Generalized controlled invariance for nonlinear systems

Generalized Controlled Invariance for Nonlinear Systems*

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Abstract

A general setting is developed which describes controlled invariance for nonlinear control systems and which incorporates the previous approaches dealing with controlled invariant (co -) distributions. A special class of controlled invariant subspaces, called controllability cospaces, is introduced. These geometric notions are shown to be useful for deriving a (geometric) solution to the dynamic disturbance decoupling problem and for characterizing the so-called fixed dynamics for noninteracting control. These fixed dynamics are a central issue in studying noninteracting control with stability. The class of quasi-static state feedbacks is used.

Key words. nonlinear systems, controlled invariance, quasi-static state feedback

AMS subject classifications. 93C10, 93B27, 93C60, 93C35

1 Introduction

During the last two decades, nonlinear control theory was developed thanks to the increasing number of researchers involved in this area. A main goal of the research in the 80's was the generalization of the so-called geometric approach which proved to be particularly efficient for linear time-invariant systems (see [48],[4] for an overview). In this linear theory, controlled invariance plays a fundamental role in both static and dynamic feedback control problems. The goal of generalizing the linear approach to the nonlinear case was only *partially* reached : the situation is quite well understood when regular static feedback synthesis problems are considered, while limits of the standard (geometric) notions became clear at the end of the 80's in the study of such problems as

- control problems involving dynamic feedback
- or the inversion of a nonlinear system, the definition of its rank,...

Alternative (algebraic) tools have been developed from 1985 on ([19]) and a definition of the rank of a system was provided by a differential algebraic theory ([20]).

The goal of this paper is to introduce a generalized notion of controlled invariance. The motivation is to clarify the geometric structure of nonlinear systems and to develop a geometric

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framework to tackle synthesis problems via dynamic feedback. In particular, we answer the two following questions :

Question 1 : Does there exist any *geometric* solution to the dynamic disturbance decoupling problem *(DDDP)* ?

Question 2 : Does there exist a *geometric* structure of nonlinear systems which displays the rank, the so-called decoupling zeros (under dynamic feedback),...?

The answer to such questions has a major importance since they motivate the search for geometric solutions to any other synthesis problem which involves dynamic feedback. Such solutions will contribute to the completion of the extension to nonlinear systems of the linear geometric theory ([48],[4]).

DDDP, considered in *Question 1*, is a special control problem involving dynamic feedback and was first stated and studied in [26], [25], [44] where an (algebraic) solution was provided based on the inversion algorithm. Also a geometric interpretation is given by using (nonintrinsic) standard controlled invariant (co-)distributions on an extended state space, and then projecting these (co-)distributions on the original state space to obtain intrinsic objects. Parallel results can be found in [42],[41],[27]. The generalization of controlled invariance which is introduced in this paper is shown to give a *natural* geometric solution to the *DDDP*, i.e., without taking recourse to nonintrinsic objects defined on an extended state space that are rendered intrinsic after projection. Of course, it goes without saying that the objects defined in this paper and the geometric objects defined in [26], [25], [44] carry the same information concerning the solvability of the *DDDP*. Recall that in the special case of linear systems, *DDDP* is equivalent to *DDP* (static feedback disturbance decoupling problem). The state of the art can be summarized in the following table, where notations are borrowed from [48],[29],[38].

	Static feedback	Dynamic feedback
Linear systems	$\mathcal{E} \subset \mathcal{V}^*$	
Nonlinear systems	$\mathcal{P} \subset \Delta^*$	?

Table 1: (Geometric) solution to the disturbance decoupling problem

One contribution of the paper is the completion of the above table.

Question 2 originated in [30]. Standard controlled invariant distributions cannot be used to characterize the rank of a system in a straightforward manner. The rank was introduced in [19] based on a differential algebraic analysis. A geometric interpretation of the rank may be found in [45], based on controllability distributions defined on a certain extended state space. Contributions which parallel the geometric and algebraic approaches can be found in [49].

Generalized controlled invariance introduced in this paper is shown to give a natural and intrinsic geometric characterization of the rank. It further displays new (geometric) structures of a nonlinear system. We focus on the structure related to the so-called decoupling zeros (under quasi-static feedback ([12],[13],[14])). We summarize once again the state of the art in Table 2, and we borrow the notations from the given references.

In this paper, Table 2 is completed thanks to the controllability cospaces introduced in the sequel. Moreover, throughout the text, the new geometric structures are compared with the standard ones. Both embody different and complementary properties.

The study of controlled invariance for nonlinear systems of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in I\!R^n$, $u \in I\!R^m$, was initiated in [8]. In this paper invariants were sought under feedback transformations of the form

$$u = \alpha(x) + v \tag{2}$$

feedback	invertible decoupling matrix	non-invertible decoupling matrix
(Quasi-) Static	dim $(\mathcal{P}^*)$ (Isidori & Grizzle [32])	?
Dynamic	dim (Δ_{mix}) (Wagner [47])	dim $(\Delta_{mix}(\Sigma_p))$ (Zhan *et al.* [50])

Table 2: Decoupling zero structure

Later on, controlled invariance was tackled by various authors ([31],[24],[36],[37]). The group of feedback transformations acting on (1) was enlarged to transformations of the form

$$u = \alpha(x) + \beta(x)v \tag{3}$$

where $\beta(x)$ is square and locally invertible. These works yielded the definition of a controlled invariant distribution. The key was found for the solution of synthesis problems, such as the disturbance decoupling problem and the noninteracting control problem, via regular (or invertible) static state feedback (see the textbooks [29],[38] for an overview). The study of controlled invariance under the class of feedbacks (3) remains active - see [10],[22],[11], [43],[49] for recent contributions. A special class of controlled invariant distributions is given by controllability distributions ([39],[33],[34]). They became a basic tool for solving the noninteracting control problem with or without stability. Indeed, the controllability distributions allow to characterize the fixed dynamics of the decoupled system via static feedback ([32]).

In this paper, a generalized notion of controlled invariance is introduced by allowing an enlarged class of feedback transformations acting on (1), namely the class of quasi-static feedbacks $u = \alpha(x, v, \dot{v}, \cdots, v^{(k)})$. This class of feedbacks describes intrinsic properties of the system with respect to the solvability of synthesis problems via dynamic feedback, as disturbance decoupling or noninteracting control. In this sense, quasi-static feedbacks are considered as a mathematical tool rather than a new class of feedbacks to be used in practical applications. The various contributions to the study of controlled invariance are summarized in Table 3. This table will be completed in this paper. Preliminary results can be found in [28].

Quasi-static feedbacks have been used in [40] to derive canonical forms (see also [46]) and were

Feedback	References
$u = \alpha(x) + v$	Brockett
$u = \alpha(x) + \beta(x)v$	Isidori *et al.* Hirschorn Nijmeijer *et al.*,...
$u = \alpha(x, v, \dot{v}, \cdots, v^{(k)})$	?

Table 3: Controlled invariance

formalized in [12], [13], [14] where the input-output decoupling problem under quasi-static state feedback was solved as well. Practical applications of quasi-static feedback can be found in [16].

In the sequel we consider a nonlinear control system (1), where the entries of $f(x)$ and $g(x)$ are meromorphic functions from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that a meromorphic function is the quotient of two analytic functions. This allows to derive properties of the system under consideration on an open and dense subset of the state space. Different classes of systems can also be treated:

- C^∞ systems, where all results should be explicitely stated as local results, valid around a *regular* point, where regularity is to be defined in an appropriate way, depending on the problem under consideration,

- analytic systems, in which case the results are also valid on some open dense submanifold of the state space.

In the rest of this paper we mainly use a function field formalism. It is assumed that rank $g(x) = m$ and that $n \geq 1$.

The organization of the paper is as follows. In Section 2 we define the generalized notion of invariance with respect to the dynamics (1). Section 3 is devoted to controlled invariance and related properties. A geometric necessary and sufficient condition for the existence of a solution to *DDDP* is obtained. Controllability cospaces and their applications as well as the fixed modes or decoupling zero dynamics under quasi-static feedback are treated in Section 4.

2 Invariant subspaces

We follow the notations and setting of [18]. Let $\mathcal{K}$ denote the field of meromorphic functions of $\{x, u^{(k)}, k \geq 0\}$. $\mathcal{E}$ is the formal vector space spanned by $\{d\eta \mid \eta \in \mathcal{K}\}$ over $\mathcal{K}$. The notation dx stands for $\{dx_1, \cdots, dx_n\}$ and $du^{(k)}$ for $\{du_1^{(k)}, \cdots, du_m^{(k)}\}$. Let $\mathcal{X} := \text{span}_{\mathcal{K}}\{dx\}$ and $\mathcal{U} := \text{span}_{\mathcal{K}}\{du, d\dot{u}, \cdots, du^{(k)} \mid k \geq 0\}$.

Throughout this paper we employ the following terminology. A vector $\omega \subset \mathcal{E}$ is called *exact* if there exists a $\phi \in \mathcal{K}$ such that $\omega = d\phi$. A subspace $\Omega \subset \mathcal{E}$ of dimension r is called *exact* if there exist functions $\phi_1, \cdots, \phi_r \in \mathcal{K}$ such that $\Omega = \text{span}_{\mathcal{K}}\{d\phi_1, \cdots, d\phi_r\}$. Given subspaces $\Omega_1 \subset \Omega_2 \subset \mathcal{E}$, (Ω_2/Ω_1) is said to be *exact* if there exist functions $\phi_1, \cdots, \phi_d \in \mathcal{E}$, with $d = \dim(\Omega_2) - \dim(\Omega_1)$, such that $\Omega_2 = \Omega_1 \oplus \text{span}_{\mathcal{K}}\{d\phi_1, \cdots, d\phi_d\}$, or, in other words, (Ω_2/Ω_1) is isomorphic to an exact subspace of $\mathcal{E}$. Consider a subspace $\Omega \subset \mathcal{E}$. Then clearly $\{0\} \subset \Omega$ is exact. Furthermore, if $\Omega_1 \subset \Omega$, $\Omega_2 \subset \Omega$ are exact, then also $\Omega_1 + \Omega_2 \subset \Omega$ is exact. Hence there exists a unique maximal exact subspace in Ω.

Consider a subspace $\Omega \subset \mathcal{X}$. Define

$$\dot{\Omega} = \text{span}_{\mathcal{K}}\{\dot{\omega} \mid \omega \in \Omega\} \tag{4}$$

where $\omega = \sum_{i=1}^{n} \omega_i(x, u, \dot{u}, \cdots, u^{(n-1)})dx_i$ and time-derivation is defined by $\dot{\omega} = \sum_{i=1}^{n}(\omega_i d\dot{x}_i + \dot{\omega}_i dx_i)$. Thus $\dot{\omega} \in \text{span}_{\mathcal{K}}\{dx, du\}$.

Definition 2.1 A subspace $\Omega \subset \mathcal{X}$ is said to be invariant with respect to (1) if

$$\dot{\Omega} \subset \Omega + \text{span}_{\mathcal{K}}\{du\} \tag{5}$$

∎

Remark 2.2 Let $\mathcal{K}_k$ be the field of meromorphic functions of $x, u, \cdots, u^{(k)}$ and define

$$\mathcal{K}' = \bigcup_{k \in I\!N} \mathcal{K}_k$$

Then (5) is equivalent to the statement that $(\Omega + \text{span}_{\mathcal{K}'}\{du^{(k)} \mid k \geq 0\})$ is a differential vector space, with thederivation defined above.

Example 2.3 Let Ω be an integrable invariant codistribution for (1) in the sense of e.g. [29], [38] and let (x_1, x_2) be a local system of coordinates such that $\Omega = \text{span}\{dx_1\}$. Then in the coordinates (x_1, x_2), (1) takes the form (cf. [29],[38])

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)u \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned} \tag{6}$$

Interpreting Ω as a subspace of $\text{span}_{\mathcal{K}}\{dx\}$, we then obtain:

$$\dot{\Omega} = \text{span}_{\mathcal{K}}\{d\dot{x}_1\} = \text{span}_{\mathcal{K}}\{d(f_1(x_1) + g_1(x_1)u)\} \subset \Omega + \text{span}_{\mathcal{K}}\{du\} \tag{7}$$

Hence Ω is invariant in the sense of Definition 2.1.

When a given subspace is not invariant, it is interesting to know whether or not there exists a feedback transformation that renders it invariant. This is the topic of the next section.

3 Controlled invariant subspaces

In this section we define and characterize the controlled invariance of subspaces $\Omega \subset \mathcal{X}$ under quasi-static state feedback. In Subsection 3.1 we first define quasi-static state feedback, based on ([12],[13],[14]). In Subsection 3.2 we give a definition of controlled invariance under quasi-static state feedback. In Subsection 3.3 some properties of controlled invariance under regular static state feedback (3) are given. Conditions for controlled invariance of subspaces $\Omega \subset \mathcal{X}$ under quasi-static state feedback are investigated in Subsection 3.4. We make some remarks about the smallest controlled invariant subspace containing some given subspace in Subsection 3.4.2. As shown in Section 3.5, this subspace allows to characterize the solvability conditions of the dynamic disturbance decoupling problem.

3.1 Quasi-static state feedback

Consider the nonlinear system (1). A *generalized static state feedback* for (1) is a feedback of the form

$$u = \phi(x, v, \cdots, v^{(r)}) \tag{8}$$

where $v \in \mathbb{R}^m$ denotes the new controls. Let $\mathcal{K}_v$ denote the field of meromorphic functions of $\{x, \{v^{(k)} \mid k \geq 0\}\}$ and define the formal vector space $\mathcal{E}_v := \text{span}_{\mathcal{K}_v}\{d\xi \mid \xi \in \mathcal{K}_v\}$. As in [12],[13], we define the following *filtrations* ([3]) of $\mathcal{E}_v$:

$$\begin{aligned} \mathcal{V}_{-1} &:= \text{span}_{\mathcal{K}_v}\{dx\} \\ \mathcal{V}_k &:= \text{span}_{\mathcal{K}_v}\{dx, dv, \cdots, dv^{(k)}\} \quad (k \geq 0) \end{aligned} \tag{9}$$

$$\begin{aligned} \mathcal{U}_{-1} &:= \text{span}_{\mathcal{K}_v}\{dx\} \\ \mathcal{U}_k &:= \text{span}_{\mathcal{K}_v}\{dx, d\phi, \cdots, d\phi^{(k)}\} \quad (k \geq 0) \end{aligned} \tag{10}$$

The filtrations $\mathcal{U}_k$ and $\mathcal{V}_k$ are said to have *bounded difference* ([3]) if there exists an $s \in \mathbb{N}$ such that for all $k \geq -1$

$$\begin{aligned} \mathcal{U}_k &\subset \mathcal{V}_{k+s} \\ \mathcal{V}_k &\subset \mathcal{U}_{k+s} \end{aligned} \tag{11}$$

Definition 3.1 ([12],[13],[14]) u given by (8) is said to be a *quasi-static state feedback* for (1) if the filtrations $\mathcal{U}_k$ and $\mathcal{V}_k$ have bounded difference.

Remark 3.2 It is easily verified that a regular static state feedback (3) is a quasi-static state feedback.

The following result is also easily proved.

Proposition 3.3 *Let u given by (8) be a quasi-static state feedback. Then there locally exists a function $\psi(x, u, \cdots, u^{(r)})$ such that*

$$v = \psi(x, u, \cdots, u^{(r)}) \tag{12}$$

■

Remark 3.4 In [17] a definition of quasi-static state feedback is given for generalized systems (systems of the form $\dot{x} = f(x, u, \dot{u}, \cdots, u^{(s)})$, $y = h(x, u, \dot{u}, \cdots, u^{(s)})$). This definition is the same as Definition 3.1 with the extra requirement that x is a state (in the sense of [17]) of the closed-loop system.

3.2 Controlled invariance

Consider the control system (1) together with a quasi-static state feedback (8) and define $\mathcal{V} := \text{span}_{\mathcal{K}_v}\{dv^{(k)} \mid k \geq 0\}$. We denote by $\Theta^{(k)}$ the time derivative of order k of Θ along the trajectories of the system (1), and by $\Theta^{[k]}$ the time derivative of order k of Θ along the trajectories of the closed loop system (1,8). We will write $\dot{\Theta}$ for $\Theta^{(1)}$.

Definition 3.5 A subspace $\Omega \subset \mathcal{X}$ is said to be controlled invariant for (1) if there exists a quasi-static state feedback (8) such that for (1,8) one has

$$\Omega^{[1]} \subset \Omega + \mathcal{V} \tag{13}$$

The definition of controlled invariance given in Definition 3.5 is in accordance with the well known definition of a controlled invariant codistribution. Recall from *e.g.* [29],[38] that a codistribution Ω is controlled invariant if there exists a regular static state feedback (3) such that

$$\begin{aligned} \mathcal{L}_{f+g\alpha}\Omega &\subset \Omega \\ \mathcal{L}_{(g\beta)_{*i}}\Omega &\subset \Omega \quad (i = 1, \cdots, m) \end{aligned} \tag{14}$$

Let $\omega \in \Omega$. Then for (1,3) we have

$$\omega^{[1]} = \mathcal{L}_{f+g\alpha}\omega + \sum_{i=1}^{m}(v_i \mathcal{L}_{(g\beta)_{*i}}\omega + \langle \omega, (g\beta)_{*i}\rangle dv_i) \in \Omega + \mathcal{V} \tag{15}$$

when we interpret Ω as a subspace of $\text{span}_{\mathcal{K}}\{dx\}$.

Example 3.6 Consider a nonlinear system given by

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_3 u_1 + x_2, \quad \dot{x}_3 = u_2$$

Let $\Omega = \text{span}_{\mathcal{K}}\{u_1 dx_3 + dx_2\}$ and

$$u_1 = v_1, \quad u_2 = (v_2 - x_3(\dot{v}_1 + v_1) - x_2)/v_1$$

where $v = (v_1, v_2)^T$ is the new input. This is a quasi-static feedback since

$$v_1 = u_1, \quad v_2 = u_1 u_2 + x_3(\dot{u}_1 + u_1) + x_2$$

This feedback renders Ω invariant, since we have

$$\Omega^{[1]} = \text{span}_{\mathcal{K}_v}\{\dot{v}_1 dx_3 + v_1 d((v_2 - x_3(\dot{v}_1 + v_1) - x_2)/v_1) + d(x_3 v_1 + x_2)\} \subset \mathcal{V}$$

■

The following theorem gives a necessary condition for controlled invariance. For (1), let $\mathcal{G}$ denote the distribution spanned by the input vector fields. Define the subspace $\mathcal{G}^\perp \subset \mathcal{X}$ by

$$\mathcal{G}^\perp = \{\omega \in \mathcal{X} \mid \langle \omega, g \rangle \equiv 0, \ \forall g \in \mathcal{G}\} \tag{16}$$

Theorem 3.7 *Let $\Omega \subset \mathcal{X}$. Then Ω is controlled invariant only if*

$$\dot{\widehat{(\Omega \cap \mathcal{G}^\perp)}} \subset \Omega \tag{17}$$

Proof By definition of $\mathcal{G}^\perp$, $\dot{\widehat{(\Omega \cap \mathcal{G}^\perp)}} \subset \mathcal{X}$. Controlled invariance of Ω then implies (17). ■

Remark 3.8 Let Ω be an involutive codistribution. Using (15), it may be shown that (17) (with Ω interpreted as a subspace of $\mathcal{X}$) is equivalent to the well known conditions $\mathcal{L}_f(\Omega \cap \mathcal{G}^\perp) \subset \Omega$, $\mathcal{L}_{g_i}(\Omega \cap \mathcal{G}^\perp) \subset \Omega \, (i = 1, \cdots, m)$ for controlled invariance of Ω (cf. [29], [38]).

3.3 Characterization of controlled invariant subspaces under regular static state feedback

In this subsection we investigate under what conditions a subspace $\Omega \subset \mathcal{X}$ is controlled invariant under regular static state feedback. Recall from Subsection 3.2 that a regular static state feedback is a special sort of quasi-static state feedback. A first result is the following.

Proposition 3.9 *Consider a d-dimensional subspace $\Omega \subset \mathcal{X}$. Assume that Ω is controlled invariant under a quasi-static state feedback of the form $u = \phi(x, v)$. Then Ω admits a basis $\omega_1, \cdots, \omega_d$ with*

$$\omega_i = \sum_{j=1}^{n} \omega_{ij}(x) dx_j \tag{18}$$

Proof Assume that $\Omega = \text{span}_{\mathcal{K}}\{\tilde{\omega}_1, \cdots, \tilde{\omega}_d\}$, with

$$\tilde{\omega}_i = \sum_{j=1}^{n} \tilde{\omega}_{ij}(x, u) dx_j \tag{19}$$

Let $A(x, u)$ be the matrix with entries $\tilde{\omega}_{ij}$ $(i = 1, \cdots, d; j = 1, \cdots, n)$. Viewing Ω as a linear subspace (over $\mathcal{K}$) of $\mathcal{X} \oplus \text{span}_{\mathcal{K}}\{du\}$, it may be characterized by

$$\Omega = \text{rowspan}_{\mathcal{K}} \left(A(x, u) \quad 0 \right) \tag{20}$$

Similarly, $\Omega + \dot{\Omega}$ is characterized by

$$\Omega + \dot{\Omega} = \text{rowspan}_{\mathcal{K}} \begin{pmatrix} A(x, u) & 0 \\ B(x, u, \dot{u}) & (Ag)(x, u) \end{pmatrix} \tag{21}$$

where

$$B(x, u, \dot{u}) = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i}(x, u) \dot{x}_i(x, u) + \sum_{j=1}^{m} \frac{\partial A}{\partial u_j} \dot{u}_j + A(x, u) \left(f_x(x) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} u \right) \tag{22}$$

with f_x the Jacobian of f. Since Ω is rendered invariant via $u = \phi(x, v)$ there exist matrices $P(x, v, \dot{v})$ and $Q(x, v)$ such that

$$B(x, \phi, \dot{\phi}) dx + (Ag)(x, \phi) d\phi = P(x, v, \dot{v}) A(x, \phi) dx + Q(x, v) dv \tag{23}$$

or

$$\begin{aligned} B(x, \phi, \dot{\phi}) &= P(x, v, \dot{v}) A(x, \phi) - (Ag)(x, \phi) \phi_x(x, v) \\ (Ag)(x, \phi) \phi_v(x, v) &= Q(x, v) \end{aligned} \tag{24}$$

Since $\phi_v(x, v)$ is invertible, this yields

$$B(x, \phi, \dot{\phi}) = P(x, v, \dot{v}) A(x, \phi) - Q(x, v) \phi_v(x, v)^{-1} \phi_x(x, v) \tag{25}$$

Since $u = \phi(x, v)$ is a quasi-static state feedback, by Proposition 3.3 there locally exists a function $\psi(x, u)$ such that $\phi(x, \psi(x, u)) = u$. This yields in particular that

$$\psi_x(x, u) = -\phi_v(x, \psi(x, u))^{-1} \phi_x(x, \psi(x, u))$$

Hence (25) yields

$$B(x, u, \dot{u}) = \tilde{P}(x, u, \dot{u}) A(x, u) + \tilde{Q}(x, u) \psi_x(x, u) \tag{26}$$

where $\tilde{P}(x,u,\dot{u}) = P(x,\psi(x,u),\dot{\psi}(x,u,\dot{u}))$ and $\tilde{Q}(x,u) = Q(x,\psi(x,u))$. Taking partial derivatives with respect to $\dot{u}_i$, we obtain

$$\frac{\partial A}{\partial u_i} = \frac{\partial \tilde{P}}{\partial \dot{u}_i} A(x,u) \quad (i = 1, \cdots, m) \tag{27}$$

Obviously,

$$\frac{\partial^2 \tilde{P}}{\partial \dot{u}_i \partial \dot{u}_j} = 0 \quad (i,j = 1, \cdots, m)$$

Hence there exist matrices $R_i(x,u)$ $(i = 1, \cdots, m)$ such that

$$\frac{\partial A}{\partial u_i} = R_i(x,u)A(x,u) \tag{28}$$

Using arguments from the theory of linear time-varying ordinary differential equations this yields that $A(x,u)$ is of the form

$$A(x,u) = \Phi(x,u)\Psi(x)$$

where $\Phi(x,u)$ is a square invertible matrix. Hence

$$\Omega = \text{rowspan}_{\mathcal{K}}\,(A(x,u) \;\; 0) = \text{rowspan}_{\mathcal{K}}\,(\Psi(x) \;\; 0) \tag{29}$$

which establishes our claim. If $\Omega = \text{rowspan}_{\mathcal{K}}\left(A(x,u,\cdots,u^{(\ell)}) \;\; 0\right)$ with $\ell > 1$, the claim is established by using the same arguments together with an induction argument. ∎

From the above proposition it follows that the set of subspaces $\Omega \subset \mathcal{X}$ that are controlled invariant under a quasi-static state feedback $u = \phi(x,v)$ may be identified with the set of "classical" controlled invariant codistributions. The following theorem gives a characterization of controlled invariance in our generalized framework.

Theorem 3.10 *Let $\Omega \subset \mathcal{X}$ be a subspace such that*

$$(\Omega + \dot{\Omega})/\Omega \text{ is exact} \tag{30}$$

and which admits a basis satisfying (18). Then Ω is controlled invariant under a quasi-static state feedback $u = \phi(x,v)$ if and only if

$$(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega \tag{31}$$

Moreover, if the conditions above are satisfied, then $\phi(x,v)$ rendering Ω invariant may be chosen of the form (3).

Proof The necessity was proven in Theorem 3.7.
Assume that (31) holds. Note that $\Omega + \dot{\Omega} \subset \text{span}_{\mathcal{K}}\{dx, du\}$. Let $\tilde{\Omega} \subset \mathcal{X}$ be such that $\Omega = (\Omega \cap \mathcal{G}^{\perp}) \oplus \tilde{\Omega}$. Assume that $\dot{\tilde{\Omega}} \cap \mathcal{X} \neq \{0\}$. This implies that there is an $\tilde{\omega} \in \tilde{\Omega}$, $\tilde{\omega} \neq 0$, such that $\dot{\tilde{\omega}} \in \mathcal{X}$ and hence $\tilde{\omega} \in (\Omega \cap \mathcal{G}^{\perp})$, which gives a contradiction. Thus

$$\dot{\tilde{\Omega}} \cap \mathcal{X} = \{0\} \tag{32}$$

By (30), there exists $v_1(x,u)$ such that

$$\Omega + \dot{\Omega} = \Omega \oplus \text{span}_{\mathcal{K}}\{dv_1\} \tag{33}$$

Since (31) and (32) hold, we must have that $(\partial v_1/\partial u)$ has full row rank. Then there exists a function $v_2(u)$ such that $(\partial v/\partial u)$ is square and invertible, where $v = (v_1^T \;\; v_2^T)^T$. By (33) we now have that

$$\Omega^{[1]} \subset \Omega + \mathcal{V} \tag{34}$$

Moreover, since $(\partial v/\partial u)$ is invertible, there exists a $\psi(x,v)$ such that $u = \psi(x,v)$. Hence ψ defines a quasi-static state feedback and thus Ω can be rendered invariant via quasi-static state feedback. Since we are dealing with a control system (1) that is affine in u, it is easily seen that v can be taken affine in u and thus ψ can be taken affine in v. This implies that Ω can be rendered invariant via a static state feedback (3). ■

Remark 3.11

(i) If Ω is exact, then clearly also $(\Omega + \dot{\Omega})/\Omega$ is exact. Hence the set of subspaces $\Omega \subset \mathcal{X}$ such that $(\Omega + \dot{\Omega})/\Omega$ is exact, incorporates the standard integrable codistributions.

(ii) The exactness of $(\Omega + \dot{\Omega})/\Omega$ is not necessary for controlled invariance. This can be seen from the following counter example. Take the system $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$, $\dot{x}_3 = 0$ and $\Omega = \text{span}_{\mathcal{K}}\{dx_1 + x_2 dx_3\}$. It is straightforward to check that $(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega$ and that $(\Omega + \dot{\Omega})/\Omega$ is not exact. However, with the regular static state feedback $u_1 = v_1 - x_3 v_2$, $u_2 = v_2$ we obtain

$$\dot{\Omega} = \text{span}_{\mathcal{K}}\{dv_1 - x_3 dv_2\} \subset \Omega + \mathcal{V}$$

and hence Ω is controlled invariant.

3.4 Some characterizations of controlled invariance

In this subsection, conditions are derived for controlled invariance of a subspace under a quasi-static state feedback.

3.4.1 The general case : a sufficient condition

Let us consider a general subspace $\Omega \subset \mathcal{X}$. Define by induction:

$$\begin{aligned}
\hat{\Omega}_0 &:= 0 \\
\Omega_0 &:= \Omega \\
\hat{\Omega}_{k+1} &:= \text{maximal exact subspace in } \frac{\Omega_k + \dot{\Omega}_k}{\Omega_k} \\
\Omega_{k+1} &:= \Omega_k + \hat{\Omega}_{k+1}
\end{aligned}$$

Furthermore, define

$$k^* := \max\{k \geq 1 \mid \dim(\hat{\Omega}_k) > \dim(\hat{\Omega}_{k-1})\}$$

Theorem 3.12 *Let $\Omega \subset \mathcal{X}$. If*

(i) $(\widehat{\Omega \cap \mathcal{G}^{\perp}}) \subset \Omega$

(ii) $\dfrac{\Omega_{k^*-1} + \dot{\Omega}_{k^*-1}}{\Omega_{k^*-1}}$ *is exact.*

then Ω is controlled invariant for (1).

Proof From the definition of k^*, there exist vector valued $dv_1, \cdots, dv_{k^*}$ in $\mathcal{E}$, where each dv_i is non-empty, such that

$$
\begin{aligned}
\hat{\Omega}_1 &= \text{span}_{\mathcal{K}}\{dv_1\} \subset \frac{\Omega_0 + \dot{\Omega}_0}{\Omega_0} \\
\hat{\Omega}_2 &= \text{span}_{\mathcal{K}}\{d\dot{v}_1, dv_2\} \subset \frac{\Omega_1 + \dot{\Omega}_1}{\Omega_1} \\
&\vdots \\
\hat{\Omega}_{k^*} &= \text{span}_{\mathcal{K}}\{dv_1^{(k^*-1)}, dv_2^{(k^*-2)}, \cdots, dv_{k^*}\} \subset \frac{\Omega_{k^*-1} + \dot{\Omega}_{k^*-1}}{\Omega_{k^*-1}}
\end{aligned}
\tag{35}
$$

Note that from *(ii)* the last inclusion in (35) is in fact an equality. We now have

$$
\begin{aligned}
\dot{\Omega} \subset\ & \Omega_0 + \hat{\Omega}_1 + \dot{\Omega}_0 + \dot{\hat{\Omega}}_1 = \Omega_1 + \dot{\Omega}_1 \subset \cdots \subset \\
& \Omega_{k^*-1} + \dot{\Omega}_{k^*-1} = \Omega_{k^*-1} + \text{span}_{\mathcal{K}}\{dv_1^{(k^*-1)}, \cdots, dv_{k^*}\} \subset \\
& \Omega + \text{span}_{\mathcal{K}}\{dv^{(k)} \mid k \geq 0\}
\end{aligned}
\tag{36}
$$

It remains to be shown that v defines a quasi-static state feedback. From the above construction, one has

$$
\begin{aligned}
v_1 &= \phi_1(x, u) \\
v_2 &= \phi_2(x, v_1, \dot{v}_1, u) \\
&\vdots \\
v_{k^*} &= \phi_{k^*}(x, \{v_i^{(j)} \mid 1 \leq i \leq k^* - 1, 0 \leq j \leq k^* - i\}, u)
\end{aligned}
\tag{37}
$$

From *(i)*, $(\partial(\phi_1, \cdots, \phi_{k^*})/\partial u)$ has full row rank on an open and dense subset of $I\!R^n \times I\!R^{(k^*-1)(k^*-i+1)} \times I\!R^m$. By the Implicit Function Theorem, for every point of this open and dense subset there exists a neighborhood of this point and a function ψ such that $u = \psi(x, v, \dot{v}, \cdots, v^{(k^*)})$. By applying this feedback, one has

$$\Omega^{[1]} \subset \Omega + \text{span}_{\mathcal{K}}\{dv^{(k)} \mid k \geq 0\}$$

■

Remark 3.13 The above theorem only gives sufficient conditions for the controlled invariance of a subspace $\Omega \subset \mathcal{X}$. In Theorem 3.7 it was shown that *(i)* is also a necessary condition. But the condition *(ii)* is not. This is shown by the following example.

Example 3.14 ([30]) We consider a nonlinear system on $I\!R^4$ with three inputs u_1, u_2, u_3 given by:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_4 + u_2, \quad \dot{x}_3 = x_3 u_1 + u_2, \quad \dot{x}_4 = u_3$$

Let $\Omega = \text{span}_{\mathcal{K}}\{dx_1 - u_1 dx_3, dx_4\}$.Then Ω is not exact, and $\dot{\Omega}$ is given by

$$\dot{\Omega} = \text{span}_{\mathcal{K}}\{(1 - u_1 x_1)du_1 - u_1 du_2 - \dot{u}_1 dx_3 - u_1^2 dx_1, du_3\}.$$

Ω is rendered invariant by $u_1 = v_1$, $u_2 = -\dfrac{\dot{v}_1}{v_1^2}x_1 - v_1 x_1 + v_2$ and $u_3 = v_3$. One obtains $k^* = 1$, but $\dfrac{\Omega + \dot{\Omega}}{\Omega}$ is not exact.

3.4.2 The smallest controlled invariant subspace containing a given subspace

Given a subspace $\Pi \subset \mathcal{X}$, it is unclear whether (or under what conditions) there exists a smallest controlled invariant subspace containing Π. This is due to the fact that for two controlled invariant subspaces $\Omega_1, \Omega_2 \subset \mathcal{X}$, we do not necessarily have that $\Omega_1 \cap \Omega_2$ is controlled invariant, so that we cannot use the "standard" arguments (as in e.g. [48],[29],[38]). In this subsection we will give some comments on this question.

We will use the following notation. Given a subspace $\Pi \subset \mathcal{X}$, we define

$$\Pi_* := \mathcal{X} \cap (\Pi + \Pi^{(1)} + \cdots + \Pi^{(n-1)}) \tag{38}$$

In what follows, we will need the following lemma.

Lemma 3.15 *Consider a subspace $\Omega \subset \mathcal{X}$ satisfying $(\Omega \cap \mathcal{G}^{\perp}) = \{0\}$. Then we have for all $k \in I\!N$:*

$$\mathcal{X} \cap (\Omega^{(1)} + \cdots + \Omega^{(k)}) = \{0\} \tag{39}$$

Proof Let $d := \dim(\Omega)$, and let $\omega_1, \cdots, \omega_d$ be a basis of Ω, with

$$\omega_i = \sum_{j=1}^{n} \omega_{ij}(x, u, \cdots, u^{(r)}) dx_j \quad (i = 1, \cdots, d) \tag{40}$$

Let $A(x, u, \cdots, u^{(r)})$ be the (d, n)-matrix with entries ω_{ij} $(i = 1, \cdots, d; j = 1, \cdots, n)$. Since $\omega_1, \cdots, \omega_d$ forms a basis of Ω, the matrix A has full row rank over $\mathcal{K}$. We may now characterize Ω by

$$\Omega = \text{rowspan}_{\mathcal{K}}(A(x, u, \cdots, u^{(r)}) \ 0 \ \cdots \ 0) \tag{41}$$

while $\Omega^{(k)}$ $(k = 1, 2, \cdots)$ may be characterized by

$$\Omega^{(k)} = \text{rowspan}_{\mathcal{K}}(X_{k0} \ X_{k1} \ \cdots \ X_{kk-1} \ (Ag) \ 0 \ \cdots \ 0) \tag{42}$$

for certain matrices $X_{k0}, \cdots, X_{kk-1}$. Now assume that (Ag) is not right-invertible over $\mathcal{K}$. This implies that there exists a non-zero row-vector $\eta^T := (\eta_1 \cdots \eta_d)$ such that

$$\eta^T (Ag) = 0 \tag{43}$$

However, this would imply that $\omega := \sum_{j=1}^{d} \eta_j \omega_j$ satisfies

$$\langle \omega, \tau \rangle = 0 \quad (\forall \tau \in \mathcal{G}) \tag{44}$$

which contradicts the fact that $(\Omega \cap \mathcal{G}^{\perp}) = \{0\}$. Hence we have that (Ag) is right-invertible over $\mathcal{K}$. Next, let $\omega \in \mathcal{X} \cap (\Omega^{(1)} + \cdots + \Omega^{(k)})$ $(k \in \{1, 2, \cdots\})$. Since $\omega \in (\Omega^{(1)} + \cdots + \Omega^{(k)})$, we may represent ω by a row-vector

$$(\eta_1^T \ \cdots \ \eta_k^T) \begin{pmatrix} X_{10} & (Ag) & 0 & \cdots & \cdots & 0 \\ X_{20} & X_{21} & (Ag) & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ X_{k0} & X_{k1} & X_{k2} & \cdots & X_{kk-1} & (Ag) \end{pmatrix}$$

The fact that $\omega \in \mathcal{X}$ implies that necessarily

$$(\eta_1^T \ \cdots \ \eta_k^T) \begin{pmatrix} (Ag) & 0 & 0 & \cdots & \cdots & 0 \\ X_{21} & (Ag) & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ X_{k1} & X_{k2} & X_{k3} & \cdots & X_{kk-1} & (Ag) \end{pmatrix} = 0$$

and thus

$$\eta_i^T (Ag) = 0$$

which give $\eta_i^T = 0$, since (Ag) is right-invertible. Thus, $\omega = 0$, which establishes our claim. ∎

Proposition 3.16 *Let $\Omega \subset \mathcal{X}$ be a subspace satisfying $(\widehat{\Omega \cap \mathcal{G}^\perp})^{\cdot} \subset \Omega$. Then*

$$\Omega_* = \Omega$$

Proof Let $\tilde{\Omega}$ be such that

$$\Omega = (\Omega \cap \mathcal{G}^\perp) \oplus \tilde{\Omega} \tag{45}$$

By hypothesis we have

$$(\widehat{\Omega \cap \mathcal{G}^\perp})^{\cdot} \subset \Omega \tag{46}$$

We now prove by induction that we have

$$(\Omega \cap \mathcal{G}^\perp)^{(k)} \subset \Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(k-1)} \quad (k = 1, 2, \cdots) \tag{47}$$

By (46), we have that (47) holds for $k = 1$. Next assume that (47) holds for $k = 1, \cdots, \ell - 1$. Then

$$\begin{aligned}(\Omega \cap \mathcal{G}^\perp)^{(\ell)} = \;& ((\Omega \cap \mathcal{G}^\perp)^{(\ell-1)})^{(1)} \overset{\text{IH}}{\subset} (\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-2)})^{(1)} = \\ & (\Omega^{(1)} + \tilde{\Omega}^{(2)} + \cdots + \tilde{\Omega}^{(\ell-1)}) \overset{(45)}{=} ((\widehat{\Omega \cap \mathcal{G}^\perp})^{\cdot} + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-1)}) \overset{(46)}{\subset} \\ & (\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(\ell-1)})\end{aligned}$$

which establishes (47). Using (47) and the modular distributive rule (see e.g. [48, Section 0.3]) we obtain

$$\begin{aligned}\Omega_* = \;& \mathcal{X} \cap (\Omega + \Omega^{(1)} + \cdots + \Omega^{(n-1)}) = \\ & \mathcal{X} \cap (\Omega + (\Omega \cap \mathcal{G}^\perp)^{(1)} + \tilde{\Omega}^{(1)} + \cdots + (\Omega \cap \mathcal{G}^\perp)^{(n-1)} + \tilde{\Omega}^{(n-1)}) \subset \\ & \mathcal{X} \cap (\Omega + \tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(n-1)}) = \Omega + \mathcal{X} \cap (\tilde{\Omega}^{(1)} + \cdots + \tilde{\Omega}^{(n-1)})\end{aligned} \tag{48}$$

Since by definition of $\tilde{\Omega}$ we have that $(\tilde{\Omega} \cap \mathcal{G}^\perp) = \{0\}$, we obtain from (48) and Lemma 3.15 that

$$\Omega_* \subset \Omega \tag{49}$$

Furthermore, we have by definition of Ω_* that

$$\Omega \subset \Omega_* \tag{50}$$

Hence we have that $\Omega_* = \Omega$, which establishes our claim. ∎

Corollary 3.17 *Consider a subspace $\Pi \subset \mathcal{X}$ and let $\Omega \subset \mathcal{X}$ be a controlled invariant subspace containing Π. Then $\Pi_* \subset \Omega$.*

Proof Using the definition of Π_*, the fact that $\Pi \subset \Omega$, and combining the results of Theorem 3.7 and Proposition 3.16, we obtain

$$\Pi_* = \mathcal{X} \cap (\Pi + \Pi^{(1)} + \cdots + \Pi^{(n-1)}) \subset \mathcal{X} \cap (\Omega + \Omega^{(1)} + \cdots + \Omega^{(n-1)}) = \Omega_* = \Omega$$

which establishes our claim. ∎

The subspace Π_* defined in (38) is, by Corollary 3.17, a candidate for being the smallest controlled invariant subspace containing Π. If Π is exact, it can be shown that indeed it *is*. This may be shown in the following way. Let $r = \dim \Pi$ and choose meromorphic functions $h_1(x), \cdots, h_r(x)$ such that $\Pi = \mathrm{span}_{\mathcal{K}}\{dh_1, \cdots, dh_r\}$. Next consider the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{51}$$

Then for this system, $\Pi_* = \mathcal{X} \cap \mathcal{Y}$, where $\mathcal{Y} = \mathrm{span}_{\mathcal{K}}\{dy, \cdots, dy^{(n-1)}\}$. (The subspace $\mathcal{X} \cap \mathcal{Y}$ was introduced in [9] for the study of the minimal order input-output decoupling problem.) If the system (51) is right-invertible, one can construct a quasi-static state feedback which renders Π_* invariant by using the construction in [41]. If (51) is not right-invertible, the same construction, together with Lemma 1 from [35] may be used to show that Π_* is controlled invariant. Summarizing, we have the following result:

Theorem 3.18 *Consider a subspace $\Pi \subset \mathcal{X}$ which is exact. Then $\Pi_* := \mathcal{X} \cap (\Pi + \cdots + \Pi^{(n-1)})$ is the smallest controlled invariant subspace containing Π.* ∎

An application of the subspace $\Omega_* = \mathcal{X} \cap \mathcal{Y}$ is given in Section 3.5, where we consider the dynamic disturbance decoupling problem.

It has been shown that a "standard" controlled invariant codistribution is a controlled invariant subspace in the sense of Definition 3.5. If Δ^* denotes the largest controlled invariant distribution contained in $ker\, dh$, then ${\Delta^*}^{\perp} \cap \mathcal{X}$ is a controlled invariant subspace containing the differential of the output. Since $\Omega_* = \mathcal{X} \cap \mathcal{Y}$ is the smallest controlled invariant subspace containing $\mathrm{span}_{\mathcal{K}}\{dy\}$, one has ${\Delta^*}^{\perp} \cap \mathcal{X} \supset \Omega_*$. These two different geometric structures are displayed in the following lattice diagram (Figure 1).

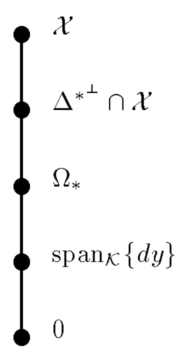

Figure 1: Lattice diagram: (geometric) structure of nonlinear systems

In the special case of linear systems, this lattice diagram is simplified since ${\Delta^*}^{\perp} \cap \mathcal{X} = \Omega_*$.

3.4.3 A special case

Let us consider a subspace $\Omega \subset \mathcal{X}$ such that

$$\Omega = \Omega \cap \mathcal{G}^{\perp} + \Phi_* \tag{52}$$

where Φ is an exact subspace of $\mathcal{X}$.

Proposition 3.19 *Let $\Omega \subset \mathcal{X}$ satisfy (52), then Ω is controlled invariant if and only if*

$$(\widehat{\Omega \cap \mathcal{G}^{\perp}})^{\cdot} \subset \Omega \tag{53}$$

Proof By Theorem 3.7 we only need to show the sufficiency. Clearly Φ_* is controlled invariant (see Theorem 3.18). Hence there exists a quasi-static feedback (8) such that

$$\Phi_*^{[1]} \subset \Phi_* + \mathcal{V}$$

Now (53) implies that

$$\Omega^{[1]} \subset \Omega + \mathcal{V}$$

and hence Ω is controlled invariant. ■

The following proposition gives conditions for the existence of a subspace $\Phi \subset \mathcal{X}$ such that (52) holds.

Proposition 3.20 *Let $\Omega \subset \mathcal{X}$ be a subspace such that (53) holds. Then there exists an exact subspace $\Phi \subset \Omega$ satisfying (52) if and only if*

$$\Omega = \Omega \cap \mathcal{G}^{\perp} + \hat{\Phi}_* \tag{54}$$

where $\hat{\Phi}$ is the largest exact subspace in Ω

Proof Assume that (54) holds. Taking $\Phi = \hat{\Phi}$, we then have (52). Conversely, assume that there exists an exact subspace $\Phi \subset \mathcal{X}$ such that (52) holds. Clearly $\Phi_* \subset \hat{\Phi}_*$. Now $\hat{\Phi} \subset \Omega$ implies by Proposition 3.16 that $\hat{\Phi}_* \subset \Omega$. Thus

$$\Omega = \Omega \cap \mathcal{G}^{\perp} + \Phi_* \subset \Omega \cap \mathcal{G}^{\perp} + \hat{\Phi}_* \subset \Omega$$

Hence (54) is verified. ■

3.5 Dynamic disturbance decoupling

A fundamental application of controlled invariance is disturbance decoupling ([48],[29],[38]). In this section, generalized controlled invariance is shown to yield a geometric condition that characterizes the solvability of the dynamic feedback disturbance decoupling problem (*DDDP*). *DDDP* is stated as follows.

Consider a perturbed system Σ_q given by

$$\Sigma_q : \left\{ \begin{array}{lcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ y & = & h(x) \end{array} \right. \tag{55}$$

where q represents a disturbance. Find, if possible, a dynamic state feedback such that the disturbance q does not affect the output y.

Let $\mathcal{P}$ denote the distribution spanned by the disturbance vector fields. Define the subspace $\mathcal{P}^{\perp}$ by

$$\mathcal{P}^{\perp} = \{\omega \in \mathcal{X} \mid \langle \omega, p \rangle = 0, \ \forall p \in \mathcal{P}\} \tag{56}$$

The following result gives a necessary and sufficient condition for *DDDP* to be solvable.

Theorem 3.21 *DDDP is solvable if and only if there exists a controlled invariant subspace Ω such that*

$$\text{span}_{\mathcal{K}}\{dy\} \subset \Omega \subset \mathcal{P}^{\perp} \tag{57}$$

Proof From Theorem 2.3 in [41] it follows that *DDDP* is solvable if and only if it is solvable by quasi-static state feedback. Thus, to prove Theorem 3.21, it suffices to show that the disturbance decoupling problem is solvable by quasi-static state feedback.

Sufficiency. Controlled invariance of Ω implies that there exists a quasi-static state feedback $u = \phi(x, v, \cdots, v^{(r)})$ such that

$$\Omega^{[1]} \subset \Omega + \mathcal{V} \tag{58}$$

By (57) and (58), one has to

$$dy^{[k]} \subset \Omega + \mathcal{V}\ , \quad \forall k \geq 0 \tag{59}$$

Thus in the closed loop system the output y is decoupled from the disturbance.

Necessity. Suppose that the quasi-static state feedback $u = \phi(x, v, \cdots, v^{(r)})$ solves the disturbance decoupling problem. Therefore, for the system Σ_q fed back with $u = \phi(x, v, \cdots, v^{(r)})$, one has

$$dy^{[k]} \subset \mathrm{span}_{\mathcal{K}_v}\{dx, dv, \cdots, dv^{(r+k-1)}\}, \quad \forall k \tag{60}$$

Define the sequence Ω_μ as

$$\begin{array}{rcl} \Omega_0 & = & \mathcal{P}^\perp \\ \Omega_{\mu+1} & = & \{\omega \in \Omega_\mu \mid \omega^{[1]} \in \Omega_\mu + \mathcal{V}\} \quad \forall \mu \geq 1 \end{array} \tag{61}$$

and

$$\Omega = \lim_{\mu \to \infty} \Omega_\mu .$$

Obviously $\Omega^{[1]} \subset \Omega + \mathcal{V}$. Thus, Ω is a controlled invariant subspace. Since $\mathrm{span}_{\mathcal{K}}\{dy\} \subset \Omega$ and $\Omega \subset \mathcal{P}^\perp$, (57) also holds. ∎

Condition in Theorem 3.21 is not constructive. The corresponding constructive condition is obtained when considering the smallest controlled invariant subspace containing the differential of the output Ω_*. From Theorem 3.18, Ω_* is given by $\mathcal{X} \cap \mathcal{Y}$. An immediate consequence of Theorem 3.21 is then

Corollary 3.22 *DDDP is solvable if and only if*

$$\Omega_* \subset \mathcal{P}^\perp \tag{62}$$

Remark Theorem 3.21 gives the nonlinear feedback analogous of Theorem 4.2 in [48] for the linear *(D)DDP*. Also, it gives the dynamic feedback analogous of condition (3.1) in [29] and Proposition 7.8 in [38] for the nonlinear *DDP*. In this way it is established that our generalized notion of controlled invariance is the natural generalization to the nonlinear dynamic feedback case of the linear notion of controlled invariance defined in [48]. It may be checked that condition (62) is equivalent to the geometric conditions (44) in [26] and (4.5) in [44]. Further, (62) is exactly the same as the condition for solvability of the *DDDP* derived in [41]. However, in [41] the concept of controlled invariance was missing.

Table 1, which displays the various solutions of the disturbance decoupling problem, is now completed in Table 4.

4 Controllability cospaces

In this section, controllability cospaces under quasi-static state feedback are studied. These controllability cospaces form a special class of the controlled invariant subspaces defined previously. They parallel the dynamic controllability distributions [45]. In Subsection 4.1 we first define controllability cospaces. An algorithm which characterizes these cospaces is then given in Subsection

	Static feedback	Dynamic feedback
Linear systems	$\mathcal{P} \subset \mathcal{V}^*$	
Nonlinear systems	$\mathcal{P} \subset \Delta^*$	$\Omega_* \subset \mathcal{P}^\perp$

Table 4: (Geometric) solution to disturbance decoupling problem (complete)

4.2 and some properties of these controllability cospaces are discussed. In Subsection 4.3 we derive an algorithm computing the smallest controllability cospace containing a given exact subspace. Applications of controllability cospaces are treated in Subsection 4.4 and Subsection 4.5. In particular, the fixed modes or decoupling zero dynamics under quasi-static feedback are characterized using controllability cospaces.

4.1 Definition of controllability cospaces

Controllability cospaces are vector spaces that are autonomous after having applied a certain quasi-static state feedback $u = \psi(x, v, \cdots, v^{(r)})$ and zeroing certain input channels v_j, where $j \in \mathcal{J} \subset \{1, \cdots, m\}$. Such nonregular transformations are not defined for every element in $\mathcal{K}_v$. One possibility to circumvent this problem is to consider the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring of analytic functions rather than the linear space over the field of meromorphic functions. Another way is chosen here; it consists in taking a particular basis of a given subspace of $\text{span}_{\mathcal{K}}\{dx\}$ so that its time derivative is well defined when applying nonregular feedback. Such a basis always exists. More precisely, let $\Theta \subset \mathcal{X}$ be a subspace which admits a basis $\theta_1, \cdots, \theta_d$ with

$$\theta_i = \sum_{k=1}^{n} \frac{\alpha_{ik}(x, v, \cdots, v^{(\nu)})}{\beta_{ik}(x, v, \cdots, v^{(\nu)})} dx_i,$$

where α_{ik} and β_{ik} are in $\mathcal{A}$, the ring of analytic functions of $\{x, v^{(k)} \mid k \geq 0\}$. Obviously, we can choose another basis for Θ, $\tilde{\theta}_1, \cdots, \tilde{\theta}_d$, in the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$ by taking

$$\tilde{\theta}_i = (\prod_{k=1}^{n} \beta_{ik})\theta_i$$

Definition 4.1 A subspace $\mathcal{C}\subset\mathcal{X}$ is said to be a controllability cospace for (1) if there exist a quasi-static state feedback (8) and a set of integers $\mathcal{J} \subset \{1, \cdots, m\}$ such that for (1, 8) one has

$$\mathcal{C}^{[1]} \subset \mathcal{C} + \mathcal{V} \tag{63}$$

and

$$\mathcal{C} = \max\{\Theta \subset \mathcal{X} \mid \text{span}_{\mathcal{K}}\{\tilde{\theta}_i^{[1]} \mid_{v_j=0, j\in\mathcal{J}}\} \subset \Theta\} \tag{64}$$

where $\tilde{\theta}_i$ is defined as above.

This means that $\mathcal{C}$ is the largest autonomous subspace in $\mathcal{X}$ of the closed loop system. Moreover, by this definition, it is clear that a controllability cospace is controlled invariant. The following example illustrates the above definition.

Example 4.2 Consider again the nonlinear system given in Example 3.14. Let $\mathcal{C} = \text{span}_{\mathcal{K}} \{dx_1, d(x_2 - x_3), dx_4 - u_1 dx_3\}$, and suppose that $u_1 = v_1 + c$ where c is a non-zero constant, $u_2 = v_2$ and $u_3 = v_3 + \dot{v}_1 x_3 + (v_1 + c)^2 x_3 + (v_1 + c)v_2$. This feedback is quasi-static since $v_1 = u_1 - c$ and $v_2 = u_2$ and $v_3 = u_3 - \dot{u}_1 x_3 - u_1^2 x_3 - u_1 u_2$.

From this, it is easy to show that

$$\mathcal{C}^{[1]} = \text{span}_{\mathcal{K}}\{dv_1, dx_4 - u_1 dx_3, dv_3 + (x_3(v_1 + c) + v_2)dv_1 + x_3 d\dot{v}_1\} \subset \mathcal{C} + \mathcal{V}$$

and

$$\mathcal{C}^{[1]}\mid_{v_1=0,v_3=0}=\operatorname{span}_{\mathcal{K}}\{dx_4-u_1dx_3\}\subset\mathcal{C}$$

Furthermore

$$\mathcal{C}=\max\{\Theta\subset\mathcal{X}\mid\Theta^{[1]}\mid_{v_1=0,v_3=0}\subset\Theta\}$$

Hence $\mathcal{C}$ is a controllability cospace in the sense of Definition 4.1.

4.2 Controllability cospace algorithm

First of all, we give an algorithm characterizing controllability cospaces, called *the controllability cospace algorithm*. Some properties of a general controllability cospace are then derived. Let $\mathcal{C}$ be a given subspace and define a sequence $\mathcal{S}_\mu$ according to

$$\begin{aligned}\mathcal{S}_0 &:= \mathcal{X}\\ \mathcal{S}_{\mu+1} &:= \operatorname{span}_{\mathcal{K}}\{\omega\in\mathcal{S}_\mu\mid\dot{\omega}\in\mathcal{S}_\mu+\dot{\mathcal{C}}\}\quad(\mu\in I\!N)\end{aligned}\tag{65}$$

The sequence $\mathcal{S}_\mu$ is decreasing. Thus, there exists a $\mu^*\in I\!N$ such that $\mathcal{S}_{\mu^*}=\mathcal{S}_{\mu^*+1}=\ldots=\mathcal{S}^*$.
The algorithm (65) yields a necessary condition for a subspace $\mathcal{C}$ of $\mathcal{X}$ to be a controllability cospace. This is shown in the following lemma.

Lemma 4.3 *Let $\mathcal{C}\subset\mathcal{X}$. If $\mathcal{C}$ is a controllability cospace, then $\mathcal{C}=\mathcal{S}^*$*

Proof Assume that $\mathcal{C}$ is a controllability cospace. Let $\{\tilde{\omega}_i\}$ be a basis for $\mathcal{C}$ in the module $\operatorname{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$. Then by definition there exists a quasi-static state feedback (8) and a set of integers $\mathcal{J}\subset\{1,\cdots,m\}$ such that $\mathcal{C}^{[1]}\subset\mathcal{C}+\mathcal{V}$ and $\mathcal{C}^{[1]}=\operatorname{span}_{\mathcal{K}}\{\tilde{\omega}_i^{[1]}\mid_{v_j=0,j\in\mathcal{J}}\}\subset\mathcal{C}$. From (65), it follows that $\mathcal{S}^*$ satisfies

$$\mathcal{S}^*=\operatorname{span}_{\mathcal{K}}\{\omega\in\mathcal{X}\mid\dot{\omega}\in\mathcal{S}^*+\dot{\mathcal{C}}\}\tag{66}$$

Let $\omega\in\mathcal{C}$. We have $\dot{\omega}\in\dot{\mathcal{C}}$ and hence $\omega\in\mathcal{S}^*$. This implies that $\mathcal{C}\subset\mathcal{S}^*$. Now, $\dot{\mathcal{S}}^*\subset\mathcal{S}^*+\dot{\mathcal{C}}$. By the feedback which yields $\mathcal{C}^{[1]}\subset\mathcal{C}$, one has $\mathcal{S}^{*[1]}\subset\mathcal{S}^*$. Since $\mathcal{C}$ is the largest subspace in $\mathcal{X}$ such that $\mathcal{C}^{[1]}\subset\mathcal{C}$, one has $\mathcal{S}^*\subset\mathcal{C}$. ∎

In the next section, we give an algorithm computing the smallest controllability cospace containing a given subspace, based on algorithm (65).

4.3 The smallest controllability cospace containing a given subspace

In general, the intersection of two controllability cospaces is not a controllability cospace. Thus it is unclear if there exists a smallest controllability cospace containing some given subspace. However, if an exact subspace $\Pi\subset\mathcal{X}$ is given, then there exists a smallest one containing Π.
Consider a nonlinear system given by (1). By Theorem 3.18, Π_* is the smallest controlled invariant subspace containing Π. The next theorem will relate Π_* to the smallest controllability cospace containing Π.

Theorem 4.4 *Define the sequence $\mathcal{D}_\mu$ by*

$$\begin{aligned}\mathcal{D}_0 &= \mathcal{X}\\ \mathcal{D}_{\mu+1} &= \operatorname{span}_{\mathcal{K}}\{\omega\in\mathcal{D}_\mu\mid\dot{\omega}\in\mathcal{D}_\mu+\dot{\Pi}_*\}\quad(\mu\in I\!N)\end{aligned}\tag{67}$$

Then $\mathcal{D}_*=\lim_{\mu\to\infty}\mathcal{D}_\mu$ is the smallest controllability cospace containing Π.

Proof Note that

$$\mathcal{D}_* = \text{span}_{\mathcal{K}}\{\omega \in \mathcal{X} \mid \dot{\omega} \in \mathcal{D}_* + \dot{\Pi}_*\} \tag{68}$$

Let $r = \dim\Pi$. The fact that Π is exact implies that there exist meromorphic functions $\varphi_1(x), \cdots, \varphi_r(x)$ such that $\Pi = \text{span}_{\mathcal{K}}\{d\varphi_1, \cdots, d\varphi_r\}$. Consider the system (1) with a "dummy" output $\varphi = (\varphi_1, \cdots, \varphi_r)^T$. We decompose the output φ as $\varphi = (\tilde{\varphi}, \hat{\varphi})^T$ so that the system (1) with the output $\tilde{\varphi}$ is right-invertible. Define $\rho := \dim(\tilde{\varphi})$.

Construct a quasi-static state feedback $u = \phi(x, v, \ldots, v^{(r)})$, by taking $v_i = \tilde{\varphi}_i^{(n_i^{'})}$ where $\{n_i^{'}\}$ is the set of orders of zeros at infinity [18], for $i = 1, \cdots, \rho$ and $v_i = w_i$ for $i = \rho + 1, \ldots, m$. This feedback always renders Π_* invariant. Thus, $\mathcal{D}_*$ is rendered invariant too, i.e., $\mathcal{D}_*^{[1]} \subset \mathcal{D}_* + \mathcal{V}$. Let now $\{\tilde{\omega}_i\}$ be a basis for $\mathcal{D}_*$ in the module $\text{span}_{\mathcal{A}}\{dx\}$ over the ring $\mathcal{A}$. If we set $v_i = 0$ for $i = 1, \ldots, \rho$ one obtains

$$\mathcal{D}_*^{[1]} = \text{span}_{\mathcal{K}}\{\tilde{\omega}_i^{[i]} \mid_{v_j=0, j=1,\cdots,\rho}\} \subset \mathcal{D}_*$$

and $\mathcal{D}_*$ is then a controllability cospace. In order to prove that $\mathcal{D}_*$ is the smallest controllability cospace containing Π, we consider another controllability cospace $\mathcal{D}$ such that $\mathcal{D} \supset \Pi$. By definition $\mathcal{D}$ is controlled invariant and according to Lemma 4.3, $\mathcal{D}$ satisfies

$$\mathcal{D} = \text{span}_{\mathcal{K}}\{\omega \in \mathcal{X} \mid \dot{\omega} \in \mathcal{D} + \dot{\mathcal{D}}\} \tag{69}$$

Since Π_* is the smallest controlled invariant subspace containing Π, this implies that $\mathcal{D} \supset \Pi_*$. From (68) and (69), one has $\mathcal{D}_* \subset \mathcal{D}$. ∎

Now we consider a nonlinear system given by (51). Clearly $\Omega_* = \mathcal{X} \cap \mathcal{Y}$ is the smallest controlled invariant subspace containing the differential of the output. Therefore the smallest controllability cospace containing the differential of the output is given by the next corollary.

Corollary 4.5 *Define the sequence $\mathcal{C}_\mu$ according to*

$$\begin{aligned} \mathcal{C}_0 &= \mathcal{X} \\ \mathcal{C}_{\mu+1} &= \text{span}_{\mathcal{K}}\{\omega \in \mathcal{C}_\mu \mid \dot{\omega} \in \mathcal{C}_\mu + \dot{\Omega}_*\} \quad (\mu \in I\!N) \end{aligned} \tag{70}$$

Then $\mathcal{C}_* = \lim_{\mu\to\infty} \mathcal{C}_\mu$ is the smallest controllability cospace containing $\text{span}_{\mathcal{K}}\{dh(x)\}$.

Remark 4.6 When specialized to linear systems, the sequence $\mathcal{C}_\mu$ (70) turns out to be equal to the dual of the sequence $\mathcal{R}_\mu$ (the sequence computing the maximal controllability subspace in the kernel of the output mapping). A proof of this can be found in the Appendix.

The geometric structure of a nonlinear system as presented in Figure 1 can now be completed. Let $\mathcal{R}^*$ be the largest controllability distribution contained in the kernel of the output. As an immediate consequence, $\mathcal{R}^{*^\perp} \cap \mathcal{X}$ is a controllability cospace in the sense of Definition 4.1. Figure 2 displays further geometric structures of nonlinear systems.

4.4 The block input-output decoupling problem

We now use the smallest controllability cospace $\mathcal{C}_*$ previously defined to solve a quasi-static state feedback input-output decoupling problem. For this, we consider the system (1) together with the partitioned output blocks y_i for $i = 1, \ldots, k$, given by:

$$y_i = h_i(x) \tag{71}$$

The problem can be stated as follows: find a quasi-static state feedback and a partition of the new control $v = (v_1^T, \ldots, v_k^T)^T$ such that the new input v_i^T affects and only affects the output y_i.

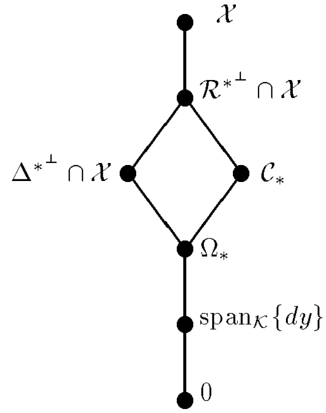

Figure 2: Lattice diagram : (Geometric) structure of nonlinear systems (continued, see Figure 1).

Define $\mathcal{C}_{i*}$ and Ω_{i*} to be the smallest controllability cospace and the smallest controlled invariant subspace respectively, both containing $\text{span}_{\mathcal{K}}\{dh_i(x)\}$.

First, let us give the following property which is derived from Theorem 5.1 in ([42]).

Property 4.7 *Consider system (51), and assume that* $\dim(\mathcal{G}^\perp) = n - m$. *Let* ρ *be its differential output rank. Then*

$$\dim(\mathcal{G}^\perp + \Omega_*) = \dim(\mathcal{G}^\perp + \mathcal{C}_*) = (n\text{-}m\text{+}\rho). \tag{72}$$

Moreover, if the system (51) is right-invertible, then

$$\dim(\mathcal{G}^\perp + \Omega_*) = \dim(\mathcal{G}^\perp + \mathcal{C}_*) = (n\text{-}m\text{+}p). \tag{73}$$

This property is a generalization of a known result on linear systems. It gives a geometric interpretation for the rank of a system. The property was also derived by Respondek in [45] using dynamic controllability distributions.

Corollary 4.8 *The block input-output decoupling problem via quasi-static (or dynamic) state feedback for the system (1),(71) is solvable if and only if*

$$\dim\left(\frac{\mathcal{G}^\perp + \mathcal{C}_*}{\mathcal{G}^\perp}\right) = \sum_{i=1}^{k} \dim\left(\frac{\mathcal{G}^\perp + \mathcal{C}_{i*}}{\mathcal{G}^\perp}\right) \tag{74}$$

Condition (74) coincides with the condition given by Di Benedetto *et al.* ([18]), in case of the dynamic block decoupling problem. Indeed, if ρ denotes the rank of the system (1),(71) and ρ_i denotes the rank of the subsystem (1) with the output y_i, then by Property 4.7, (74) is equivalent to

$$\rho = \sum_{i=1}^{k} \rho_i \tag{75}$$

By applying the structure algorithm to the system (1),(71), a quasi-static feedback which decouples the system is obtained ([14]).

Further, controllability cospaces also allow to characterize the fixed dynamics with respect to any quasi-static feedback. This will be the topic of the next section.

4.5 Fixed modes by quasi-static state feedback

The problem of noninteraction with stability of nonlinear systems by means of static feedback has first been considered by Isidori and Grizzle [32]. They have shown that there exists a fixed internal dynamics, called P^* *dynamics* whose stability is a necessary condition to solve the noninteracting control problem with stability via static feedback. In the case where the P^* *dynamics* are unstable, Wagner has shown in [47] that there exists a well-defined dynamics, called Δ_{mix} *dynamics*, which cannot be eliminated by any regular dynamic feedback that renders the considered system noninteractive. The Δ_{mix} *dynamics* must then be asymptotically stable if noninteracting control with stability is to be achieved by means of dynamic state feedback. Glumineau *et al.* ([21]) used a dynamic compensator to remove a one-dimensional interconnection zero dynamics and showed that such compensator is able to cancel only the fixed dynamics which have a certain linearity property. A sufficient condition to solve the problem of noninteracting control with stability by means of dynamic state feedback was given in ([5],[6],[7]). In these references, the problem of dynamic feedback noninteracting control with stability is solved if some regularity assumptions are satisfied, the Δ_{mix} *dynamics* are asymptotically stable and each decoupled subsystem is asymptotically stabilizable.

All results above are valid under the assumption that the decoupling matrix $A(x)$ is nonsingular. In the case where $A(x)$ is singular and the system is square and invertible, Zhan *et al.* [50] introduced the so-called Canonical Dynamic Decoupling Algorithm to construct a canonical dynamic extension (Σ_p). They have shown that the dynamically decoupled system is stable only if the Δ_{mix} *dynamics* of the canonical dynamic extension is stable, which is an intrinsic property of the given system. These different contributions are summarized in Table 2.

In this section, we investigate the case where the decoupling matrix is not necessarily invertible and study the noninteracting control problem with stability by means of quasi-static feedback. The goal is to show that the controllability cospaces introduced before are able to describe intrinsic geometric conditions with respect to quasi-static feedbacks, analogous to the above ones. Preliminary results may be found in [1].

Let us consider a square invertible nonlinear affine system (Σ) of the form

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + \sum_{i=1}^{m} g_i(x) u_i, \ x \in I\!R^n, = \ u_i \in I\!R \\ \\ y_i & = & h_i(x), \ i = 1, \cdots, m, \ y_i \in I\!R \end{array} \right. \tag{76}$$

Let $\{n_i^{'}\}$ be the set of orders of zeros at infinity [18], where $n_1^{'} > n_2^{'} > \cdots > n_m^{'}$. Permute if necessary y_i such that the corresponding order of zero at infinity is $n_i^{'}$. Let $\mathcal{C}_{i*}$ be the smallest controllability cospace containing $\text{span}_{\mathcal{K}}\{dh_i(x)\}$. A first result is the following.

Lemma 4.9 *Suppose that the system (76) is decouplable by a quasi-static state feedback* $u = \psi(x, v, \cdots, v^{(s)})$. *Then, there always exist coordinates* $\xi = (\xi_0, \xi_1, \cdots, \xi_m, \hat{\xi})$ *such that the system (76) is presented in the following form:*

$$\begin{array}{rcl} \dot{\xi}_0 & = & f_0(\xi_0) \\ \dot{\xi}_1 & = & f_1(\xi_0, \xi_1, v_1) \\ \vdots & & \\ \dot{\xi}_m & = & f_m(\xi_0, \xi_m, v_m) \\ \dot{\hat{\xi}} & = & \hat{f}(\xi, v, \dot{v}, \cdots, v^{(s)}) \\ \\ y_i & = & h_i(\xi_0, \xi_i) \end{array} \tag{77}$$

System (77) will be referred to as a *standard decomposed system*, in analogy to [23]. To prove Lemma 4.9, we first need the following property of $\mathcal{C}_{i*}$.

Lemma 4.10 *For a scalar output* $y_i = h_i(x)$, $\mathcal{C}_{i*}$ *is an exact subspace.*

Proof Let Ω_{i*} be the smallest controlled invariant subspace containing $\text{span}_{\mathcal{K}}\{dh_i(x)\}$. If Δ_i^* is the maximal controlled invariant distribution in $\ker\{dh_i(x)\}$, we have $\Omega_{i*} = \Delta_i^{*^\perp}$. Let now $\mathcal{R}_i^*$ be the maximal controllability distribution in $\ker\{dh_i(x)\}$. Clearly $\mathcal{R}_i^{*^\perp}$ is a controllability cospace containing $\text{span}_{\mathcal{K}}\{dh_i(x)\}$, and thus $\mathcal{C}_{i*} \subset \mathcal{R}_i^{*^\perp}$. From [29], we have

$$\mathcal{R}_i^* = \Delta_i^* \cap ([f, \mathcal{R}_i^*] + \sum_{j=1}^{m}[g_j, \mathcal{R}_i^*] + \mathcal{G}) \tag{78}$$

and thus

$$\begin{aligned}\mathcal{R}_i^{*^\perp} &= \Omega_{i*} + [f, \mathcal{R}_i^*]^\perp \cap (\bigcap_{j=1}^{m}[g_j, \mathcal{R}_i^*]^\perp) \cap \mathcal{G}^\perp \\ &= \{\omega \in \mathcal{X} \mid \exists \omega_1 \in \Omega_{i*}, \exists \omega_2 \in \mathcal{G}^\perp \text{ such that } \omega = \omega_1 + \omega_2 \\ &\qquad \text{and } (\forall \tau \in \mathcal{R}_i^*)(\forall \sigma \in \{f, g_1, \cdots, g_m\})(\langle [\sigma, \tau], \omega_2 \rangle = 0)\}\end{aligned} \tag{79}$$

Let $\omega \in \mathcal{R}_i^{*^\perp}$. Then there exist $\omega_1 \in \Omega_{i*}$ and $\omega_2 \in \mathcal{G}^\perp$ such that $\omega = \omega_1 + \omega_2$, and $\forall \tau \in \mathcal{R}_i^*, \forall \sigma \in \{f, g_1, \cdots, g_m\}$, one has $\langle [\sigma, \tau], \omega_2 \rangle = 0$. Compute

$$\dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2$$

Clearly $\dot{\omega}_1 \in \dot{\Omega}_{i*}$. Furthermore

$$\begin{aligned}\dot{\omega}_2 &= \mathcal{L}_f \omega_2 + \sum_{j=1}^{m} (u_j \mathcal{L}_{g_j} \omega_2 + \langle \omega_2, g_j \rangle du_j) \\ &= \mathcal{L}_f \omega_2 + \sum_{j=1}^{m} u_j \mathcal{L}_{g_j} \omega_2\end{aligned} \tag{80}$$

Now, let $\tau \in \mathcal{R}_i^*$ and $\sigma \in \{f, g_1, \cdots, g_m\}$. Then

$$\begin{aligned}\langle \tau, \mathcal{L}_\sigma \omega_2 \rangle &= \mathcal{L}_\sigma \langle \tau, \omega_2 \rangle - \langle [\sigma, \tau], \omega_2 \rangle \\ &= \mathcal{L}_\sigma \langle \tau, \omega_2 \rangle = \mathcal{L}_\sigma \langle \tau, (\omega - \omega_1) \rangle = 0\end{aligned} \tag{81}$$

where the last equality follows from the fact that $\omega \in \mathcal{R}_i^{*^\perp}$ and $\omega_1 \in \Omega_{i*} \subset \mathcal{R}_i^{*^\perp}$. By (80),(81), we then have $\dot{\omega}_2 \in \mathcal{R}_i^{*^\perp}$, and hence

$$\dot{\widehat{\mathcal{R}_i^{*^\perp}}} \subset \dot{\Omega}_{i*} + \mathcal{R}_i^{*^\perp}$$

By construction, $\mathcal{C}_{i*}$ is the largest subspace in $\mathcal{X}$ which verifies $\dot{\mathcal{C}}_{i*} \subset \mathcal{C}_{i*} + \dot{\Omega}_{i*}$. This implies $\mathcal{R}_i^{*^\perp} \subset \mathcal{C}_{i*}$. So $\mathcal{C}_{i*}$ is the annihilator of $\mathcal{R}_i^*$, which is defined to be involutive ([39],[29]). Hence $\mathcal{C}_{i*}$ is exact, which establishes our claim. ■

Proof of Lemma 4.9 By Lemma 4.10, $\mathcal{C}_{i*}$ is an exact subspace. Thus, $\dot{\mathcal{C}}_{i*}$ as well as $\sum_{j=0}^{n_i'-1} \mathcal{C}_{i*}^{(j)}$ are also exact. Let us define $\mathcal{C}_0$ as the uncontrollable subspace of (Σ) which is the subspace $\mathcal{H}_\infty$ introduced in [2]. It is obvious that for each $i = 1, \cdots, m$

$$\mathcal{C}_0 = \sum_{j=0}^{n_i'-1} \mathcal{C}_{i*}^{(j)} \cap \sum_{k=\neq i} \sum_{j=0}^{n_k'-1} \mathcal{C}_{k*}^{(j)}$$

Let $d\xi_0$ to be a basis of $\mathcal{C}_0$, thus $\dot{\xi}_0 = f_0(\xi_0)$. For an invertible system, we can construct a quasi-static state feedback which decouples system (Σ) by taking $v_i = y_i^{(n_i')}$. For $i = 1, \cdots, m$, then choose $d\xi_i$ such that $\{d\xi_0, d\xi_i\}$ is a basis of $\sum_{j=0}^{n_i'-1} \mathcal{C}_{i*}^{(j)}$. Then one has

$$\dot{\xi}_i = f_i(\xi_0, \xi_i, v_i)$$

Complete the new coordinates by choosing $\hat{\xi}$ such that $\{d\xi_0, d\xi_1, \cdots, d\xi_m, d\hat{\xi}\}$ is linearly independent. Without loss of generality, $\hat{\xi}$ can be chosen so that span$\{d\hat{\xi}\} \subset \mathcal{X}$). Thus, one has

$$\dot{\hat{\xi}} = \hat{f}(\xi, v, \dot{v}, \cdots, v^{(s)}),$$

and (77) is established. ∎

Now we may state the following theorem.

Theorem 4.11 *For a square invertible nonlinear system, the dimension of the fixed dynamics with respect to any quasi-static state feedback is*

$$n - \dim\,(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j \geq 0} \mathcal{C}_{i*}^{(j)}) \tag{82}$$

Moreover, if the origin is an equilibrium point for Σ and the quasi-static state feedback rendering (76) noninteractive preserves this equilibrium point, then the induced fixed dynamics are

$$\dot{\hat{\xi}} = \hat{f}(0, \cdots, 0, \hat{\xi}, 0, \cdots, 0). \tag{83}$$

where $\hat{\xi}$ is as defined in Lemma 4.9.

Proof From the proof of Lemma 4.9, the dimension of the fixed dynamics with respect to any quasi-static feedback which decouples the system, is

$$n - \dim\,(\sum_{i=1}^{m} \sum_{j=0}^{n_i'-1} \mathcal{C}_{i*}^{(j)}). \tag{84}$$

From the definition of the structure at infinity, one gets

$$\dim\,(\sum_{i=1}^{m} \sum_{j=0}^{n_i'-1} \mathcal{C}_{i*}^{(j)}) = \dim\,(\mathcal{X} \cap \sum_{i=1}^{m} \sum_{j \geq 0} \mathcal{C}_{i*}^{(j)}). \tag{85}$$

Thus (82) is established. Once the system is in standard decomposed form (77) and analogously to [23], any decoupling quasi-static state feedback is of the form $v_i = \alpha_i(\xi_0, \xi_i, w_i, \ldots, w_i^{(\nu)})$. Hence, if the α_i's preserve the equilibrium, the second statement in Theorem 4.11 is immediate. ∎

The asymptotic stability of dynamics (83) is a necessary condition for noninteracting control with internal stability by quasi-static state feedback.

The next example illustrates Theorem 4.11.

Example 4.12 Let us consider a nonlinear system given by:

$$\dot{x}_1 = u_1,\ \dot{x}_2 = x_4 + x_3 u_1,\ \dot{x}_3 = x_3 + x_4\ ,\ \dot{x}_4 = u_2,\ \dot{x}_5 = x_1 + x_2$$

$$y_1 = x_1,\ y_2 = x_2$$

We have $\{n_i'\} = \{2, 1\}$. Permute then y_i, and thus $\mathcal{C}_{1*} = \{dx_2\}$ and $\mathcal{C}_{2*} = \{dx_1\}$. The quasi-static feedback which decouples the system is $u_1 = v_1$ and $u_2 = v_2 - (x_3 + x_4)v_1 - x_3\dot{v}_1$, where (v_1, v_2) is a new input vector.

It is clear that $\mathcal{C}_0 = 0$. We choose $d\xi_1 = \{dx_2, d(x_4 + x_3 u_1)\}$ as a basis of $\{\mathcal{C}_{1*} + \dot{\mathcal{C}}_{1*}\}$, and thus

$$\dot{\xi}_1 = \begin{pmatrix} \dot{\xi}_{11} \\ \dot{\xi}_{12} \end{pmatrix} = \begin{pmatrix} \xi_{12} \\ v_2 \end{pmatrix} \tag{86}$$

Choose now $\{d\xi_2\} = \{dx_1\}$ as a basis of $\mathcal{C}_{2*}$, and one has

$$\dot{\xi}_2 = v_1 \tag{87}$$

We complete our coordinate transformation by taking $\hat{\xi} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}$. So in the new coordinates $(\xi_1, \xi_2, \hat{\xi})$, the considered system becomes

$$\begin{aligned} \dot{\xi}_{11} &= \xi_{12} \\ \dot{\xi}_{12} &= v_2 \\ \dot{\xi}_2 &= v_1 \\ \dot{\hat{\xi}}_1 &= v_2 - (\xi_{12} - \hat{\xi}_1) - \hat{\xi}_1 v_1 - (\xi_{12} - \hat{\xi}_1)\dot{v}_1 / v_1 \\ \dot{\hat{\xi}}_2 &= \xi_2 + \xi_{11} \\ \\ y_1 &= \xi_2 \\ y_2 &= \xi_{11} \end{aligned} \tag{88}$$

Clearly dim $(\hat{\xi}) = 2$. It equals $n -$ dim $(\mathcal{X} \cap (\sum_{i=1}^{m} \sum_{j \geq 0} \mathcal{C}_{i*}^{(j)})) = n -$ dim $(dx_1, dx_2, dx_4 + u_1 dx_3)$. Thus, the dimension of the fixed dynamics is two. Since the origin is an equilibrium point, the fixed dynamics are then

$$\begin{aligned} \dot{\hat{\xi}}_1 &= \hat{\xi}_1 \\ \dot{\hat{\xi}}_2 &= 0 \end{aligned} \tag{89}$$

Similarly to Wagner's and Battilotti's results, in the case where no quasi-static state feedback can render the system simultaneously noninteractive and stable, a suitable dynamic feedback may still solve the problem. This reduces to the results in Zhan *et al.* [50].

Table 2 which displays the dimension of the various decoupling zero dynamics is now completed in Table 5.

Feedback	$A(x)$ invertible	$A(x)$ non-invertible
(Quasi) Static	dim $(\mathcal{P}^*)$ (Isidori & Grizzle [32])	$n -$ dim $(\mathcal{X} \cap (\sum_{i=1}^{m} \sum_{j \geq 0} \mathcal{C}_i^{*(j)}))$
Dynamic	dim (Δ_{mix}) (Wagner [47])	dim $(\Delta_{mix}(\Sigma_p))$ (Zhan *et al.* [50])

Table 5: Decoupling zero structure (complete)

5 Conclusions

A generalized notion of controlled invariance under quasi-static state feedback for nonlinear systems was introduced. It was shown that this notion coincides with the standard notion of a controlled invariant distribution under regular static state feedback. Using the generalized notion of controlled invariance, a condition for the controlled invariance of not necessarily integrable codistributions was derived. For a subspace $\Omega \subset \mathcal{X}$, we gave sufficient conditions for controlled invariance under quasi-static state feedback. Furthermore, a necessary and sufficient condition for controlled invariance was also given for a special class of subspaces Ω. The generalized controlled invariance was applied to the disturbance decoupling problem by dynamic feedback. A necessary and sufficient condition for solvability of this *DDDP* was obtained.

For a controllability cospace $\mathcal{C} \subset \mathcal{X}$, some properties were derived by means of the controllability cospace algorithm. Moreover the smallest controllability cospace containing the differential of the output mapping allowed to solve the block input-output decoupling problem. It also characterized the dimension of the fixed dynamics with respect to any quasi-static state feedback in the case of one to one decoupling.

This paper leaves some interesting open questions, which are the topic for further research. A first question is related to necessary and sufficient conditions for controlled invariance for a general class of subspaces. A second question is whether (or under what conditions) there exists a smallest controlled invariant subspace containing some given subspace. It seems that for the answer to both questions a better understanding of quasi-static state feedback is needed.

Finally, let us remark that throughout the paper we have restricted ourselves to "Kalmanian" systems and to subspaces $\Omega \subset \mathcal{X}$. However, the definition of controlled invariance and the characterizations of controlled invariance in this paper can, *mutatis mutandis*, be translated to non-Kalmanian systems and subspaces $\Omega \subset \mathcal{X} \times \mathcal{U}$.

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Appendix

According to Remark 4.6, we will prove that the sequence (70) computing $\mathcal{C}_*$ is the same as the one computing ${\mathcal{R}^*}^\perp$ (the dual of $\mathcal{R}^*$, the maximal controllability subspace in kernel of the output) for linear time invariant systems. We proceed by induction. First, we recall some basic operations that we need.

Consider a linear system given by :

$$\begin{aligned} \dot{x} &= Ax + Bu, \; x \in I\!\!R^n, u \in I\!\!R^m \\ y &= Cx \end{aligned} \tag{90}$$

Identify elements of $I\!\!R^n$ with column vectors, while elements of ${I\!\!R^n}^\perp$, its dual, are identified with row-vectors. Thus, $\omega = \sum_{i=1}^{n} \alpha_i dx_i \in {I\!\!R^n}^\perp$ is identified with the row-vector $\alpha := (\alpha_1, \ldots, \alpha_n) \in I\!\!R^n$. With this notation,

$$\dot{\omega} = \alpha d\dot{x} = \alpha A dx + \alpha B du \in (I\!\!R^n \times I\!\!R^m)^\perp \tag{91}$$

is identified with the row-vector $(\alpha A \;\; \alpha B)$.

Let a subspace $V \subset I\!\!R^n$ be given. Then

$$\begin{aligned} (AV)^\perp &= \{\omega \in \mathrm{span}\{dx\} \,| < \omega, Av >= 0, \forall v \in V\} \\ &= \{\alpha \in I\!\!R^n \mid \alpha A v = 0, \forall v \in V\} = \{\alpha \in I\!\!R^n \mid \alpha A \in V^\perp\} \\ &=: \;{}^{-1}AV^\perp \end{aligned} \tag{92}$$

if $\omega = \alpha dx \in (AV)^\perp \cap \mathcal{B}^\perp$ where $\mathcal{B} = \mathrm{Im}B$, then

$$\dot{\omega} = \alpha A dx + \alpha B du = \alpha A dx \simeq \alpha A \in V^\perp \tag{93}$$

The two sequences to be compared are:

$$\left\{ \begin{array}{lcl} \mathcal{R}_0^\perp &:=& \mathcal{X} \\ \mathcal{R}_{\mu+1}^\perp &:=& {\mathcal{V}^*}^\perp + \;{}^{-1}A\mathcal{R}_\mu^\perp \cap \mathcal{B}^\perp \quad (\mu \in I\!\!N) \end{array} \right. \tag{94}$$

and

$$\left\{ \begin{array}{lcl} \mathcal{C}_0 &:=& \mathcal{X} \\ \mathcal{C}_{\mu+1} &:=& \{\omega \in \mathcal{C}_\mu \mid \dot{\omega} \in \mathcal{C}_\mu + \dot{\mathcal{V}^*}^\perp\} \quad (\mu \in I\!\!N) \end{array} \right. \tag{95}$$

where $\mathcal{V}^*$ is the maximal controlled invariant subspace in Ker$C$ for the system (90). For step 0, it is obvious that $\mathcal{R}_0^\perp = \mathcal{C}_0$. Suppose that $\mathcal{R}_\mu^\perp = \mathcal{C}_\mu$ for $\mu = 0, \cdots, \ell$. Let $\omega \in \mathcal{R}_{\ell+1}^\perp$, thus there exist $\omega_1 \in {\mathcal{V}^*}^\perp$ and $\omega_2 \in \;{}^{-1}A\mathcal{R}_\ell^\perp \cap \mathcal{B}^\perp$ such that $\omega = \omega_1 + \omega_2$. By (93) , $\dot{\omega}_2 \in \mathcal{R}_\ell^\perp = \mathcal{C}_\ell$ and hence $\mathcal{R}_{\ell+1}^\perp \subset \mathcal{C}_{\ell+1}$. To show the other inclusion, let $\omega \in \mathcal{C}_{\ell+1}$, then

$$\dot{\omega} \in \mathcal{C}_\ell + \dot{\mathcal{V}^*}^\perp = \mathcal{R}_\ell^\perp + \dot{\mathcal{V}^*}^\perp \tag{96}$$

Thus, there exists $\omega_1 \in {\mathcal{V}^*}^\perp$ and $\omega_2 \in \mathcal{R}_\ell^\perp$ such that $\dot{\omega} = \dot{\omega}_1 + \omega_2$. Let now $\dot{\omega}_0 = \dot{\overbrace{\omega - \omega_1}} = \omega_2$. So, $\dot{\omega}_0 \in \mathcal{R}_\ell^\perp$. This implies that

$$\begin{aligned} \omega_0 &\in \{\omega = \alpha dx \mid \dot{\omega} \in \mathcal{R}_\ell^\perp\} = \{\alpha dx \mid \alpha A dx + \alpha B du \in \mathcal{R}_\ell^\perp\} \\ &= \{\alpha \mid \alpha A \in \mathcal{R}_\ell^\perp\} \cap \mathcal{B}^\perp = \;{}^{-1}A\mathcal{R}_\ell^\perp \cap \mathcal{B}^\perp \end{aligned} \tag{97}$$

So, $\omega = \omega_1 + \omega_0 \in \mathcal{R}_{\ell+1}^\perp$, which yields that $\mathcal{C}_{\ell+1} \subset \mathcal{R}_{\ell+1}^\perp$. Thus, we have that $\mathcal{C}_\mu = \mathcal{R}_\mu^\perp$ for all $\mu \in I\!\!N$, which establishes our claim. ∎