

Hilding's theorem for Banach spaces

Citation for published version (APA):

Eijndhoven, van, S. J. L. (1996). *Hilding's theorem for Banach spaces*. (RANA : reports on applied and numerical analysis; Vol. 9612). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/1996

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 96-12
August 1996

Hilding's theorem for Banach spaces

by

S.J.L. van Eijndhoven



Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507

Hilding's theorem for Banach spaces

by

S.J.L. van Eijndhoven

In this note, we establish sufficient conditions for invertibility of bounded linear operators on Banach spaces. Herewith, we extend and reprove a result of Hilding [Hi] presented in the context of Hilbert spaces.

For a bounded linear operator T on a complex Banach space X by $\sigma(T)$ we denote the spectrum of T and by $\rho(T)$ its resolvent set. So $\rho(T) = \mathbb{C} \setminus \sigma(T)$. We recall that $\sigma(T)$ is non-empty compact subset of \mathbb{C} . By $B(X, Y)$, we denote the collection of all bounded linear operators from the Banach space X into the Banach space Y . Also, we write $B(X)$ instead of $B(X, X)$.

Lemma 1. For $T \in B(X)$, suppose there are $\alpha_0 \in \mathbb{R}$ and $\beta_0 > 0$ such that

$$\|Tx - \alpha x\| \geq \beta_0 \|x\|$$

for all $x \in X$ and all $\alpha \leq \alpha_0$. Then the half-infinite interval $(-\infty, \alpha_0]$ is contained in $\rho(T)$.

Proof. We may assume that $\sigma(T) \cap \mathbb{R} \neq \emptyset$. Let

$$\lambda_0 := \min\{\lambda \in \mathbb{R} \mid \lambda \in \sigma(T)\} .$$

Now suppose $\lambda_0 \leq \alpha_0$. Put $\alpha_n = \lambda_0 - \frac{1}{n}$, $n \in \mathbb{N}$. Then for all n , $\alpha_n \in \rho(T)$. Since

$$\|Tx - \lambda_0 x\| \geq \beta_0 \|x\| , \quad x \in X ,$$

$T - \lambda_0$ is injective (and has closed range). To arrive at a contradiction we shall prove that $T - \lambda_0$ is surjective. Let $y \in X$, and define the sequence (x_n) in X by

$$x_n = (T - \alpha_n)^{-1} y , \quad n \in \mathbb{N} .$$

By the second resolvent identity, for all $n, m \in \mathbb{N}$

$$\|x_n - x_m\| = \left| \frac{1}{n} - \frac{1}{m} \right| \|(T - \alpha_n)^{-1}(T - \alpha_m)^{-1}y\| \leq \frac{1}{\beta_0^2} \left| \frac{1}{n} - \frac{1}{m} \right| \|y\| .$$

So the sequence $(x_n)_{n \in \mathbb{N}}$ converges, to $x \in X$ say. Then for all $n \in \mathbb{N}$

$$\|(T - \lambda_0)x - y\| \leq \|(T - \lambda_0)(x - x_n)\| + \frac{1}{n}\|x_n\| ,$$

so that $y = (T - \lambda_0)x$.

We conclude that $\lambda_0 > \alpha_0$. □

Theorem 2. Let $T \in B(X)$. Suppose there are $\alpha_1 \in \mathbb{R}$ and $\beta_1 > 0$ such that

$$\|Tx - \alpha x\| \geq \beta_1 \|x\|$$

for all $x \in X$ and $\alpha \leq \alpha_1$. Then $(-\infty, \alpha_1 + \beta_1) \subset \rho(T)$.

Proof. Let $\alpha_0 = \alpha_1 + \eta$ and $\beta_0 = \beta_1 - \eta$, where $0 < \eta < \beta_1$. Then for all α with $\alpha_1 \leq \alpha \leq \alpha_0$ and all $x \in X$

$$\|Tx - \alpha x\| \geq \|Tx - \alpha_1 x\| - (\alpha - \alpha_1)\|x\| \geq (\beta_1 - \eta)\|x\|$$

We conclude that

$$\|Tx - \alpha x\| \geq \beta_0 \|x\|$$

for all $\alpha \leq \alpha_0$ and $x \in X$. Hence $(-\infty, \alpha_0] \subset \rho(T)$. The result follows, since

$$(-\infty, \alpha_1 + \beta_1) = \bigcup_{\eta < \beta_1} (-\infty, \alpha_1 + \eta] \subset \rho(T) .$$

□

Theorem 3. Let $T : X \rightarrow X$ be a linear operator. Suppose there are $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 \leq \lambda_1, \lambda_2 < 1$ such that

$$\|Tx - x\| \leq \lambda_1 \|x\| + \lambda_2 \|Tx\| .$$

Then

a) T is bounded with $\|T\| \leq \frac{1 + \lambda_1}{1 - \lambda_2}$

b) $\left(-\infty, \frac{1 - \lambda_1}{1 + \lambda_2}\right) \subset \rho(T)$.

c) T is invertible with $\|T^{-1}\| \leq \frac{1 + \lambda_2}{1 - \lambda_1}$.

Proof.

a) Let $x \in X$. Then

$$\|Tx\| \leq \|x\| + \|Tx - x\| \leq (1 + \lambda_1)\|x\| + \lambda_2 \|Tx\|$$

so that $\|Tx\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \|x\|$.

b) Let $\alpha \leq 0$, and $x \in X$. Then

$$\begin{aligned} \|Tx - \alpha x\| &= \|Tx - x + (1 - \alpha)x\| \geq (1 - \alpha)\|x\| - \lambda_1 \|x\| - \lambda_2 \|Tx\| \\ &\geq (1 - \alpha - \lambda_1 + \lambda_2 \alpha)\|x\| - \lambda_2 \|Tx - \alpha x\| . \end{aligned}$$

Hence for all $x \in X$ and $\alpha \leq 0$

$$\|Tx - \alpha x\| \geq \frac{1 - \lambda_1 - (1 - \lambda_2)\alpha}{1 + \lambda_2} \|x\| \geq \frac{1 - \lambda_1}{1 + \lambda_2} \|x\| .$$

By Theorem 2 with $\alpha_1 = 0$ and $\beta_1 = \frac{1 - \lambda_1}{1 + \lambda_2}$, we obtain

$$\left(-\infty, \frac{1 - \lambda_1}{1 + \lambda_2}\right) \subset \rho(T) .$$

c) Since $0 \in \rho(T)$ by b), T is invertible. Moreover

$$\|Tx\| \geq \frac{1 - \lambda_1}{1 + \lambda_2} \|x\| , \quad x \in X ,$$

so that $\|T^{-1}\| \leq \frac{1 + \lambda_2}{1 - \lambda_1}$.

□

We mention some consequences of the above theorem.

Corollary 4. Let $T : X \rightarrow X$ be a *bounded* linear operator. Suppose there is $\lambda \in \mathbb{R}$ with $0 \leq \lambda < 1$ such that

$$\|Tx - x\| \leq \lambda\|x\| + \|Tx\| .$$

Then

a) $\left(-\infty, \frac{1 - \lambda}{2}\right) \subset \rho(T)$.

b) T is invertible with $\|T^{-1}\| \leq \frac{2}{1 - \lambda}$.

Proof. Let $\varepsilon_0 := \min\left(1, \frac{\lambda}{\|T\|}\right)$. Then for $0 < \varepsilon \leq \varepsilon_0$,

$$0 \leq 1 - \varepsilon < 1 \text{ and } 0 \leq \lambda - \varepsilon\|T\| < 1 ,$$

and

$$\|Tx - x\| \leq (\lambda - \varepsilon\|T\|)\|x\| + (1 - \varepsilon)\|Tx\| .$$

Hence by the preceding theorem

$$\left(-\infty, \frac{1 - \lambda + \varepsilon\|T\|}{2 - \varepsilon}\right) \subset \rho(T)$$

and

$$\|T^{-1}\| \leq \frac{2 - \varepsilon}{1 - \lambda - \varepsilon\|T\|} .$$

By letting $\varepsilon \downarrow 0$ the assertions a) and b) follow. \square

Corollary 5. Let X and Y be Banach spaces and let $U : X \rightarrow Y$ be a bounded invertible operator. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 \leq \lambda_1 < 1$ and $0 \leq \lambda_2 \leq 1$. Then for all $V \in B(X, Y)$ satisfying that for all $x \in X$

$$(*) \quad \|Ux - Vx\| \leq \lambda_1 \|Ux\| + \lambda_2 \|Vx\| .$$

a) $V - \alpha U$ is invertible for all $\alpha \in \left(-\infty, \frac{1 - \lambda_1}{1 + \lambda_2}\right)$.

b) V is invertible and $\|V^{-1}\| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \|U^{-1}\|$

Proof. Let $V \in B(X, Y)$ satisfy condition (*). Put $S = VU^{-1}$. Then for all $y \in Y$

$$\|y - Sy\| \leq \lambda_1 \|y\| + \lambda_2 \|Sy\| .$$

We conclude from the above results that $S - \alpha I$ is invertible for all $\alpha \in \left(-\infty, \frac{1 - \lambda_1}{1 + \lambda_2}\right)$ and that $\|S^{-1}\| \leq \frac{1 + \lambda_2}{1 - \lambda_1}$. Now the assertions a) and b) follow from the observation that $V - \alpha U = (S - \alpha I)U$. \square

Definition 6. For X and Y Banach spaces, $U \in B(X, Y)$ is said to be right invertible if there is $U^+ \in B(Y, X)$ such that $UU^+y = y$ for all $y \in Y$. U^+ is called a right inverse of U .

Remark. Recall that for X and Y Hilbert spaces the following are equivalent

- 1) $U \in B(X, Y)$ is right invertible.
- 2) $U \in B(X, Y)$ is surjective.
- 3) $U \in B(X, Y)$ and $UU^* \in B(Y)$ invertible.

Corollary 7. Let X and Y be Banach spaces and $U \in B(X, Y)$ be right invertible with right inverse U^+ . Then all $V \in B(X, Y)$ for which there are $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that for all $x \in X$,

$$\|Ux - Vx\| \leq \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \lambda_3 \|x\|$$

where $0 \leq \lambda_1 + \lambda_3 \|U^+\| < 1$ and $0 \leq \lambda_2 \leq 1$, are right invertible.

Proof. Let V satisfy the condition, stated in the assertion. Then for all $y \in Y$,

$$\begin{aligned} \|VU^+y - y\| &\leq \lambda_1 \|y\| + \lambda_2 \|VU^+y\| + \lambda_3 \|U^+y\| \\ &\leq (\lambda_1 + \lambda_3 \|U^+\|) \|y\| + \lambda_2 \|VU^+y\|. \end{aligned}$$

By Theorem 3, VU^+ is invertible in $B(Y)$. Hence $V^+ := U^+(VU^+)^{-1}$ is a right inverse of V . Observe that

$$\|V^+\| \leq \left(\frac{1 + \lambda_2}{1 - \lambda_1 - \lambda_3 \|U^+\|} \right) \|U^+\|.$$

□

Remark. Let X and Y be Banach spaces, and $U \in B(X, Y)$ be right invertible. Let X_0 be a dense subspace of X and let $V_0 : X_0 \rightarrow X$ satisfy for all $x \in X_0$

$$\|Ux - V_0x\| \leq \lambda_1 \|Ux\| + \lambda_2 \|V_0x\| + \mu \|x\|,$$

where $0 \leq \lambda_2 < 1$. Then V_0 extends to a bounded operator $V \in B(X, Y)$ with

$$\|V\| \leq \frac{\mu + \lambda_1 \|U\|}{1 - \lambda_2}$$

and for all $x \in X$

$$\|Ux - Vx\| \leq \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \mu \|x\|.$$

[Hi] Hilding, S.; Note on completeness theorems of Paley-Wiener type. Ann. of Math. 49, no. 4 (1948), pp. 953-955.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
96-04	J.P.E. Buskens M.J.D. Slob	Prototype of the Numlab program. A laboratory for numerical engineering	March '96
96-05	J. Molenaar	Oscillating boundary layers in polymer extrusion	March '96
96-06	S.W. Rienstra	Geometrical Effects in a Joule Heating Problem from Miniature Soldering	April '96
96-07	A.F.M. ter Elst D.W. Robinson	On Kato's square root problem	April '96
96-08	H.J.C. Huijberts	On linear subsystems of nonlinear control systems	May '96
96-09	H.J.C. Huijberts	Combined partial feedback and input-output linearization by static state feedback for nonlinear control systems	May '96
96-10	A.F.M. ter Elst D.W. Robinson	Analytic elements of Lie groups	May '96
96-11	S.L. de Snoo R.M.M. Mattheij G.A.L. van de Vorst	Modelling of glass moulding, in particular small scale surface changes	July '96
96-12	S.J.L. van Eijndhoven	Hilding's theorem for Banach spaces	August '96

