

Explicit formulas for the solutions of piecewise linear networks

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Explicit Formulas for the Solutions of Piecewise Linear Networks

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Abstract—A methodology is presented for obtaining explicit formulas for the solution of class-P piecewise linear (PL) networks and, inherently, for the linear complementary problem (LCP). The method uses the $[\cdot]$ operator, which has been previously defined in the literature to extend the explicit PL model descriptions of Chua. An important consequence of the methodology is that it proves that class-P networks have explicit solutions.

Index Terms—Constrained optimization, explicit solution algorithm, linear complementary problem, piecewise linear networks.

I. INTRODUCTION

IN ELECTRICAL simulation, component models are often defined by piecewise linear (PL) mappings and used in so-called PL simulators to obtain the response of nonlinear electrical networks for a given value of the input sources [1]. For these PL-models, various formulations or model descriptions have been devised [2]–[4]. However, it has recently been shown that they all are special cases of a so-called state-model description [5]. This state-model description is equivalent to an electrical network consisting of linear (including negative) resistors, (controlled) sources, and ideal diodes only [1]. Each ideal diode represents a single linear segment in the PL mapping described by the state model. The more segments that are used to approximate the behavior, the more ideal diodes are implicitly used in the network.

For an ideal diode, the voltage across the diode is complementary to the current through the ideal diode. This means that the value of voltage and current is nonnegative, but their product is zero. This complementary property results in the fact that the PL electrical network and its state-model description are based on the linear complementary problem (LCP) as a central element in the mathematical description [6].

In mathematics, the LCP is a fundamental problem that has drawn much attention in the last decades, partly due to its wide area of applications. The solution of the general problem is known to require an exponential amount of operations when using algorithms that globally search through the solution space by using appropriate pivoting strategies. Inherently, this means that to obtain the solution of a PL network, an exponential amount of operations with respect to the number of ideal diodes in the network is required.

For certain classes of LCP's, proofs for the existence and obtainability of the solutions of the LCP are known. It is well

known that if the PL network or the LCP belongs to class P, the problem has a unique solution [6]. This property was used in [9] to show that any class-P LCP mapping can be reformulated in a PL model description, based on the absolute-sign operator. Consequently, for a given input vector the solutions for the mapping can be obtained explicitly.

The fact that for a class-P LCP the solution can be obtained has been known for long time. However, it was not known how these solutions could be given in an explicit formula. The approaches proposed in the literature up until now solve this problem numerically for a given fixed value of the input sources. Regarding the numerical solution of PL resistive circuits, methods as proposed in [7] and [8] can be used.

In this paper we will extend the results from [9], starting with a strict proof to obtain the explicit formulas for the solution of a LCP (in [9] only the outline of the proof was given). The proof yields a concrete algorithm which constructs the representations. The expressions make use of the operation of taking the nonnegative part. Furthermore, it will be shown that the nesting of this operator is at most n for an n -dimensional LCP. The results will then be discussed from the viewpoint of PL electrical networks and their properties.

One of the advantages to the approach is that an analytical expression for the solution arises, that can be used in further analytic operations. In contrast, homotopy (like Katzenelson [10]) and iterative algorithms (e.g., the modulus algorithm [1]) known up to now do not yield an explicit formula for the solution, but apply numerical algorithms that find a particular solution corresponding to a given value of the input sources. Because the presented methodology is algorithmic, the procedure can be used in symbolic analysis to obtain relationships between network elements and their large signal behaviors. Up to now, symbolic analysis could only be applied to small-signal behavior.

In Section II preliminary information will be provided. In Section III, a small network example will be discussed. In Section IV, the complete proof to obtain explicit expressions for the solutions of class-P LCP's will be given. This result will be discussed from a network point of view in section V. Some network examples are provided in Section VI. In Section VII it will be shown that the results can also be of advantage when used in synthesis problems, e.g., constrained optimization. Some conclusions are given in Section VIII.

II. PRELIMINARY

The PL function or mapping f is assumed to be continuous. The function or mapping can be deduced from a PL electrical

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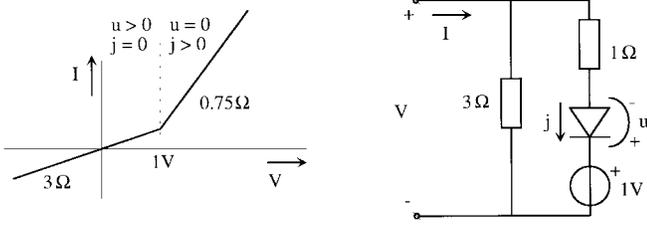


Fig. 1. A PL function and its electrical network equivalent based on the ideal diode.

network or vice versa. Such an electrical network is realized with PL components, linear resistors, (controlled) sources, and ideal diodes. A simple example is provided in Fig. 1, where the ideal diode determines the boundary between the two line segments in the characteristic. The definition of an ideal diode in terms of the current j through the diode and the voltage u across the diode is given as

$$j \geq 0, \quad u \geq 0, \quad uj = 0. \quad (1)$$

The orientation of the diode voltage u is reversed in polarity, compared to what normally is used in network theory. Take into account that with this limited set of elements, very complex circuits can be modeled. For instance, a transistor can be accurately modeled with resistors, controlled voltage sources, and ideal diodes only [11].

Most often, the function or mapping f can be written in a certain PL model description and is inherently confronted with the LCP [6]. This problem is defined as obtaining the solution for u and j of

$$j = Du + q \quad (2)$$

under the restriction $u \geq 0, j \geq 0$ and $u^T \cdot j = 0$, known as the complementary condition. Consider again the network in Fig. 1. The currents in the network are equal to

$$\begin{aligned} I &= \frac{1}{3}V + 1 \cdot (V + u - 1) \\ j &= V + u - 1. \end{aligned} \quad (3)$$

The second equation of (3) together with (1) defines the state of the diode. When V is at least 1 V, u has to be equal to zero. The current I can then be obtained from the first equation of (3). When V is less than 1 V, j is equal to zero. Defining $q = V - 1$, the second equation in (3) becomes $j = u + q$ and is therefore a special case of the LCP (2).

The LCP (2) yields a unique solution for any q under the restriction that $D \in P$, where class P is defined as in [12].

Definition 1: Matrix D is of class P if and only if $\forall_{x \neq 0} \exists_k$ such that $x_k \cdot (Dx)_k > 0$. Class-P matrices can also be recognized as matrices having positive determinants for every principal submatrix.

In the example, $D = 1$ and, thus, is of class-P and the unique solution is $j = q, u = 0$ in case $q \geq 0$; else $j = 0, u = -q$. Normally this unique solution can be obtained using solution strategies as developed by, e.g., Lemke and Katzenelson [10], [13]. If a PL network, reformulated as (2), has $D \in P$, we will refer to this network as a class-P network.

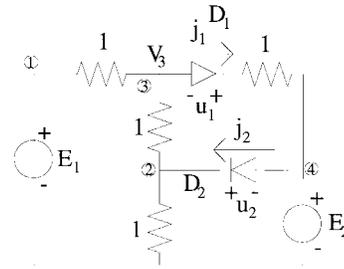


Fig. 2. A PL network with two ideal diodes, D_1 and D_2 , several identical resistors, and two voltage sources.

In this sequel the $[\cdot]$ operator¹ is defined as

$$y = [x] \rightarrow \begin{cases} y = x, & x \geq 0 \\ y = 0, & x < 0 \end{cases} \quad (4)$$

and was previously used to extend the PL model descriptions of Chua [9], [14]. A conclusion of the network example is that the unique solution is given as $j = [q], u = [-q]$. Note also that $[x] = \frac{1}{2}(|x| + x)$ and, therefore, a link does exist to the explicit model descriptions, e.g., the ones proposed by Chua. Chua's model descriptions are normally expressed using the absolute-value operator [2]–[4].

III. A NETWORK EXAMPLE

Before introducing the mathematical concepts, an example is discussed to outline the approach. To this purpose, consider the network of Fig. 2, consisting of two ideal diodes, several resistors, and two voltage sources. Each resistor has a value of 1 Ω. Taking the orientation of the diode voltages and currents, as indicated in the figure, the circuit behavior is defined by the following LCP:

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3}E_1 + E_2 \\ \frac{1}{3}E_1 - E_2 \end{pmatrix}, \\ u, j &\geq 0, \quad u^T j = 0. \end{aligned} \quad (5)$$

Note that the LCP matrix is of class P and, consequently, the network is of class P and therefore has a unique solution for any value of voltage sources. Suppose that diode D_1 is off, i.e., $j_1 = 0$ and $u_1 \geq 0$. Then $u_1 = V_4 - V_3$ and u_1 is given in terms of the current through D_2 as

$$u_1 = -\frac{2}{3}E_1 + E_2 - \frac{1}{3}j_2 \geq 0 \quad (6)$$

which follows directly from (5) as well as from the network. However, the values E_1 and E_2 might be chosen such that $-\frac{2}{3}E_1 + E_2 - \frac{1}{3}j_2 < 0$ holds. According to the definitions of the ideal diode, it follows immediately that $j_1 > 0$ and, hence, $u_1 = 0$. Consequently, using (4) u_1 can be expressed as

$$u_1 = [-\frac{2}{3}E_1 + E_2 - \frac{1}{3}j_2]. \quad (7)$$

The problem is now to determine the value of j_2 . This value is however only of importance if $-\frac{2}{3}E_1 + E_2 - \frac{1}{3}j_2 \geq 0$, otherwise the value for u_1 is known, namely, $u_1 = 0$ independently of the value of $-\frac{2}{3}E_1 + E_2 - \frac{1}{3}j_2$. Hence,

¹Note that the definition implies $[x] = \max(x, 0)$ and that in the LCP-literature (see, e.g., [6]) the symbol x^+ is used to denote $[x]$.

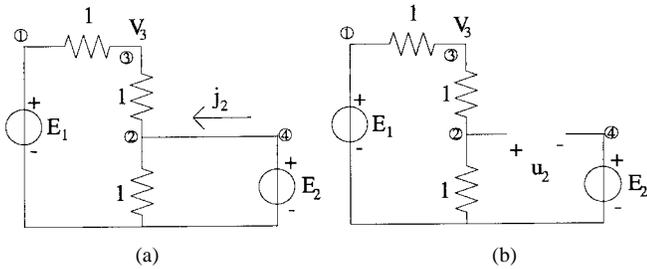


Fig. 3. The network of Fig. 2 for the situation with $j_1 = 0$, i.e., D_1 is replaced by an open circuit. Shown are the situations (a) in which D_2 is conducting $u_2 = 0$ and (b) in which D_2 is blocked $j_2 = 0$.

j_2 can be determined at $j_1 = 0$, thus leaving out diode D_1 . The expression for j_2 is then easily found from the network, yielding (see also Fig. 3)

$$j_2 = \frac{3}{2} E_2 - \frac{1}{2} E_1 \geq 0, \quad u_2 = 0$$

$$j_2 = 0, \quad u_2 > 0$$

or

$$j_2 = \lfloor \frac{3}{2} E_2 - \frac{1}{2} E_1 \rfloor. \tag{8}$$

We recall that (8) is only valid under the restriction $j_1 = 0$ and thus is an intermediate result, only valid within the context of (7), i.e., only valid to produce the solution for (5). Combining (7) with (8) yields finally the explicit solution for u_1 in terms of the values of the voltage sources E_1 and E_2

$$u_1 = \lfloor -\frac{2}{3} E_1 + E_2 - \frac{1}{3} \lfloor \frac{3}{2} E_2 - \frac{1}{2} E_1 \rfloor \rfloor. \tag{9}$$

In a similar way the expression for u_2 can be obtained

$$u_2 = \lfloor \frac{1}{3} E_1 - E_2 - \frac{1}{3} \lfloor \frac{2}{3} E_1 - E_2 \rfloor \rfloor. \tag{10}$$

Note that many electrical PL networks are class-P networks [2]–[4], [11]. Exceptions are, for instance, networks with hysteretic behavior. In the following section the underlying methodology will be discussed.

IV. EXPLICIT FORMULAS

In this section we will present a construction for solutions in explicit form for any class-P LCP. Let us start with the one-dimensional (1-D) situation, i.e., $P \in \mathbb{R}$ and the LCP looks like

$$j = du + q$$

$$u, j \geq 0, \quad uj = 0 \tag{11}$$

with d and q scalars. Because $d \in P$, we know that $d > 0$ and we easily observe that

$$q > 0 \Rightarrow j = q, \quad u = 0$$

$$q \leq 0 \Rightarrow j = 0, \quad u = -\frac{q}{d}. \tag{12}$$

This equation can be written in a single explicit formula

$$j = \lfloor q \rfloor, \quad u = \lfloor -\frac{q}{d} \rfloor \tag{13}$$

which is a basic step in the construction of the solution. Now the following theorem is given:

Theorem 1: For any n -dimensional LCP of the form $j = Du + q$, u and j fulfilling the nonnegative and complementary conditions and $D \in P$, the solution can be given in terms of solutions for the corresponding $(n - 1)$ -dimensional LCP.

Define the symbol α for the index set $\{2, \dots, n\}$ and let us rewrite the n -dimensional LCP into the format

$$j_1 = D_{11}u_1 + D_{1\alpha}u_\alpha + q_1$$

$$j_\alpha = D_{\alpha 1}u_1 + D_{\alpha\alpha}u_\alpha + q_\alpha \tag{14}$$

with

$$j_\alpha = (j_2 \ \dots \ j_n)^T, \quad u_\alpha = (u_2 \ \dots \ u_n)^T$$

$$D_{\alpha\alpha} = \begin{pmatrix} D_{22} & \dots & D_{2n} \\ \vdots & \ddots & \vdots \\ D_{n2} & \dots & D_{nn} \end{pmatrix}, \quad D_{\alpha 1} = \begin{pmatrix} D_{21} \\ \vdots \\ D_{n1} \end{pmatrix},$$

$$q_\alpha = (q_2 \ \dots \ q_n)^T$$

where the first entry is taken as the leading entry without loss of generality. The unique solution of (14) will be partitioned and denoted as $j = (j_1 \ j_\alpha)^T$ and $u = (u_1 \ u_\alpha)^T$. Let us also define another $(n - 1)$ -dimensional LCP to be derived from (14) by taking $u_1 = 0$,

$$\tilde{j} = D_{\alpha\alpha}\tilde{u} + q_\alpha \tag{15}$$

also having a unique solution because $D_{\alpha\alpha} \in P$, as well.

The key issue in the proof of this theorem is to show that

$$j_1 = \lfloor D_{1\alpha}\tilde{u} + q_1 \rfloor. \tag{16}$$

Suppose $D_{1\alpha}\tilde{u} + q_1 \geq 0$ then from (4) $j_1 = D_{1\alpha}\tilde{u} + q_1$, $u_1 = 0$. Furthermore, this result implies $u_\alpha = \tilde{u}$ and $j_\alpha = D_{\alpha\alpha}\tilde{u} + q_\alpha$. Because of the assumption that D is of class P, this is a unique solution of the LCP. Now suppose $D_{1\alpha}\tilde{u} + q_1 < 0$, then there can be no solution of the LCP with $u_1 = 0$ because it would imply that $j_1 = D_{1\alpha}\tilde{u} + q_1 < 0$. This is in contradiction to the nonnegative and complementary assumption for j . So, in this case, the solution of the LCP must have $j_1 = 0$. It follows that, indeed, (16) holds.

A straightforward way to find u_1 is then given by the formula $u_1 = -(D^{-1}j)_1 - q_1$. Therefore, with (16) the solution of the n -dimensional LCP is expressed in terms of solutions of the corresponding $(n - 1)$ -dimensional problem (15). ■

From the proof we obtain the remarkable property

$$\lfloor D_{1\alpha}u_\alpha + q_1 \rfloor = \lfloor D_{1\alpha}\tilde{u} + q_1 \rfloor \tag{17}$$

i.e., the value of $\lfloor D_{1\alpha}u_\alpha + q_1 \rfloor$ is fully determined by the value of the corresponding $(n - 1)$ -dimensional subproblem. Furthermore, this property also means that $\text{sign}(D_{1\alpha}u_\alpha + q_1) = \text{sign}(D_{1\alpha}\tilde{u} + q_1)$, which is graphically depicted in Fig. 4.

Theorem 2: For any n -dimensional LCP of the form $j = Du + q$, u and j fulfilling the nonnegative and complementary conditions and $D \in P$, the solution can be given in explicit form and obtained in a recursive way.

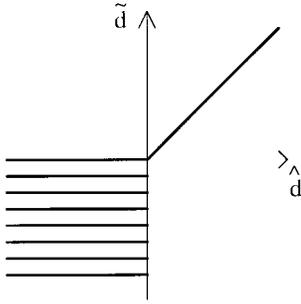


Fig. 4. The relation between $J = D_{1\alpha}u_\alpha + q_1$ and $\tilde{J} = D_{1\alpha}\tilde{u}_\alpha + q_1$ for the 2-D situation.

Proof: For $n = 1$, the solution is given in an explicit form by (13). Consider the case $n = 2$. Then, using (16), we have $j_1 = [D_{12}\tilde{u}_2 + q_1]$ and $j_2 = [D_{21}\tilde{u}_1 + q_2]$. Scalar \tilde{u}_2 is obtained from the corresponding 1-D LCP $\tilde{j}_2 = D_{22}\tilde{u}_2 + q_2$, leading to $\tilde{u}_2 = \lfloor -(q_2/D_{22}) \rfloor$ and, in the same way, $\tilde{u}_1 = \lfloor -(q_1/D_{11}) \rfloor$. Consequently

$$\begin{aligned} j_1 &= \left[q_1 + D_{12} \left\lfloor -\frac{q_2}{D_{22}} \right\rfloor \right] \\ j_2 &= \left[q_2 + D_{21} \left\lfloor -\frac{q_1}{D_{11}} \right\rfloor \right] \\ u &= -D^{-1}j - q. \end{aligned} \quad (18)$$

Therefore, the solutions j for the two-dimensional (2-D) case can be found by making use of the expressions found for u in the $n = 1$ case and are again written in an explicit form. Consider now the case $n = 3$ and, with α representing the index set $\{2, 3\}$, we have $j_1 = [D_{1\alpha}\tilde{u}_\alpha + q_1]$ and

$$\tilde{j} = \begin{pmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{pmatrix} \tilde{u} + \tilde{q}. \quad (19)$$

The solutions of (19) can be given in an explicit form similar to (18). So, the solution j_1 for the three-dimensional (3-D) case has an explicit form and is obtained via the solution of the 2-D subproblem. In a similar way, the explicit expressions for j_2 and j_3 can be obtained from their corresponding subLCP's. The solutions for u can be obtained from $u = -D^{-1}j - q$.

According to Theorem 1, the solution of the n -dimensional LCP can be expressed in terms of solutions for the corresponding $(n - 1)$ -dimensional LCP. Because of the recursion, the solutions for the n -dimensional LCP are found in an explicit form. The expressions are described in terms of explicit formulas starting with the solutions of the 1-D subproblem. ■

To obtain the explicit solutions, the procedure has common elements with Cramer's rule for solving linear algebraic equations, using Laplace expansion of determinants, in which each determinant is computed in terms of all its minors. In our case, system (14) is solved in a similar way, by solving all its principal subsystems (15).

Theorem 2 leads to the following corollary.

Corollary 3: The nesting of the $\lfloor \cdot \rfloor$ -operator is at maximum n and obtaining the explicit formulas demands an exponential amount of operations in n .

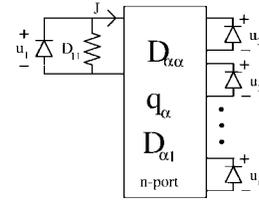


Fig. 5. The n port loaded with $(n - 1)$ ideal diodes and at port one loaded with a small network.

Proof: For $n = 1$, the nesting is at maximum one and for $n = 2$, according to (18), the maximum is two. Because the solution for case n is recursively obtained from lower dimensional LCP's, the maximum nesting of the operator is n .

Concerning the complexity, for $n = 3$ three 2-D LCP's have to be solved. Each 2-D LCP is solved by means of solving two scalar LCP's. In the same way, the n -dimensional LCP is solved by n LCP's of dimension $(n - 1)$, each $(n - 1)$ -dimensional LCP by solving $(n - 1)$ LCP's of dimension $n - 2$, and so forth. Therefore, the solution is obtained after an exponential amount of operations in n . ■

Due to the recursive nature of the process, the solution for each component of u and j consists of a linear combination of nested $\lfloor \cdot \rfloor$ operators of increasing depth. This property was already observed in [14] for lower triangular matrices D . Obviously, in running the algorithm some subsystems may show up in a form which allows for a direct explicit solution, thus reducing the operation count.

Note that any reordering of the equations will produce a different but equivalent solution. Of course, the method, as such, can also be used as a solution algorithm for numerically given D and q but, due to the high complexity, this does not seem to appear to be an advantage as compared to using other well-known algorithms. For low-order problems, however, explicit solutions might be efficient as well.

V. NETWORK VIEWPOINT

It is possible to explain the main steps in the methodology in terms of network properties. In general, a PL network can be treated as a linear multiport, loaded with an ideal diode at each port [1]. This multiport is then described by the general LCP. Equation (14) can be treated as an n port, loaded with $(n - 1)$ ideal diodes and, at one port, loaded with a small network, as depicted in Fig. 5. The diode current in the small network is equal to $j_1 = D_{11}u_1 + J$. Here, J represents a current pulled into port one and completely determined by the remaining network $J = D_{1\alpha}u_\alpha + q_1$. The port behavior with respect to the $(n - 1)$ other ideal diodes is described by matrix $D_{\alpha\alpha}$ and vector q_α , while the influence of diode one to the other diodes is described by $D_{\alpha 1}$. The complete network is therefore described by (14). Note that the diagonal elements of $D_{\alpha\alpha}$ represent the impedances of the $(n - 1)$ ports. These impedances must be positive because the n -port is of class P. Entry D_{i1} represents the impedance seen between port one and port i . If $u_1 = 0$, then clearly $j_1 = J$ and, obviously, if $J \geq 0$, then $u_1 = 0$.

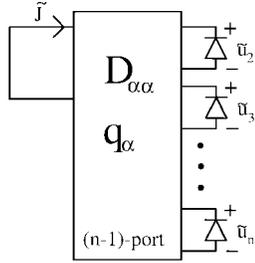


Fig. 6. The $((n - 1))$ port as constructed from the n port by setting $u_1 = 0$. The behavior of the port is completely determined by $D_{\alpha\alpha}$ and q_α .

Consider also the $(n - 1)$ -port system, derived from the n port by setting $u_1 = 0$. The $(n - 1)$ -port behavior is completely determined by matrix $D_{\alpha\alpha}$ and vector q_α (see also Fig. 6). For convenience, the branch with current $J = D_{1\alpha}\tilde{u} + q_1$ is drawn outside the box.

Consider the situation that $J \geq 0$, yielding $j_1 \geq 0$ and, thus, $u_1 = 0$. For this situation port one is short circuited and current J becomes equal to current \tilde{J} and the two systems become equal. The behavior of the n port is complete described by the behavior of the $(n - 1)$ port. As long as the sign of J remains positive, the value is not important. Suppose that $\tilde{J} \geq 0$, then an additional ideal diode may be added parallel to the branch in which \tilde{J} is flowing. This is true as long as, for this additional diode, we have $u = 0$. The $(n - 1)$ port in this situation is equal to an n port. The conclusion is that the behavior of the two systems is identical if $J \geq 0$ or $\tilde{J} \geq 0$. Consider the $(n - 1)$ port and assume that $\tilde{J} < 0$. Knowing that the n port is constructed from the $(n - 1)$ port by simple replacement of the branch with \tilde{J} with an ideal diode and a resistor, current J must be negative as well. However, the value of J may differ from that of \tilde{J} due to the influence of the impedance of the port to the remaining network. This is exactly described by (17).

The consequence is that voltage u_1 and current j_1 of the n port are completely determined by current \tilde{J} in the $(n - 1)$ port. Now we can repeat the procedure by removing diode D_2 of the $(n - 1)$ port, yielding a $(n - 2)$ port. Finally, a one port will remain, consisting of a parallel network of one ideal diode D_n , one resistor with value D_{nn} , and one current source with value q_n . The solution for this one port can be obtained directly as $j_n = \lfloor q_n \rfloor$, $u_n = \lfloor -(q_n/D_{nn}) \rfloor$ and is used as an intermediate result to describe the solutions for u_1 and j_1 in explicit formulas. This yields the same procedure as the recursive construction in Theorem 2.

VI. NETWORK EXAMPLES

Let us return to the example of Section III, for which the network was depicted in Fig. 2, having the LCP according to

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} E_1 + E_2 \\ \frac{1}{3} E_1 - E_2 \end{pmatrix}$$

$$u, j \geq 0, \quad u^T j = 0. \tag{20}$$

Following Section IV, the subsystem is defined as a 1-D LCP according to

$$\tilde{u}_2 = \frac{2}{3} \tilde{j}_2 + \frac{1}{3} E_1 - E_2. \tag{21}$$

For the systems (20) and (21) we have

$$\begin{aligned} D_{1\alpha} j_\alpha + q_1 &= -\frac{2}{3} E_1 + E_2 - \frac{1}{3} j_2 \\ D_{1\alpha} \tilde{j} + q_1 &= -\frac{2}{3} E_1 + E_2 - \frac{1}{3} \tilde{j}_2. \end{aligned} \tag{22}$$

Applying (16) the expression for u_1 yields

$$u_1 = \lfloor -\frac{2}{3} E_1 + E_2 - \frac{1}{3} \tilde{j}_2 \rfloor. \tag{23}$$

The expression for \tilde{j}_2 follows from the 1-D subsystem (21) and is given as

$$\begin{aligned} \tilde{j}_2 &= \frac{3}{2} (-\frac{1}{3} E_1 + E_2) + \frac{3}{2} \tilde{u}_2 \\ \tilde{j}_2 \geq 0, \tilde{u}_2 &\geq 0, \tilde{j}_2 \cdot \tilde{u}_2 = 0 \end{aligned} \tag{24}$$

having the solution

$$\tilde{j}_2 = \frac{3}{2} \lfloor -\frac{1}{3} E_1 + E_2 \rfloor = \lfloor \frac{3}{2} E_2 - \frac{1}{2} E_1 \rfloor. \tag{25}$$

Combining (25) with (23) yields, finally, the explicit solution

$$u_1 = \lfloor -\frac{2}{3} E_1 + E_2 - \frac{1}{3} \lfloor \frac{3}{2} E_2 - \frac{1}{2} E_1 \rfloor \rfloor$$

exactly the same as (9), obtained via straightforward network analysis. The expression for node voltage V_3 can be given as

$$\begin{aligned} V_3 &= \frac{1}{3} (E_1 + 2E_2 - \lfloor -\frac{2}{3} E_1 + E_2 - \frac{1}{3} \lfloor \frac{3}{2} E_2 - \frac{1}{2} E_1 \rfloor \rfloor \\ &\quad + \lfloor \frac{1}{3} E_1 - E_2 - \frac{1}{3} \lfloor \frac{3}{2} E_1 - E_2 \rfloor \rfloor). \end{aligned}$$

The $\lfloor \cdot \rfloor$ operator is nested two times, as was to be expected, because in [4] it was proven that a 2-D PL mapping needs a double nesting of the absolute-sign operator. Fig. 7 shows the dc-dc transfer characteristic $V_3 = f(E_1, E_2)$ for a certain interval of E_1 and E_2 .

As an other example, consider the PL network of Fig. 8, yielding the following network equations:

$$\begin{aligned} j_1 &= q_1 + b_1 u_1 + \sum_{i=2}^n b_i (u_1 - u_i) \\ \forall_{i \in \{2, \dots, n\}} j_i &= q_i + c_i u_i + b_i (u_i - u_1) \end{aligned} \tag{26}$$

where the nonnegativity and complementary conditions should still be obeyed. Reformulating (26) leads to the following class-P LCP:

$$\begin{pmatrix} j_1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & & \\ \vdots & & \ddots & \\ d_{n1} & & & d_{nn} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$j \geq 0, \quad u \geq 0, \quad u^T j = 0 \tag{27}$$

with

$$\begin{aligned} d_{11} &= b_1 + b_2 \\ \forall_{i \in \{2, \dots, n\}} d_{1i} &= d_{i1} = -b_i \\ \forall_{i \in \{2, \dots, n\}} d_{ii} &= c_i + b_i \\ \forall_{i \in \{2, \dots, n\}} d_{ij} &|_{j \neq 1, i \neq 1, j \neq i} = 0. \end{aligned}$$

The $(n - 1)$ -dimensional subsystem is defined as

$$\begin{pmatrix} j_2 \\ \vdots \\ j_n \end{pmatrix} = \begin{bmatrix} d_{22} & & \\ & \ddots & \\ & & d_{nn} \end{bmatrix} \begin{pmatrix} u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$j \geq 0, \quad u \geq 0, \quad u^T j = 0. \tag{28}$$

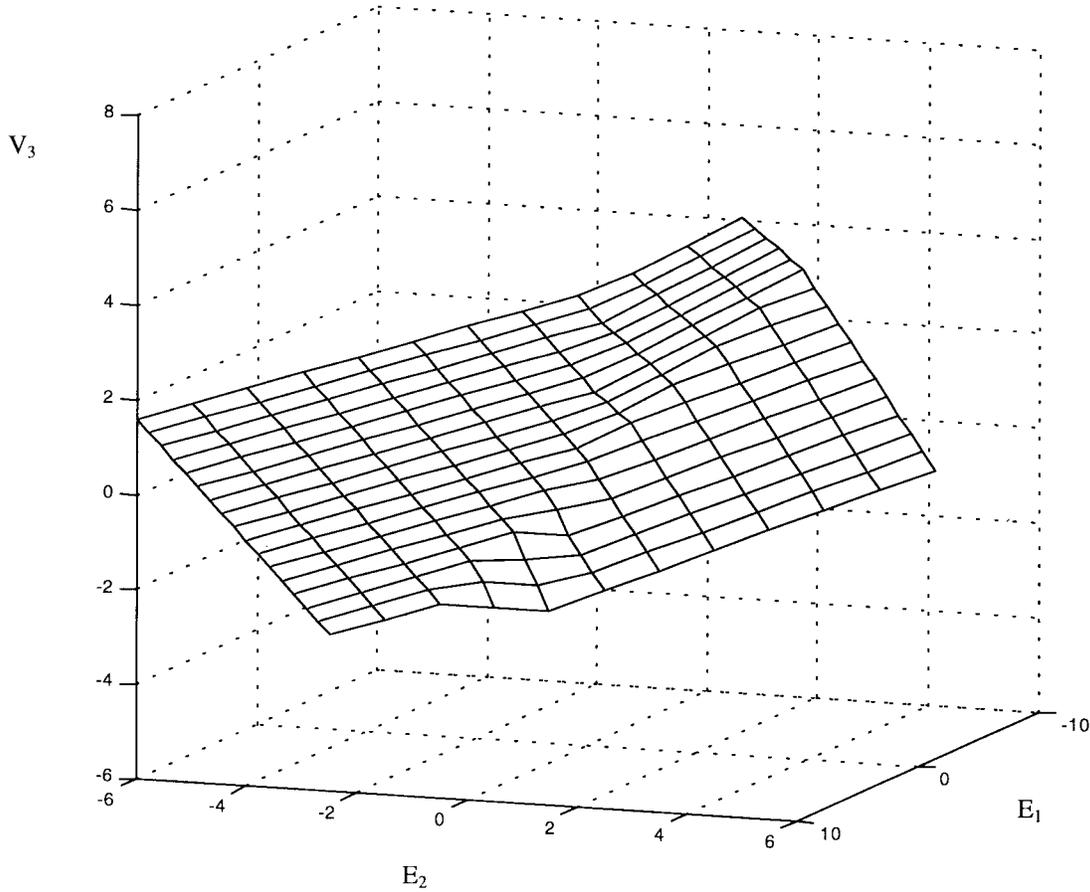


Fig. 7. The dc-dc transfer characteristic $V_3 = f(E_1, E_2)$ for a certain interval of E_1 and E_2 .

This subsystem has a diagonal matrix structure and, therefore, the equations are independent of each other. This property follows also from the network, each subnetwork (one ideal diode, resistor c_i , and current source q_i) behaves independently from the others. The solution of a single subsystem is easily found and expressed as

$$\forall_{i \in \{2, \dots, n\}} j_i = \lfloor q_i \rfloor, u_i = \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor \quad (29)$$

meaning that if diode i is a short-circuit, current j_i is equal to the value of the current source and if the diode is an open circuit, voltage u_i is equal to $-(q_i/d_{ii})$. Note that (29) is only valid within the frame of the complete solution. From (27) combined with (29) we obtain

$$\begin{aligned} j_1 &= d_{11}u_1 + \sum_{i=2}^n d_{1i}u_i + q_1 \\ j_1 &= d_{11}u_1 + \sum_{i=2}^n d_{1i} \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor + q_1. \end{aligned} \quad (30)$$

Completing the procedure to obtain the explicit formula leads to

$$\begin{aligned} j_1 &= \left\lfloor \sum_{i=2}^n d_{1i} \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor + q_1 \right\rfloor \\ u_1 &= \left\lfloor -\sum_{i=2}^n \frac{d_{1i}}{d_{11}} \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor - \frac{q_1}{d_{11}} \right\rfloor. \end{aligned} \quad (31)$$

The remaining currents and voltages in (27) are expressed in terms of u_1

$$\forall_{i \in \{2, \dots, n\}} j_i = d_{i1}u_1 + d_{ii}u_i + q_i \quad (32)$$

yielding

$$\begin{aligned} \forall_{i \in \{2, \dots, n\}} j_i &= \lfloor q_i + d_{i1}u_1 \rfloor \\ &= \left\lfloor q_i + d_{i1} \left\lfloor -\sum_{i=2}^n \frac{d_{1i}}{d_{11}} \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor - \frac{q_1}{d_{11}} \right\rfloor \right\rfloor \end{aligned} \quad (33)$$

and

$$\forall_{i \in \{2, \dots, n\}} u_i = \left\lfloor -\frac{q_i}{d_{ii}} - \frac{d_{i1}}{d_{ii}} \left\lfloor -\sum_{i=2}^n \frac{d_{1i}}{d_{11}} \left\lfloor -\frac{q_i}{d_{ii}} \right\rfloor - \frac{q_1}{d_{11}} \right\rfloor \right\rfloor. \quad (34)$$

Expression (31) yields a level-two nesting of the $\lfloor \cdot \rfloor$ operator. In Fig. 8, we can connect subcircuit two to the other subcircuits, using additional resistors. This means that the second row and second column in (27) contain nonzero elements. Repeating the procedure yields a level-three nesting for the currents and voltages of diode one, and a level-two nesting for diode two. We may continue this strategy until the $(n-1)$ subcircuit is connected with all other subnetworks. At this moment, we have a full matrix D and we need a level- n

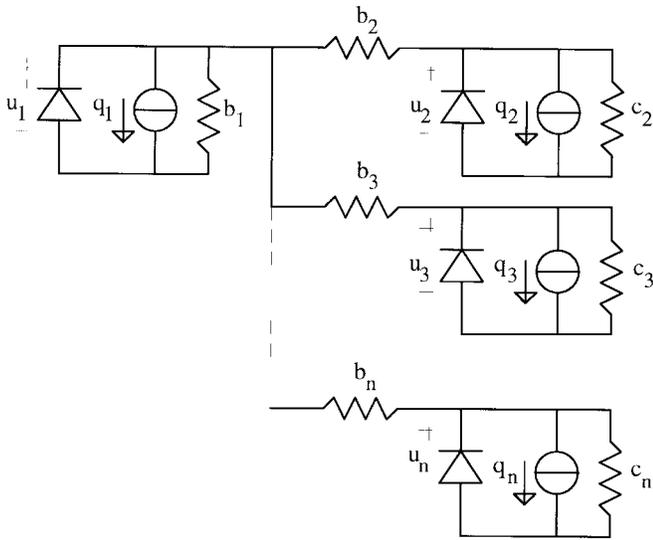


Fig. 8. A PL network with a special LCP matrix D . Note that each subnetwork i , consisting of resistor c_i , current source q_i , and ideal diode i is only connected with subnetwork one via resistors b_i . Setting $u_1 = 0$ separates all diode circuits from each other, eliminating the interactions and yielding a diagonal matrix $D_{\alpha\alpha}$.

nesting to describe the explicit formula for the voltages and currents for any diode voltage or current.

This result is in agreement with [14] where for a lower triangular matrix $D \in R^{n \times n}$ it was already shown that to express the solutions, the $[\cdot]$ operator is nested up to level n . For a lower triangular matrix D , it is clear that the unique solution can be obtained explicitly in a top-down recursive manner. Therefore, it is not surprising that for a class-P matrix, having also a unique solution, the solutions can be expressed with a nesting of the $[\cdot]$ operator n -levels deep at maximum. Obviously, if the network is degenerated, the degree of nesting will decrease.

VII. CONSTRAINED OPTIMIZATION

Constrained quadratic minimization is an important research topic in mathematical programming. In circuit synthesis and analysis, the problem arises when, for instance, several component values must be obtained while minimizing a certain parameter, e.g., power dissipation. This type of problem can be defined as

$$\text{minimize } \frac{1}{2} x^T M x + q^T x, \quad \text{under the condition } x \geq 0 \quad (35)$$

(see for instance [15] and [16]).

The solution for x is known to satisfy the general LCP $y = Mx + q$ with the nonnegative and complementary conditions $y \geq 0$, $x \geq 0$, $y^T \cdot x = 0$. Matrix M must be symmetric positive definite to yield a solution for the constrained problem. Note that the class of symmetric positive definite matrices is a subclass of class-P matrices.

As example, consider the problem

$$\text{min}\{x_1^2 + x_1 x_2 + x_2^2 + a x_1 + b x_2\}, \quad x_1, x_2 \geq 0 \quad (36)$$

yielding the LCP

$$y = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} a \\ b \end{pmatrix} \\ y \geq 0, \quad x \geq 0, \quad x^T y = 0. \quad (37)$$

The solution of (37) along the lines of the methodology can be found as

$$x_1 = \left[-\frac{2}{3} a + \frac{1}{3} b - \frac{1}{3} \left[-\frac{1}{2} a + b \right] \right] \\ x_2 = \left[\frac{1}{3} a - \frac{2}{3} b - \frac{1}{3} \left[a - \frac{1}{2} b \right] \right]. \quad (38)$$

VIII. CONCLUSIONS

In this paper a methodology is presented to obtain explicit formulas for the solutions of PL networks. The methodology is proved for the case where the network is of class P. Consequently, for any LCP of class P, explicit formulas can be obtained. Because the LCP is known in many more research fields, such as constrained optimization and linear programming, the applications of the proposed method can be used in a wider area than only electrical network analysis.

An important consequence of the proof is that for an n -dimensional system (in terms of the number of ideal diodes) the solution can be expressed in terms of, at maximum, n nested $[\cdot]$ operators. To obtain the solutions will require an exponential amount of operations in terms of the dimension of the problem.

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