Solution to problem 97-16 : A parametric integral arising from a mixed boundary value problem for the Laplacian

By making the substitution $x^2 = -1 + 1/(4t(1-t))$, this becomes

$$J(a,b) = \frac{1}{2}e^{-4b}\int_0^\infty \left\{\left[\sqrt{x^2+1}-x\right]^{a-1} + \left[\sqrt{x^2+1}+x\right]^{a-1}\right\} e^{-4bx^2}(x^2+1)^{-1-a/2}\,dx$$

$$= \frac{1}{2}e^{-4b}\int_{-\infty}^\infty \left[\sqrt{x^2+1}-x\right]^{a-1} e^{-4bx^2}(x^2+1)^{-a/2-1}\,dx.$$

By expanding

$$\left[\sqrt{x^2+1}-x\right]^{a-1} = (x^2+1)^{(a-1)/2}\sum_{k=0}^\infty \binom{a-1}{k}(-1)^k x^k (x^2+1)^{-k/2},$$

we get

$$J(a,b) = \frac{1}{2}e^{-4b}\sum_{n=0}^\infty \binom{a-1}{2n}\int_0^\infty \frac{t^{n-1/2}}{(t+1)^{n+3/2}}\,e^{-4bt}\,dt.$$

Symmetry has been used to eliminate the odd terms in the sum and $t = x^2$. The remaining integral is a tabulated Laplace transform, yielding

$$J(a,b) = \frac{1}{2^a}e^{-2b}\sum_{n=0}^\infty \Gamma\left(n+\frac{1}{2}\right)\binom{a-1}{2n} W_{-n-\frac{1}{2},-\frac{1}{2}}(4b).$$

When a is an integer, the series terminates. This yields a closed expression as a sum of Whittaker functions, as pointed out by the proposers.

Also solved by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium).

A Parametric Integral Arising from a Mixed Boundary Value Problem for the Laplacian

Problem 97-16, *by* LUCIO R. BERRONE (Instituto de Matemática "Beppo Levi," Rosario, Argentina).

Prove that, for every $0 < \alpha < 2\pi$,

$$\int_0^\alpha \ln\left(\frac{\sin\frac{\alpha-\theta}{2} + 2\sin\frac{\alpha}{4}\sqrt{\sin\frac{\alpha-\theta}{2}\sin\frac{\theta}{2}} + \sin\frac{\theta}{2}}{\sin\frac{\alpha-\theta}{2} - 2\sin\frac{\alpha}{4}\sqrt{\sin\frac{\alpha-\theta}{2}\sin\frac{\theta}{2}} + \sin\frac{\theta}{2}}\right)d\theta = -4\pi\ln\cos\frac{\alpha}{4}.$$

The integral of the left-hand side arises in the analysis of the solution to the equation $\triangle u = 0$ in the unitary ball $B_1(0) \subset \mathbb{R}^2$ with mixed boundary conditions given by $(\partial u/\partial n)|_{\Gamma_1} \equiv 1$, $u|_{\Gamma_0} \equiv 0$ when Γ_1 is an arc of length α, and $\Gamma_0 = \partial B_1(0)\backslash\Gamma_1$.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, The Netherlands).

Replace α by 2α, and change the integration variable θ into $\alpha+\theta$. Then the integrand reduces to

$$\ln\left(\frac{\sin\frac{\alpha-\theta}{2} + 2\sin\frac{\alpha}{2}\sqrt{\sin\frac{\alpha-\theta}{2}\sin\frac{\alpha+\theta}{2}} + \sin\frac{\alpha+\theta}{2}}{\sin\frac{\alpha-\theta}{2} - 2\sin\frac{\alpha}{2}\sqrt{\sin\frac{\alpha-\theta}{2}\sin\frac{\alpha+\theta}{2}} + \sin\frac{\alpha+\theta}{2}}\right) = \ln\left(\frac{\cos\frac{\theta}{2} + \sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}{\cos\frac{\theta}{2} - \sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}\right).$$

Thus it is to be proved that

$$I(\alpha) = \int_{-\alpha}^{\alpha} \ln \left(\frac{\cos\frac{\theta}{2} + \sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}{\cos\frac{\theta}{2} - \sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \right) d\theta = -4\pi \ln\cos\frac{\alpha}{2}$$

for $0 < \alpha < \pi$.

We evaluate the derivative $I'(\alpha)$ as follows:

$$I'(\alpha) = \int_{-\alpha}^{\alpha} \frac{2\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} - \sin^2\frac{\alpha}{2} + \sin^2\frac{\theta}{2}} \frac{\frac{1}{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}\, d\theta$$

$$= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \int_{-\alpha}^{\alpha} \frac{\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}\, d\theta$$

$$= \frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \int_{-\sin\frac{\alpha}{2}}^{\sin\frac{\alpha}{2}} \frac{dt}{\sqrt{\sin^2\frac{\alpha}{2} - t^2}} = 2\pi\,\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}.$$

By integration of $I'(\alpha)$, starting from $I(0) = 0$, we obtain the desired result

$$I(\alpha) = \int_0^{\alpha} 2\pi \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\, d\theta = -4\pi\ln\cos\frac{\alpha}{2}.$$

Solution by H.-J. SEIFFERT (Berlin).

We substitute $\theta = 2\phi + \alpha/2$, $-\alpha/4 \le \phi \le \alpha/4$. Then,

$$\sin\frac{\alpha-\theta}{2} + \sin\frac{\theta}{2} = \sin\left(\frac{\alpha}{4} - \phi\right) + \sin\left(\frac{\alpha}{4} + \phi\right) = 2\sin\left(\frac{\alpha}{4}\right)\cos\phi$$

and

$$\sin\frac{\alpha-\theta}{2}\sin\frac{\theta}{2} = \sin^2\left(\frac{\alpha}{4}\right)\cos^2\phi - \cos^2\left(\frac{\alpha}{4}\right)\sin^2\phi = \cos^2\phi - \cos^2\frac{\alpha}{4}.$$

Hence, if $I(\alpha)$ denotes the integral in question, we have $I(\alpha) = 2J(\alpha) - 2K(\alpha)$, where

$$J(\alpha) = \int_{-\alpha/4}^{\alpha/4} \ln\left(\cos\phi + \sqrt{\cos^2\phi - \cos^2\frac{\alpha}{4}}\right) d\phi$$

and

$$K(\alpha) = \int_{-\alpha/4}^{\alpha/4} \ln\left(\cos\phi - \sqrt{\cos^2\phi - \cos^2\frac{\alpha}{4}}\right) d\phi.$$

From $J(\alpha) + K(\alpha) = \alpha\ln\cos(\alpha/4)$ and from [1, p. 563],

$$J(\alpha) = 2\int_0^{\alpha/4} \ln\left(\cos\phi + \sqrt{\cos^2\phi - \cos^2\frac{\alpha}{4}}\right) d\phi = \left(\frac{\alpha}{2} - \pi\right)\ln\cos\frac{\alpha}{4};$$

we then obtain $I(\alpha) = -4\pi\ln\cos(\alpha/4)$.

REFERENCE

[1] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, 4th edition, A. Jeffrey, ed., Academic Press, Orlando, 1994.

Also solved by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium) *and the proposer.*

The Asymptotic Sum of a Kapteyn Series

Problem 97-18*, *by* D. H. WOOD *and* H. GUANG (University of Newcastle, Callaghan, NSW, Australia).

Show that for positive p and ε,

$$\sum_{m=1}^{\infty} mK_m'\left(\frac{m}{p}\right) I_m\left(\frac{(1-\varepsilon)m}{p}\right) \sim \frac{p^2}{2\sqrt{1+p^2}}\left(\frac{1}{\varepsilon}-\frac{1}{2}\log\varepsilon - c\right), \qquad \varepsilon \downarrow 0,$$

where I and K are modified Bessel functions, the prime indicates differentiation with respect to the argument, and c is a constant. The series is a Kapteyn series which arises in the solution for the inviscid flow within an infinite helical vortex of constant radius; see equation (8) of Hardin [1]. This flow models the wake of horizontal-axis wind turbines, propellers, and helicopter rotors in vertical flight or hover, where p is the pitch and ε represents the radial distance from the vortex whose radius has been used to normalize p and ε, so that $\varepsilon < 1$. The limiting sum is suggested by the singularity that occurs in the immediate vicinity of any curved line vortex, as described, for example, in section 2.3 of Saffman [2], especially equation (2.3.9). The first term is consistent with the results of our numerical evaluations of the series for small ε, but our technique does not appear to be sufficiently accurate to check the second term or to evaluate c.

REFERENCES

[1] J. C. HARDIN, *The velocity field induced by a helical vortex filament*, Phys. Fluids, 25 (1982), pp. 1949–1952.
[2] P. G. SAFFMAN, *Vortex Dynamics*, Cambridge University Press, Cambridge, UK, 1992.

Solution by J. BOERSMA *and* S. B. YAKUBOVICH (Eindhoven University of Technology, Eindhoven, The Netherlands).

Introduce the notation

$$S(a,b) = \sum_{m=1}^{\infty} K_m(ma)I_m(mb),$$

where the series is convergent for $a > b \geq 0$. Then the Kapteyn series under consideration is equal to the derivative

$$\frac{\partial S(a,b)}{\partial a} = \sum_{m=1}^{\infty} mK_m'(ma)I_m(mb),$$

with $a = 1/p,\ b = a(1-\varepsilon)$.

By inversion of the Fourier cosine transform in [4, form. 1.12(47)] we have

$$K_\nu(ax)I_\nu(bx) = \frac{1}{\pi\sqrt{ab}}\int_0^\infty Q_{\nu-1/2}\left(\frac{t^2+a^2+b^2}{2ab}\right)\cos(xt)dt,$$