

Solution to problem 97-16 : A parametric integral arising from a mixed boundary value problem for the Laplacian

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By making the substitution $x^2 = -1 + 1/(4t(1 - t))$, this becomes

$$\begin{aligned}
 J(a, b) &= \frac{1}{2} e^{-4b} \int_0^\infty \left\{ \left[\sqrt{x^2 + 1} - x \right]^{a-1} + \left[\sqrt{x^2 + 1} + x \right]^{a-1} \right\} e^{-4bx^2} (x^2 + 1)^{-1-a/2} dx \\
 &= \frac{1}{2} e^{-4b} \int_{-\infty}^\infty \left[\sqrt{x^2 + 1} - x \right]^{a-1} e^{-4bx^2} (x^2 + 1)^{-a/2-1} dx.
 \end{aligned}$$

By expanding

$$\left[\sqrt{x^2 + 1} - x \right]^{a-1} = (x^2 + 1)^{(a-1)/2} \sum_{k=0}^\infty \binom{a-1}{k} (-1)^k x^k (x^2 + 1)^{-k/2},$$

we get

$$J(a, b) = \frac{1}{2} e^{-4b} \sum_{n=0}^\infty \binom{a-1}{2n} \int_0^\infty \frac{t^{n-1/2}}{(t+1)^{n+3/2}} e^{-4bt} dt.$$

Symmetry has been used to eliminate the odd terms in the sum and $t = x^2$. The remaining integral is a tabulated Laplace transform, yielding

$$J(a, b) = \frac{1}{2^a} e^{-2b} \sum_{n=0}^\infty \Gamma\left(n + \frac{1}{2}\right) \binom{a-1}{2n} W_{-n-\frac{1}{2}, -\frac{1}{2}}(4b).$$

When a is an integer, the series terminates. This yields a closed expression as a sum of Whittaker functions, as pointed out by the proposers.

Also solved by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium).

A Parametric Integral Arising from a Mixed Boundary Value Problem for the Laplacian

Problem 97-16, by LUCIO R. BERRONE (Instituto de Matemática “Beppo Levi,” Rosario, Argentina).

Prove that, for every $0 < \alpha < 2\pi$,

$$\int_0^\alpha \ln \left(\frac{\sin \frac{\alpha-\theta}{2} + 2 \sin \frac{\alpha}{4} \sqrt{\sin \frac{\alpha-\theta}{2} \sin \frac{\theta}{2} + \sin \frac{\theta}{2}}}{\sin \frac{\alpha-\theta}{2} - 2 \sin \frac{\alpha}{4} \sqrt{\sin \frac{\alpha-\theta}{2} \sin \frac{\theta}{2} + \sin \frac{\theta}{2}}} \right) d\theta = -4\pi \ln \cos \frac{\alpha}{4}.$$

The integral of the left-hand side arises in the analysis of the solution to the equation $\Delta u = 0$ in the unitary ball $B_1(0) \subset \mathbb{R}^2$ with mixed boundary conditions given by $(\partial u / \partial n)|_{\Gamma_1} \equiv 1$, $u|_{\Gamma_0} \equiv 0$ when Γ_1 is an arc of length α , and $\Gamma_0 = \partial B_1(0) \setminus \Gamma_1$.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, The Netherlands).

Replace α by 2α , and change the integration variable θ into $\alpha + \theta$. Then the integrand reduces to

$$\ln \left(\frac{\sin \frac{\alpha-\theta}{2} + 2 \sin \frac{\alpha}{2} \sqrt{\sin \frac{\alpha-\theta}{2} \sin \frac{\alpha+\theta}{2} + \sin \frac{\alpha+\theta}{2}}}{\sin \frac{\alpha-\theta}{2} - 2 \sin \frac{\alpha}{2} \sqrt{\sin \frac{\alpha-\theta}{2} \sin \frac{\alpha+\theta}{2} + \sin \frac{\alpha+\theta}{2}}} \right) = \ln \left(\frac{\cos \frac{\theta}{2} + \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}}{\cos \frac{\theta}{2} - \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \right).$$

Thus it is to be proved that

$$I(\alpha) = \int_{-\alpha}^{\alpha} \ln \left(\frac{\cos \frac{\theta}{2} + \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}}{\cos \frac{\theta}{2} - \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} \right) d\theta = -4\pi \ln \cos \frac{\alpha}{2}$$

for $0 < \alpha < \pi$.

We evaluate the derivative $I'(\alpha)$ as follows:

$$\begin{aligned} I'(\alpha) &= \int_{-\alpha}^{\alpha} \frac{2 \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\alpha}{2} + \sin^2 \frac{\theta}{2}} \cdot \frac{\frac{1}{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} d\theta \\ &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \int_{-\alpha}^{\alpha} \frac{\cos \frac{\theta}{2}}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}} d\theta \\ &= \frac{2 \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \int_{-\sin \frac{\alpha}{2}}^{\sin \frac{\alpha}{2}} \frac{dt}{\sqrt{\sin^2 \frac{\alpha}{2} - t^2}} = 2\pi \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}. \end{aligned}$$

By integration of $I'(\alpha)$, starting from $I(0) = 0$, we obtain the desired result

$$I(\alpha) = \int_0^{\alpha} 2\pi \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} d\theta = -4\pi \ln \cos \frac{\alpha}{2}.$$

Solution by H.-J. SEIFFERT (Berlin).

We substitute $\theta = 2\phi + \alpha/2$, $-\alpha/4 \leq \phi \leq \alpha/4$. Then,

$$\sin \frac{\alpha - \theta}{2} + \sin \frac{\theta}{2} = \sin \left(\frac{\alpha}{4} - \phi \right) + \sin \left(\frac{\alpha}{4} + \phi \right) = 2 \sin \left(\frac{\alpha}{4} \right) \cos \phi$$

and

$$\sin \frac{\alpha - \theta}{2} \sin \frac{\theta}{2} = \sin^2 \left(\frac{\alpha}{4} \right) \cos^2 \phi - \cos^2 \left(\frac{\alpha}{4} \right) \sin^2 \phi = \cos^2 \phi - \cos^2 \frac{\alpha}{4}.$$

Hence, if $I(\alpha)$ denotes the integral in question, we have $I(\alpha) = 2J(\alpha) - 2K(\alpha)$, where

$$J(\alpha) = \int_{-\alpha/4}^{\alpha/4} \ln \left(\cos \phi + \sqrt{\cos^2 \phi - \cos^2 \frac{\alpha}{4}} \right) d\phi$$

and

$$K(\alpha) = \int_{-\alpha/4}^{\alpha/4} \ln \left(\cos \phi - \sqrt{\cos^2 \phi - \cos^2 \frac{\alpha}{4}} \right) d\phi.$$

From $J(\alpha) + K(\alpha) = \alpha \ln \cos(\alpha/4)$ and from [1, p. 563],

$$J(\alpha) = 2 \int_0^{\alpha/4} \ln \left(\cos \phi + \sqrt{\cos^2 \phi - \cos^2 \frac{\alpha}{4}} \right) d\phi = \left(\frac{\alpha}{2} - \pi \right) \ln \cos \frac{\alpha}{4};$$

we then obtain $I(\alpha) = -4\pi \ln \cos(\alpha/4)$.

REFERENCE

- [1] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, 4th edition, A. Jeffrey, ed., Academic Press, Orlando, 1994.

Also solved by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium) and the proposer.

The Asymptotic Sum of a Kapteyn Series

*Problem 97-18**, by D. H. WOOD and H. GUANG (University of Newcastle, Callaghan, NSW, Australia).

Show that for positive p and ε ,

$$\sum_{m=1}^{\infty} mK'_m\left(\frac{m}{p}\right) I_m\left(\frac{(1-\varepsilon)m}{p}\right) \sim \frac{p^2}{2\sqrt{1+p^2}} \left(\frac{1}{\varepsilon} - \frac{1}{2} \log \varepsilon - c\right), \quad \varepsilon \downarrow 0,$$

where I and K are modified Bessel functions, the prime indicates differentiation with respect to the argument, and c is a constant. The series is a Kapteyn series which arises in the solution for the inviscid flow within an infinite helical vortex of constant radius; see equation (8) of Hardin [1]. This flow models the wake of horizontal-axis wind turbines, propellers, and helicopter rotors in vertical flight or hover, where p is the pitch and ε represents the radial distance from the vortex whose radius has been used to normalize p and ε , so that $\varepsilon < 1$. The limiting sum is suggested by the singularity that occurs in the immediate vicinity of any curved line vortex, as described, for example, in section 2.3 of Saffman [2], especially equation (2.3.9). The first term is consistent with the results of our numerical evaluations of the series for small ε , but our technique does not appear to be sufficiently accurate to check the second term or to evaluate c .

REFERENCES

- [1] J. C. HARDIN, *The velocity field induced by a helical vortex filament*, Phys. Fluids, 25 (1982), pp. 1949–1952.
 [2] P. G. SAFFMAN, *Vortex Dynamics*, Cambridge University Press, Cambridge, UK, 1992.

Solution by J. BOERSMA and S. B. YAKUBOVICH (Eindhoven University of Technology, Eindhoven, The Netherlands).

Introduce the notation

$$S(a, b) = \sum_{m=1}^{\infty} K_m(ma)I_m(mb),$$

where the series is convergent for $a > b \geq 0$. Then the Kapteyn series under consideration is equal to the derivative

$$\frac{\partial S(a, b)}{\partial a} = \sum_{m=1}^{\infty} mK'_m(ma)I_m(mb),$$

with $a = 1/p$, $b = a(1 - \varepsilon)$.

By inversion of the Fourier cosine transform in [4, form. 1.12(47)] we have

$$K_\nu(ax)I_\nu(bx) = \frac{1}{\pi\sqrt{ab}} \int_0^\infty Q_{\nu-1/2}\left(\frac{t^2 + a^2 + b^2}{2ab}\right) \cos(xt) dt,$$