribution function of an individual spectrum which is proportional to the signal power is therefore exponential, or, more generally speaking, after averaging individual spectra, a Gamma or Erlang distribution function.20

Note that this is not a Gaussian distribution function, but because of the central limit theorem it will start to resemble a Gaussian distribution function as the number of averages increases. The normalized Gamma distribution function of a spectrum after averaging \( N_{\text{Avg}} \) time blocks is:

\[
pd \left( \frac{S_{FF}(f)}{S_{FF}(f)} \right, N_{\text{Avg}}) = \frac{1}{(N_{\text{Avg}} - 1)!} S_{FF}(f) \times \left( \frac{S_{FF}(f)}{S_{FF}(f)} \right)^{N_{\text{Avg}}} \\
\times e^{-N_{\text{Avg}} \frac{S_{FF}(f)}{S_{FF}(f)}}.
\]

Based on this distribution function the relative variance in a spectral value can be calculated:

\[
\frac{\Delta S_{FF}^2(f)}{S_{FF}(f)^2} = \int_{0}^{\infty} \left( \frac{S_{FF}(f)}{S_{FF}(f)} - 1 \right)^2 pd \left( \frac{S_{FF}(f)}{S_{FF}(f)} \right, N_{\text{Avg}}) \\
\times dS_{FF}(f) = \frac{1}{N_{\text{Avg}}}.
\]

A Gaussian distribution function will give the same result. The relative fourth-order moment of the spectrum can be calculated in the same way as the variance. The result is:

\[
\Delta S_{FF}^4(f) = 3 \frac{N_{\text{Avg}}}{N_{\text{Avg}}} + 2 \Delta S_{FF}^2(f)^2.
\]

Relations (2) and (3) are also valid when a total spectrum is the result of two or more uncorrelated subspectra for instance 1/f noise and thermal noise. It would seem that the variance in the total spectrum is equal to the variance in the individual parts without any cross terms, but this is not true. The latter only holds for the ensemble average. In this case the total spectrum is a result of a multiplication of two types of Fourier-transformed signals, which brings about that all individual pulses will be mixed regardless of whether they originate from the one or the other noise source, as shown by the result of the Carson–Campbell theorem. This gives the uncertainty correlation of the spectrum between all individual pulses, and hence also between two subspectra of which the average spectra are uncorrelated.

B. Cross-correlation measurements

The principle of a cross-correlation spectral measurement lies in the fact that the noise of a sample is measured through two different noise channels and only the common noise of the channels is extracted through averaging of spectra. Suppose, for instance, that the signal in the first channel is called \( X \) and consists of the signal of the sample \( F \) and an extra signal \( A \), for example, due to an amplifier. So: \( X = F + A \). In the same way the signal of the other channel is called \( Y = F + B \), where \( B \) is the signal of another amplifier. The cross-correlation spectrum based on the signals \( X \) and \( Y \) after averaging \( N_{\text{Avg}} \) time blocks, will then be:

\[
S_{XY}(f, N_{\text{Avg}}) = \frac{1}{N_{\text{Avg}}} \sum_{i=1}^{N_{\text{Avg}}} S_{FF}(f) \\
+ S_{AF}(f) + S_{FB}(f) + S_{AB}(f).
\]

The first part, \( S_{FF}(f) \), is the real signal we want. The other parts have both a real and an imaginary component whose ensemble averages are zero if \( A, B, \) and \( F \) are uncorrelated. So when enough \( S_{XY}(f) \) are averaged one is left with the real function \( S_{FF}(f) \), because the other parts average out.

We want to calculate the ensemble average and the relative variance of this spectrum as a function of the number of averaged time blocks \( N_{\text{Avg}} \). To calculate the variance we need both the average value and the power of \( S_{XY} \). The ensemble average of the power can be calculated with the probability density function of the individual noise sources, Eq. (1). The result is

\[
[S_{XY}(f, N_{\text{Avg}})]^2 = \frac{N_{\text{Avg}} + 1}{N_{\text{Avg}}} S_{FF}(f)^2 + \frac{S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f)}{N_{\text{Avg}}}.
\]

Note that the over lines represent an ensemble average and not a time average of different spectra as represented by \( N_{\text{Avg}} \).

Due to the fact that the magnitude of \( S_{XY} \) is the square root of the real and the imaginary power, which are different due to \( S_{FF} \), we are not able to calculate the ensemble average directly. Therefore have to examine two limiting cases: the correlated case where the influence of the signals \( A \) and \( B \) on \( S_{XY} \) has become very small and the opposite situation where the signal \( F \) has little influence on \( S_{XY} \), this case gives two uncorrelated channels.

In the first situation, \( S_{FF} \) dominates \( S_{XY} \) so one can eliminate the square root term with a Taylor-series approximation.
round $S_{FF}$. The average values of the real and imaginary part can now be determined individually. The resulting ensemble average of $S_{XY}$ can be approximated as

$$[S_{XY}(f,N_{Avg})] \approx \sqrt{S_{FF}(f)^2 + \frac{1}{2N_{Avg}} (S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f))}.$$  

(6)

In this equation we disregarded higher order terms than $1/N_{Avg}$ under the square root because we are only interested in a first-order approximation in $1/N_{Avg}$ for the variance of $S_{XY}$.

In the second situation where $S_{FF}$ is a minor contribution to $S_{XY}$, we calculate the ensemble average of each of the four terms individually as if the other three are zero and combine the four results. This is a good approximation if one of the four terms dominates. In general $S_{AA}S_{BB}$ will be the dominating term if $S_{FF} \ll S_{XY}$. Combining the individual terms gives an ensemble average of

$$[S_{XY}(f,N_{Avg})] \approx \sqrt{S_{FF}(f)^2 + \frac{\pi G(N_{Avg})}{4N_{Avg}} (S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f))}.$$  

(7)

With

$$G(N_{Avg}) = \frac{1}{N_{Avg}} \left( \frac{\Gamma(N_{Avg} + \frac{1}{2})}{\Gamma(N_{Avg})} \right)^2,$$

(8)

$G(1) = \pi/4$ and rises quickly to 1 as $N_{Avg}$ increases. For instance, $G(10) \approx 0.975$ and generally $N_{Avg} \geq 10$ so $G(N_{Avg})$ is, for practical purposes, 1.

The ensemble average of the magnitude of $S_{XY}$ in the two limiting situations [Eqs. (6) and (7)] can be combined with a mathematical smoothing function $\beta$. The result is

$$[S_{XY}(f,N_{Avg})] \approx \sqrt{S_{FF}(f)^2 + \frac{\beta}{N_{Avg}} (S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f))},$$

(9)

where the mathematical smoothing function $\beta$ is equal to.

$$\beta = \frac{\pi}{4} \frac{1}{N_{Avg}} \left( \frac{\Gamma(N_{Avg} + \frac{1}{2})}{\Gamma(N_{Avg})} \right)^2 \left( 1 - \frac{S_{FF}(f)^2}{|S_{XY}(f,N_{Avg})|^2} \right) + \frac{1}{2} \frac{S_{FF}(f)^2}{|S_{XY}(f,N_{Avg})|^2}.$$  

(10)

If the number of averages is high enough for the average value of $S_{XY}$ to be roughly equal to $S_{FF}$, $\beta$ will become 1/2. In general, $\beta$ will always be between 1/2 and $\pi/4$.

Based on relations (5) and (9) the ensemble average of the relative variance in a spectral value can be calculated. The result is:

$$\frac{\Delta S_{XY}(f,N_{Avg})^2}{S_{XY}(f,N_{Avg})^2} \approx \frac{S_{FF}(f)^2 + \beta(S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f))}{N_{Avg} S_{FF}(f)^2 + \beta(S_{FF}(f) S_{AA}(f) + S_{FF}(f) S_{BB}(f) + S_{AA}(f) S_{BB}(f))}.$$  

(11)

The number of averages needed to measure $S_{FF}$ can be estimated with Eq. (9), under the assumption that $S_{AA} \gg S_{BB} \gg S_{FF}$. The number of averages needed to get to the point where $S_{XY}$ changes from being dominated by $S_{AA}$ and $S_{BB}$ to being dominated by $S_{FF}$, is equal to

$$N_{Avg} = \frac{S_{AA}(f) S_{BB}(f)}{S_{FF}(f)^2} \approx \frac{1}{\gamma^2}.$$  

(12)

With $\gamma^2$ as the coherence function, defined as

$$\gamma^2 = \sqrt{\frac{S_{FF}(f)^2}{S_{XX}(f) S_{YY}(f)}},$$

(13)

$N_{Avg} = 1/\gamma^2$ is the minimum amount of averages needed to estimate $S_{FF}$ from $S_{XY}$. The level of $S_{XY}$, when $N_{Avg} = 1/\gamma^2$ and $S_{AA} \gg S_{BB} \gg S_{FF}$, is

$$S_{XY}(f) = \sqrt{\frac{20 + 2\pi}{4}} S_{FF}(f) \approx 1.28 S_{FF}(f).$$  

(14)
III. EXPERIMENTAL RESULTS USING THE CROSS-CORRELATION ANALYSIS

The measurements were done using three 1% metal film resistors which are Gaussian white noise sources. We retrieved both the ensemble average and the relative variance of $S_{XY}$ as a function of the number of averages $N_{\text{Avg}}$.

A 100 $\Omega$ resistor was used as the correlated signal $F$ we wished to measure and the other two were 1 k$\Omega$ resistors, used for the uncorrelated signals $A$ and $B$. The amplifiers we used had an equivalent thermal voltage noise at the inputs. Converting this voltage noise to an extra resistor at the inputs of a noise free amplifier gives an extra uncorrelated resistance of roughly 90 $\Omega$ for each amplifier. The measurement was done using a $T$ structure, where each amplifier was short circuited through one of the 1 k$\Omega$ resistors and both through the same 100 $\Omega$ resistance. We measured the white noise in the range of $1–10$ kHz using the 721 spectral values in this region to estimate the ensemble average and the variance of the white noise.

The results can be seen as dots in Figs. 1 and 2. We performed different measurements by first leaving both 1 k$\Omega$ resistors in the circuit as described above, then removing one of the two resistors to create an imbalance, and finally leaving both 1 k$\Omega$ resistors out of the setup. We also measured the two limiting cases. First, the uncorrelated noise is negligible. This was done by measuring the same 1 k$\Omega$ resistance with the two amplifiers directly in parallel, thereby approximating an autocorrelation measurement. And second: the noise is fully uncorrelated. We used the same $T$ structure as described above, but instead of the 100 $\Omega$ resistance we added a short circuit. The lines in the figures represent the expected behavior based on Eqs. (9) and (12). The measurements of both the ensemble average and the variance can be excellently predicted. This proves the equations to be good references for Gaussian noise in correlation measurements in general, because the autocorrelation is just a special kind of cross correlation.

Figure 3 shows the influence of the mathematical smoothing function $\beta$. We used one of the measurements and also plotted the results based on the individual limiting cases of the average spectrum as explained in the previous section.

![Figure 1](image1.png)  
**FIG. 1.** Experimental results (dots) of the ensemble average of a cross-correlation spectrum as a function of the number of time blocks: $S_{XY}(N_{\text{Avg}})$ gathered from metal film resistors. All results were measured with two amplifiers that had equivalent uncorrelated thermal noise resistances at the inputs of 87 and 97 $\Omega$. The amplifiers are connected to the left and right hand inputs of the test setup shown in the inset. We tried to measure a common correlated resistance $R_f$ of 100 $\Omega$ in measurements 1–3. The other resistances for measurements 1–3 are (1) $R_A = R_B = 0$ $\Omega$, (2) $R_A = 1$ k$\Omega$ $R_B = 0$ $\Omega$, (3) $R_A = R_B = 1$ k$\Omega$. Measurements 4 and 5 represent the extreme cases: measurement 4 is the uncorrelated case where $R_A = R_B = 1$ k$\Omega$ and $R_f = 0$ $\Omega$. Measurement 5 is the opposite case: $R_A = R_B = 0$ $\Omega$ and $R_f = 1$ k$\Omega$, leading to a mainly correlated signal. Each measurement was repeated ten times and all results are given. The lines represent the calculations with Eq. (9). Note that no fit parameters were used for these calculations.

![Figure 2](image2.png)  
**FIG. 2.** Experimental results (dots) of the relative variance of a cross-correlation spectrum times the number of time blocks: $N_{\text{Avg}} S_{XY}(N_{\text{Avg}})$. These results were calculated from the same data that were used for Fig. 1. The explanation of measurements 1–5 can therefore be found with Fig. 1. Again, the lines calculated with Eq. (11) show an excellent agreement with the experiments while no fit parameters were used.

![Figure 3](image3.png)  
**FIG. 3.** Two limiting cases on which the calculations for the cross-correlation signal are based, are plotted here as dotted lines. The relative variance is used to demonstrate the approximation. The solid line is the resulting approximation of the variance [Eq. (11)] based on these two limiting cases and the mathematical smoothing function $\beta$, given in Eq. (10). The measured variance (dots) used for comparison are from condition 3 in Fig. 2.
IV. CONCLUSIONS

We have calculated the ensemble average and variance of a spectral value of Gaussian noise that is measured with the use of the cross-correlation analysis as a function of the number of time averages. We compared the results with experimental results of known Gaussian noise sources. An excellent agreement is achieved between the calculations and the measurements. The results were tested with white noise but the calculations are independent of the sort of spectrum measured and will therefore also hold for Gaussian 1/f noise or other types of Gaussian noise. This Gaussian reference makes it possible to identify non-Gaussian noise sources in situations where a cross-correlation analysis is needed. The minimum number of averages needed to measure with the cross-correlation analysis is inversely proportional to the coherence function:

\[ N_{\text{avg min}} = 1/\gamma^2. \]