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by

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Some topological aspects of the convolution algebra $\mathcal{E}'(\mathbb{R})$ and its representing space of Fourier transforms

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Abstract

By describing the Fréchet topology on the space $\mathcal{E}(\mathbb{R})$ of all infinitely continuously differentiable functions as a projective limit of Banach spaces, its topological dual, the space $\mathcal{E}'(\mathbb{R})$ of all Schwartz distributions of compact support, is naturally equipped with an inductive limit topology of Banach spaces. Algebraically, $\mathcal{E}'(\mathbb{R})$ is isomorphic to an operator algebra of shift-invariant operators on $\mathcal{E}(\mathbb{R})$, and to the Paley-Wiener algebra $PW(\mathbb{C})$ of holomorphic functions of exponential type that are polynomially bounded on the real axis. In this paper, we study the topological relationship between $\mathcal{E}'(\mathbb{R})$ and $PW(\mathbb{C})$. The function space $PW(\mathbb{C})$ is equipped with a natural inductive limit topology, and the properties of this space are investigated. It is shown that the Fourier transformation from $\mathcal{E}'(\mathbb{R})$ to $PW(\mathbb{C})$ is a homeomorphism: it preserves both the algebraic and topological characteristics of its domain $\mathcal{E}'(\mathbb{R})$ in its codomain $PW(\mathbb{C})$.

1 The Heine-Borel property of the space $\mathcal{E}(\mathbb{R})$

Let $C(\mathbb{R})$ denote the vector space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$. The space $C(\mathbb{R})$, endowed with the seminorms

$$q_n(f) = \max_{t\in[-n,n]} |f(t)|, \tag{1}$$

is a Fréchet space (see e.g. [7]).

Definition 1.1 A subset S in $C(\mathbb{R})$ is said to be *locally equicontinuous* if for every compact subset $K \subset \mathbb{R}$, the set

$$S|_K = \{f|_K \mid f \in S\}$$

is equicontinuous in the Banach space $C(K)$ of continuous functions on K.

The theorem of Arzela-Ascoli states that if $K \subset \mathbb{R}$ is compact, a subset $S \subset C(K)$ is pre-compact if and only if it is bounded and equicontinuous. There is the following generalization.

Theorem 1.2 *Let S be a bounded, locally equicontinuous subset of the Fréchet space $C(\mathbb{R})$. Then S is pre-compact.*

Proof: Let (f_j) denote a sequence in S. Then there are subsequences $(f_{n,j})$, $n \in \mathbb{N}$, such that

(i) $(f_{n+1,j})$ is a subsequence of $(f_{n,j})$,

(ii) $(f_{n,j})$ is convergent in $C([-n,n])$.

It follows that for all $n \in \mathbb{N}$ and all $x \in [-n,n]$:

$$\lim_{j\to\infty} f_{n,j}(x) = \lim_{j\to\infty} f_{n+1,j}(x).$$

Let (g_j) denote the diagonal sequence $g_j = f_{j,j}$. Then for all $n \in \mathbb{N}$, $(g_j)_{j\geq n}$ is a subsequence of $(f_{n,j})_{j\geq n}$ and (g_j) is a convergent subsequence in $C(\mathbb{R})$ of the sequence (f_j). ∎

Now for all $k \in \mathbb{N}_0$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let $C^k(\mathbb{R})$ denote the vector space of all k-times continuously differentiable functions on $\mathbb{R}$. Then $C^k(\mathbb{R})$, endowed with the seminorms

$$q_{k,n}(f) = \max_{t\in[-n,n]} \sum_{j=0}^{k} |f^{(j)}(t)|, \qquad (n \in \mathbb{N}_0), \tag{2}$$

is a Fréchet space. Obviously, $C^{k+1}(\mathbb{R})$ is contained in $C^k(\mathbb{R})$, and the canonical injection is continuous. By Theorem 1.2, this injection is *compact*:

Lemma 1.3 *Let S be a bounded subset in $C^{k+1}(\mathbb{R})$. Then S is a pre-compact subset of $C^k(\mathbb{R})$.*

Proof: First we consider the case $k = 0$. So let S be bounded in $C^1(\mathbb{R})$. Then for $f \in S$ and $s, t \in [-n,n]$,

$$|f(t) - f(s)| = \left| \int_s^t f'(\tau)\, d\tau \right| \leq |t-s| \cdot \sup_{g\in S} q_{1,n}(g).$$

So S is a locally equicontinuous, bounded subset of $C(\mathbb{R})$, and therefore pre-compact.

Assume the assertion is true for all k with $k < k_0$. Let S be a bounded subset of $C^{k_0+1}(\mathbb{R})$. Then $S' = \{f' \mid f \in S\}$ is bounded in $C^{k_0}(\mathbb{R})$, and therefore compact in $C^{k_0-1}(\mathbb{R})$. Let (f_j) be a sequence in S. Then the conclusion follows from the fact that

$$f_j(t) = f_j(0) + \int_0^t f_j'(\tau)\, d\tau, \qquad (t \in \mathbb{R}),$$

where $(f_j(0))$ is a bounded sequence. ∎

Let $\mathcal{E}(\mathbb{R})$ be the vector space of all infinitely continuously differentiable functions from $\mathbb{R}$ into $\mathbb{C}$, endowed with the seminorms $q_{k,n}$, $(k,n \in \mathbb{N}_0)$. $\mathcal{E}(\mathbb{R})$ is a Fréchet space. Since the collection of Fréchet spaces $\{C^k(\mathbb{R}) \mid k \in \mathbb{N}_0\}$ is a projective system, with each canonical injection $i_k : C^{k+1}(\mathbb{R}) \to C^k(\mathbb{R})$ continuous and compact, $\mathcal{E}(\mathbb{R}) = \bigcap_{k=0}^{\infty} C^k(\mathbb{R})$, and its topology may be considered as the corresponding projective limit topology. Combining the results of this section, we obtain:

Theorem 1.4 *The Fréchet space $\mathcal{E}(\mathbb{R})$ is a Montel space, in the sense that it satisfies the Heine-Borel property: every closed and bounded subset of $\mathcal{E}(\mathbb{R})$ is compact.* ∎

2 Topologies on $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$

The topological dual $\mathcal{E}'(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$ is identified mostly as the space of Schwartz distributions of compact support. A natural topology for $\mathcal{E}'(\mathbb{R})$ is the weak-star-topology $\sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R}))$ (see e.g. [8, Chapter IV]). But for the topological considerations we have in mind, this topology is not appropriate. Therefore we present a collection of seminorms on $\mathcal{E}(\mathbb{R})$, that leads to a convenient description of $\mathcal{E}'(\mathbb{R})$, and a useful topology for this space.

Recalling (2), we define the seminorms p_n on $\mathcal{E}(\mathbb{R})$ by

$$p_n = q_{n,n}, \qquad (n \in \mathbb{N}_0). \tag{3}$$

Then the projective limit topology of $\mathcal{E}(\mathbb{R})$, as defined in Section 1, is brought about by these seminorms; observe that for all n and k:

$$q_{k,n}(f) \leq p_m(f), \qquad (f \in \mathcal{E}(\mathbb{R})),$$

with $m = \max(n, k)$. Consider for $n \in \mathbb{N}_0$ the quotient spaces $X_n = \mathcal{E}(\mathbb{R})/p_n^{\leftarrow}(\{0\})$. Then X_n is a normed space with the quotient norm $\tilde{p}_n$. Since $p_{n+1} \geq p_n$, the Banach completions $\overline{X_n}$ establish a projective system, and $\mathcal{E}(\mathbb{R})$ can be considered the projective limit of these Banach spaces. The canonical completion of X_n is the Banach space $C^n([-n, n])$, consisting of all n-times continuously differentiable functions f on $[-n, n]$ such that $f^{(n)} \in C([-n, n])$.

It follows that for $n \in \mathbb{N}_0$ the space

$$\mathcal{E}'(\mathbb{R}; n) := \{G \in \mathcal{E}'(\mathbb{R}) \mid \exists C > 0 \; \forall f \in \mathcal{E}(\mathbb{R}) : |G(f)| \leq C p_n(f)\} \tag{4}$$

is a Banach space with norm

$$\|G\|'_n = \sup\{|G(f)| \mid f \in \mathcal{E}(\mathbb{R}), p_n(f) \leq 1\}. \tag{5}$$

Since $p_{n+1} \geq p_n$ for all $n \in \mathbb{N}_0$, we see that

$$\mathcal{E}'(\mathbb{R}; n) \subset \mathcal{E}'(\mathbb{R}; n+1),$$

with

$$\|G\|'_{n+1} \leq \|G\|'_n, \qquad (G \in \mathcal{E}'(\mathbb{R}; n)).$$

So, the collection $(\mathcal{E}'(\mathbb{R}; n))_{n \in \mathbb{N}_0}$ is an inductive system of Banach spaces. We observe that the system is not strict. Since

$$\mathcal{E}'(\mathbb{R}) = \bigcup_{n=0}^{\infty} \mathcal{E}'(\mathbb{R}; n),$$

we endow $\mathcal{E}'(\mathbb{R})$ with the corresponding inductive limit topology. Thus $\mathcal{E}'(\mathbb{R})$ becomes an (LB)-space. It is the intention of this paper to prove some properties of the present inductive limit description of $\mathcal{E}'(\mathbb{R})$. Note that this inductive limit topology on $\mathcal{E}'(\mathbb{R})$ is stronger than the weak-star-topology $\sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R}))$.

Remark 2.1 If $\mathcal{E}(\mathbb{R})$ is considered as a projective limit of Fréchet spaces, as in Section 1, its dual space $\mathcal{E}'(\mathbb{R})$ may be characterized as an inductive limit of Fréchet spaces, endowed with a (non-strict) inductive limit topology. Although this approach leads to an interesting classification of the elements of $\mathcal{E}'(\mathbb{R})$, it is inconvenient for our topological considerations.

3 Linear translation invariant operators on $\mathcal{E}(\mathbb{R})$ and convolution

Let $(\sigma_t)_{t\in\mathbb{R}}$ denote the translation group on $\mathcal{E}(\mathbb{R})$, where

$$(\sigma_t f)(\tau) = f(t+\tau), \qquad t \in \mathbb{R}, \tau \in \mathbb{R}, f \in \mathcal{E}(\mathbb{R}). \tag{6}$$

Then $(\sigma_t)_{t\in\mathbb{R}}$ is a one parameter c_∞-group on $\mathcal{E}(\mathbb{R})$, in the sense that for all $f \in \mathcal{E}(\mathbb{R})$, the $\mathcal{E}(\mathbb{R})$-valued function

$$t \mapsto \sigma_t f, \qquad t \in \mathbb{R},$$

is infinitely differentiable as function from $\mathbb{R}$ to $\mathcal{E}(\mathbb{R})$, with

$$\left(\frac{d}{dt}\right)^k \{\sigma_t f\} = \left\{\sigma_t \frac{d^k f}{dt^k}\right\}.$$

In particular, $(\sigma_t)_{t\in\mathbb{R}}$ is a one parameter c_0-group on $\mathcal{E}(\mathbb{R})$ with everywhere defined infinitesimal generator $\frac{d}{dt}$.

It follows that for all $G \in \mathcal{E}'(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$, the function $\sigma[G]f$ defined by

$$(\sigma[G]f)(t) = G(\sigma_t f) \tag{7}$$

belongs to $\mathcal{E}(\mathbb{R})$, with $\left(\frac{d}{dt}\right)^k (\sigma[G]f) = \sigma[G]\frac{d^k f}{dt^k}$. The linear operator $\sigma[G] : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is continuous, which can be proved, using the Closed-Graph-Theorem for Fréchet spaces (see e.g. [8, p. 78]). Clearly, for all $G \in \mathcal{E}'(\mathbb{R})$,

$$\sigma_t \sigma[G] = \sigma[G]\sigma_t, \qquad t \in \mathbb{R},$$

so $\sigma[G]$ is a translation invariant operator. If on the other hand $L : \mathcal{E}(\mathbb{R}) \longrightarrow \mathcal{E}(\mathbb{R})$ is a continuous linear operator, satisfying $L\sigma_t = \sigma_t L$ for all $t \in \mathbb{R}$, then $L = \sigma[G]$, with $G \in \mathcal{E}'(\mathbb{R})$ defined by $G(f) = (Lf)(0)$.

Theorem 3.1 *Let $B_\sigma(\mathcal{E}(\mathbb{R}))$ denote the algebra of all continuous translation invariant linear operators from $\mathcal{E}(\mathbb{R})$ into $\mathcal{E}(\mathbb{R})$. Then $\sigma : \mathcal{E}'(\mathbb{R}) \to B_\sigma(\mathcal{E}(\mathbb{R}))$ is an algebra isomorphism, where the product $G_1 * G_2$ of G_1, G_2 in the vector space $\mathcal{E}'(\mathbb{R})$ is defined by*

$$\sigma(G_1 * G_2) = \sigma[G_1]\sigma[G_2].$$

■

The product $*$ is the classical convolution product in $\mathcal{E}'(\mathbb{R})$, as introduced by Schwartz (see e.g. [9, Chapter VI]). Since this convolution product $*$ is known to be commutative, $B_\sigma(\mathcal{E}(\mathbb{R}))$ is a *commutative* algebra too. In this paper, we shall show that the convolution algebra $\mathcal{E}'(\mathbb{R})$ is algebra-homeomorphic to a topological algebra of holomorphic functions with (of course) an (LB)-structure.

4 Fourier transformation and the Paley-Wiener-Theorem

Let $e_z \in \mathcal{E}(\mathbb{R})$, with $z \in \mathbb{C}$, denote the exponential function

$$e_z(t) = e^{-izt}, \qquad t \in \mathbb{R}. \tag{8}$$

Then $z \mapsto e_z$ is an $\mathcal{E}(\mathbb{R})$-valued holomorphic function on $\mathbb{C}$. Indeed, for each $z_0 \in \mathbb{C}$,

$$e_z = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} e_{z_0,n},$$

where $e_{z_0,n}(t) = (-it)^n e^{-iz_0 t}$, with convergence in $\mathcal{E}(\mathbb{R})$. So for each $G \in \mathcal{E}'(\mathbb{R})$, the Fourier transform $\mathcal{F}G$ defined by

$$\mathcal{F}G : \mathbb{C} \longrightarrow \mathbb{C} : \qquad \mathcal{F}G(z) = G(e_z) \tag{9}$$

is a holomorphic function on $\mathbb{C}$. The mapping $G \mapsto \mathcal{F}G$ is called the *Fourier transformation* on $\mathcal{E}'(\mathbb{R})$, and is denoted by $\mathcal{F}$. The Paley-Wiener-Theorem characterizes the Fourier transforms $\mathcal{F}G$ for $G \in \mathcal{E}'(\mathbb{R})$ (see e.g. [9, p. 271] or [3, p. 156]).

Definition 4.1 The *Paley-Wiener class* $PW(\mathbb{C})$ is the vector space of all holomorphic functions ϕ on $\mathbb{C}$, with the property that

$$\exists C > 0 \, \exists N \in \mathbb{N} \, \exists a > 0 \, \forall z \in \mathbb{C} : |\phi(z)| \leq C(1+|z|)^N e^{a|\operatorname{Im} z|}. \tag{10}$$

Theorem 4.2 (Paley-Wiener) *The Fourier transformation $\mathcal{F}$ defined on $\mathcal{E}'(\mathbb{R})$, is a vector space isomorphism from $\mathcal{E}'(\mathbb{R})$ onto $PW(\mathbb{C})$. So $\mathcal{F}$ is linear and*

(i) $\forall G \in \mathcal{E}'(\mathbb{R}) : \mathcal{F}G \in PW(\mathbb{C})$,

(ii) $\forall \phi \in PW(\mathbb{C}) \, \exists H \in \mathcal{E}'(\mathbb{R}) : \mathcal{F}H = \phi$. ∎

The definition of $PW(\mathbb{C})$ implies that it is a subalgebra of the algebra $H(\mathbb{C})$ of all holomorphic functions on $\mathbb{C}$. Since for $G \in \mathcal{E}'(\mathbb{R})$ and $z \in \mathbb{C}$: $\sigma[G]e_z = \mathcal{F}G(z)e_z$, it follows that for $G_1, G_2 \in \mathcal{E}'(\mathbb{R})$:

$$\mathcal{F}(G_1 * G_2) = (\mathcal{F}G_1)(\mathcal{F}G_2).$$

Hence $\mathcal{F}$ is an algebra isomorphism from the commutative convolution algebra $\mathcal{E}'(\mathbb{R})$, onto the function algebra $PW(\mathbb{C})$.

So we presented three mutually isomorphic commutative algebras, namely the operator algebra $B_\sigma(\mathcal{E}(\mathbb{R}))$, the convolution algebra $\mathcal{E}'(\mathbb{R})$, and the function algebra $PW(\mathbb{C})$. Further, we presented an (LB)-topology for $\mathcal{E}'(\mathbb{R})$. In the next section we shall endow $PW(\mathbb{C})$ with an (LB)-topology, so that the Fourier transformation is a homeomorphism.

5 An inductive limit topology on $PW(\mathbb{C})$

We recall that $H(\mathbb{C})$, the vector space of all holomorphic functions, is a Fréchet space for the seminorms

$$s_k(\phi) = \sup\{|\phi(z)| \mid z \in \mathbb{C}, |z| \leq k\}. \tag{11}$$

These seminorms describe uniform convergence on compact subsets of $\mathbb{C}$.

For each $n \in \mathbb{N}_0$, let $PW(\mathbb{C}; n)$ denote the subspace of $PW(\mathbb{C})$, consisting of all $\phi \in PW(\mathbb{C})$ such that

$$\exists C_\phi > 0 \, \forall z \in \mathbb{C} : |\phi(z)| \leq C_\phi (A(z))^n, \tag{12}$$

where A is the function on $\mathbb{C}$ with $A(z) = (1+|z|)\cdot\exp(|\operatorname{Im} z|)$. Then $PW(\mathbb{C};n)$ is a Banach space of holomorphic functions for the norm

$$\|\phi\|_{PW,n} = \sup_{z\in\mathbb{C}} \left(A(z)^{-n}|\phi(z)|\right). \tag{13}$$

We see that

$$PW(\mathbb{C}) = \bigcup_{n=0}^{\infty} PW(\mathbb{C};n),$$

and the sequence of Banach spaces $(PW(\mathbb{C};n))_{n\in\mathbb{N}_0}$ is a (non-strict) inductive system, i.e.

(1) $PW(\mathbb{C};n) \subset PW(\mathbb{C};n+1)$,

(2) $\forall\phi \in PW(\mathbb{C};n) : \|\phi\|_{PW,n+1} \leq \|\phi\|_{PW,n}$.

We endow $PW(\mathbb{C})$ with the corresponding inductive limit. Then the canonical injection $PW(\mathbb{C}) \hookrightarrow H(\mathbb{C})$ is continuous because for each $n \in \mathbb{N}_0$, the canonical injection $PW(\mathbb{C};n) \hookrightarrow H(\mathbb{C})$ is continuous. Indeed, if $k > 0$ and $\varepsilon > 0$, then for every $\phi \in PW(\mathbb{C};n)$, satisfying $\|\phi\|_{PW,n} < \frac{\varepsilon}{((1+k)e^k)^n}$, we have

$$s_k(\phi) = \sup\left\{\frac{|\phi(z)|}{A(z)^n}A(z)^n \mid z\in\mathbb{C}, |z| < k\right\} \leq \|\phi\|_{PW,n}((1+k)e^k)^n < \varepsilon,$$

i.e. the seminorm s_k is continuous on $PW(\mathbb{C};n)$. It follows that the (LB)-space $PW(\mathbb{C})$ is a *Hausdorff topological space.* We shall prove that the (LB)-space $PW(\mathbb{C})$ exhibits two important properties:

(I) $PW(\mathbb{C})$ is *regular*, i.e. every bounded set in $PW(\mathbb{C})$ is bounded in some $PW(\mathbb{C};n)$,

(II) $PW(\mathbb{C})$ is *sequentially retractive*, i.e. every convergent sequence in $PW(\mathbb{C})$ is contained in $PW(\mathbb{C};n)$ for certain $n \in \mathbb{N}_0$, and converges with respect to the topology on $PW(\mathbb{C};n)$.

For this terminology, see also [4].

6 A result on positive double sequences

In the proof of properties (I) and (II), we need a mechanism to keep track of the exponential growth of a $PW(\mathbb{C})$-function in the direction of the imaginary axis, and of its polynomial growth along the real axis. For this purpose we use a result from [2] (see also [1], [6]) on positive 2-D sequences. Let $\omega_+(\mathbb{N}_0^2)$ denote the collection of all positive functions from $\mathbb{N}_0^2$ into $\mathbb{R}^+$, partially ordered by the usual pointwise ordering. All operations such as scalar multiplication, addition, and multiplication are taken pointwise. Let $a_0 \in \omega_+(\mathbb{N}_0^2)$ be defined by

$$a_0(k,l) = (1+\sqrt{k^2+\ell^2})e^{\ell}, \qquad (k,\ell)\in\mathbb{N}_0^2, \tag{14}$$

and $\rho \subset \omega_+(\mathbb{N}_0^2)$ by the countable collection

$$\rho = \{a_0^n \mid n \in \mathbb{N}_0\}. \tag{15}$$

Then it is obvious that $a_0(k,\ell) \geq 1$ for all $(k,\ell)\in\mathbb{N}_0^2$, and $a_0^0 \leq a_0^1 \leq a_0^2 \leq a_0^3 \leq \cdots$ with respect to the partial ordering on $\omega_+(\mathbb{N}_0^2)$.

The next lemma is a restricted version of a far more general result proved in [2].

Lemma 6.1 ([1], [2], [6]) *Define*

$$\rho^{\#} = \{b \in \omega_+(\mathbb{N}_0^2) \mid \forall n \in \mathbb{N}_0 : \sup_{(k,\ell)\in\mathbb{N}_0^2} (a_0(k,\ell)^n b(k,\ell)) < \infty\}, \tag{16}$$

and

$$\rho^{\#\#} = \{a \in \omega_+(\mathbb{N}_0^2) \mid \forall b \in \rho^{\#} : \sup_{(k,\ell)\in\mathbb{N}_0^2} (a(k,\ell)b(k,\ell)) < \infty\}. \tag{17}$$

Then

$$\forall a \in \rho^{\#\#}\ \exists n \in \mathbb{N}_0\ \exists \gamma > 0 : a \leq \gamma a_0^n. \tag{18}$$

Proof: (by contradiction)

Assume that (18) does not hold, and let $a \in \rho^{\#\#}$ be such that for all $n \in \mathbb{N}_0$ and $\gamma > 0$: $\neg(a \leq \gamma a_0^n)$. For $n \in \mathbb{N}_0$ we define the index set I_n by

$$I_n = \{(k,\ell) \in \mathbb{N}_0^2 \mid a(k,\ell) > n \cdot a_0(k,\ell)^n\}.$$

Then $I_n \neq \varnothing$, $I_{n+1} \subset I_n$, and since $a_0(k,\ell) \geq 1$ for all $(k,\ell) \in \mathbb{N}_0^2$, also $\cap_{n\in\mathbb{N}_0} I_n = \varnothing$. This implies that every index set I_n is infinite.

Let $(k_n,\ell_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathbb{N}_0^2$ such that $(k_n,\ell_n) \in I_n$, and $(k_n,\ell_n) \neq (k_{n'},\ell_{n'})$ for $n \neq n'$. Define $c \in \omega_+(\mathbb{N}_0^2)$ by

$$c(k,\ell) = \begin{cases} 0 & \text{if } (k,\ell) \notin \{(k_n,\ell_n) \mid n \in \mathbb{N}_0\}, \\ \frac{1}{a_0(k,\ell)^n} & \text{if } (k,\ell) = (k_n,\ell_n). \end{cases}$$

Then $c \in \rho^{\#}$. Indeed, if $n_0 \in \mathbb{N}_0$ is fixed, then for all $n > n_0$:

$$a_0(k_n,\ell_n)^{n_0} c(k_n,\ell_n) = a_0(k_n,\ell_n)^{n_0} \frac{1}{a_0(k_n,\ell_n)^n} \leq 1,$$

and thus

$$\sup_{(k,\ell)\in\mathbb{N}_0^2} (a_0(k,\ell)^{n_0} c(k,\ell)) < \infty.$$

On the other hand,

$$a(k_n,\ell_n)c(k_n,\ell_n) > n \cdot a_0(k_n,\ell_n)^n \frac{1}{a_0(k_n,\ell_n)^n} = n,$$

and therefore $a \notin \rho^{\#\#}$. This yields a contradiction. ∎

7 Some properties of the (LB)-topology on $PW(\mathbb{C})$

In this section, we want to use Lemma 6.1 for our study of the topological properties of the (LB)-space $PW(\mathbb{C})$. For this purpose, we define for $(k,\ell) \in \mathbb{N}_0^2$ the subset $V_{(k,\ell)}$ of the complex plane by

$$V_{(k,\ell)} = \{z \in \mathbb{C} \mid k \leq |\operatorname{Re} z| < k+1,\ \ell \leq |\operatorname{Im} z| < \ell+1\} \tag{19}$$

(see Figure 1). It is obvious that $\mathbb{C} = \cup_{(k,\ell)\in\mathbb{N}_0^2} V_{(k,\ell)}$. Due to the structure of $V_{(k,\ell)}$, an upper

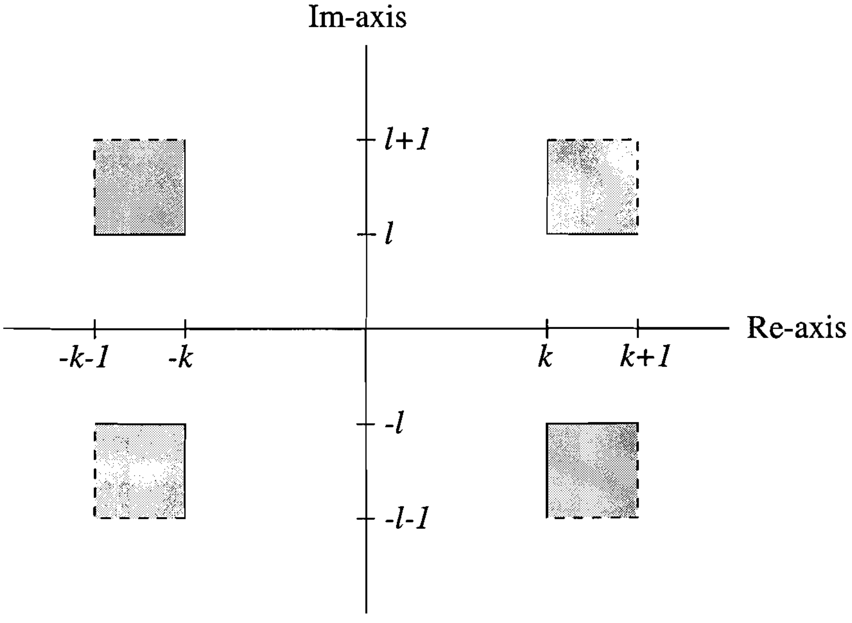

Figure 1: The set $V_{(k,\ell)}$

and lower bound of the function $A(z) = (1+|z|)\exp(|\operatorname{Im} z|)$ on $V_{(k,\ell)}$ is easily obtained, using the definition of the double sequence a_0 in (14):

$$\forall z \in V_{(k,\ell)} : a_0(k,\ell) \leq |A(z)| \leq (1+\sqrt{2})e \cdot a_0(k,\ell). \tag{20}$$

In order to study the topologies on $PW(\mathbb{C})$ and $H(\mathbb{C})$, we define for every $(k,\ell) \in \mathbb{N}_0^2$ the seminorm $s_{(k,\ell)}$ on $H(\mathbb{C})$ by

$$s_{(k,\ell)}(\psi) := \sup\{|\psi(z)| \mid z \in V_{(k,\ell)}\}, \qquad \psi \in H(\mathbb{C}). \tag{21}$$

Let $n \in \mathbb{N}_0$, and $\phi \in PW(\mathbb{C}; n)$. Then

$$\|\phi\|_{PW,n} = \sup_{z\in\mathbb{C}} \left(A(z)^{-n}|\phi(z)|\right) \leq \sup_{(k,\ell)\in\mathbb{N}_0^2} \left(a_0(k,\ell)^{-n} s_{(k,\ell)}(\phi)\right),$$

and also

$$\|\phi\|_{PW,n} = \sup_{z\in\mathbb{C}} \left(A(z)^{-n}|\phi(z)|\right) \geq \left(\frac{1}{(1+\sqrt{2})e}\right)^n \cdot \sup_{(k,\ell)\in\mathbb{N}_0^2} \left(a_0(k,\ell)^{-n} s_{(k,\ell)}(\phi)\right).$$

So, if we define for $\phi \in PW(\mathbb{C}; n)$ and $n \in \mathbb{N}_0$:

$$\|\phi\|_{\widetilde{PW},n} := \sup_{(k,\ell)\in\mathbb{N}_0^2} \left(a_0(k,\ell)^{-n} s_{(k,\ell)}(\phi)\right), \tag{22}$$

then the norms $\|\cdot\|_{\widetilde{PW},n}$ and $\|\cdot\|_{PW,n}$ on $PW(\mathbb{C}; n)$ are equivalent:

$$\forall \phi \in PW(\mathbb{C}; n) : \left(\frac{1}{(1+\sqrt{2})e}\right)^n \|\phi\|_{\widetilde{PW},n} \leq \|\phi\|_{PW,n} \leq \|\phi\|_{\widetilde{PW},n}.$$

Lemma 7.1 *Let $b \in \rho^{\#}$, as defined in (16), and define q_b on $PW(\mathbb{C})$ by*

$$q_b(\phi) = \sup_{(k,\ell)\in\mathbb{N}_0^2} \left(b(k,\ell) s_{(k,\ell)}(\phi)\right), \qquad \phi \in PW(\mathbb{C}). \tag{23}$$

Then q_b is a continuous seminorm on $PW(\mathbb{C})$.

Proof: We have to prove that $q_b\,|_{PW(\mathbb{C},n)}$ is continuous on the Banach space $PW(\mathbb{C};n)$ for each $n \in \mathbb{N}_0$. Now indeed for $\phi \in PW(\mathbb{C};n)$:

$$\begin{aligned} q_b(\phi) &= \sup_{(k,\ell)\in\mathbb{N}_0^2} \big(b(k,\ell)s_{(k,\ell)}(\phi)\big) \leq \\ &\leq \sup_{(k,\ell)\in\mathbb{N}_0^2} (a_0(k,\ell)^n b(k,\ell)) \cdot \sup_{(k,\ell)\in\mathbb{N}_0^2} \big(a_0(k,\ell)^{-n} s_{(k,\ell)}(\phi)\big) = \\ &= \left(\sup_{(k,\ell)\in\mathbb{N}_0^2} (a_0(k,\ell)^n b(k,\ell)) \right) \cdot \|\phi\|_{\widetilde{PW},n}. \end{aligned}$$ ■

Using the seminorms q_b with $b \in \rho^{\#}$, one may translate the growth conditions on a $PW(\mathbb{C})$-function ϕ (cf. (10)), to a boundedness condition on the corresponding double sequence $s_{(k,\ell)}(\phi)$.

Lemma 7.2 *Let $\phi \in H(\mathbb{C})$. Then $\phi \in PW(\mathbb{C})$ if and only if for all $b \in \rho^{\#}$*

$$\sup_{(k,\ell)\in\mathbb{N}_0^2} \big(b(k,\ell)s_{(k,\ell)}(\phi)\big) < \infty. \tag{24}$$

Proof: We only need to prove necessity. So, let $\phi \in H(\mathbb{C})$ satisfy

$$\sup_{(k,\ell)\in\mathbb{N}_0^2} \big(b(k,\ell)s_{(k,\ell)}(\phi)\big) < \infty,$$

for all $b \in \rho^{\#}$. Then $s_{(k,\ell)}(\phi) \in \rho^{\#\#}$, and according to Lemma 6.1, there exist $n \in \mathbb{N}_0$ and $\gamma > 0$ such that

$$s_{(k,\ell)}(\phi) \leq \gamma \cdot a_0(k,\ell)^n, \qquad ((k,\ell) \in \mathbb{N}_0^2).$$

Hence

$$\sup_{z\in\mathbb{C}} \big(A(z)^{-n}|\phi(z)|\big) \leq \sup_{(k,\ell)\in\mathbb{N}_0^2} \big(a_0(k,\ell)^{-n} s_{(k,\ell)}(\phi)\big) \leq \gamma,$$

and $\phi \in PW(\mathbb{C};n)$ with $\|\phi\|_{\widetilde{PW},n} \leq \gamma$. ■

In the next theorem we use the seminorms $\{q_b \mid b \in \rho^{\#}\}$ to characterize the bounded subsets of $PW(\mathbb{C})$, thus proving that $PW(\mathbb{C})$ is a regular (LB)-space.

Theorem 7.3 *Let $B \subset H(\mathbb{C})$. Then the following statements are equivalent:*

(i) $B \subset PW(\mathbb{C})$ *is* bounded,

(ii) For every $b \in \rho^{\#}$:

$$\sup_{\phi\in B} \left(\sup_{(k,\ell)\in\mathbb{N}_0^2} (b(k,\ell)s_{(k,\ell)}(\phi)) \right) < \infty,$$

(iii) B *is a* bounded *subset of* $PW(\mathbb{C};n)$ *for some* $n \in \mathbb{N}_0$.

Proof: *(i)* $\Longrightarrow$ *(ii)*: According to Lemma 7.1, the seminorms q_b, with $b \in \rho^{\#}$, are continuous on $PW(\mathbb{C})$.

(ii) $\Longrightarrow$ *(iii)*: Combining *(ii)* and Lemma 7.2, we know that B is a subset of $PW(\mathbb{C})$. The double sequence a defined by

$$a(k,\ell) = \sup_{\phi\in B} s_{(k,\ell)}(\phi)$$

belongs to $\rho^{\#\#}$. So Lemma 6.1 implies that there exist $n \in \mathbb{N}_0$ and $\gamma > 0$ such that $a \leq \gamma \cdot a_0^n$. It follows that $B \subset PW(\mathbb{C}; n)$, and B is bounded: $\forall \phi \in B : \|\phi\|_{PW,n}^{\sim} \leq \gamma$.

(iii) $\Longrightarrow$ *(i)*: This is obvious because $PW(\mathbb{C}; n)$ is continuously embedded in $PW(\mathbb{C})$. ∎

Next, we investigate the notion of convergence for sequences in the (LB)-space $PW(\mathbb{C})$.

Theorem 7.4 *Let $(\phi_m)_{m\in\mathbb{N}}$ denote a sequence in $PW(\mathbb{C})$. Then the following statements are equivalent:*

(i) (ϕ_m) is a convergent sequence in the (LB)-space $PW(\mathbb{C})$;

(ii) (ϕ_m) is a Cauchy sequence in the (LB)-space $PW(\mathbb{C})$;

(iii) (ϕ_m) is a bounded sequence in the (LB)-space $PW(\mathbb{C})$ and (ϕ_m) is a convergent sequence in the Fréchet space $H(\mathbb{C})$;

(iv) There is an $n \in \mathbb{N}_0$ such that (ϕ_m) is a convergent sequence in the Banach space $PW(\mathbb{C}; n)$.

Proof: *(i)* $\Longrightarrow$ *(ii)*: Obvious, because every convergent sequence in the (LB)-space $PW(\mathbb{C})$ is a Cauchy sequence.

(ii) $\Longrightarrow$ *(iii)*: (ϕ_m) is a Cauchy sequence in $PW(\mathbb{C})$, hence (ϕ_m) is bounded in $PW(\mathbb{C})$. Since the canonical injection from $PW(\mathbb{C})$ into $H(\mathbb{C})$ is continuous, (ϕ_m) is a Cauchy sequence in the Fréchet space $H(\mathbb{C})$, and therefore convergent.

(iii) $\Longrightarrow$ *(iv)*: Since (ϕ_m) is a bounded sequence in $PW(\mathbb{C})$, Theorem 7.3 implies that there exists an $n \geq 1$, such that (ϕ_m) is a bounded sequence in the Banach space $PW(\mathbb{C}; n-1)$. We will show that (ϕ_m) is a Cauchy sequence in $PW(\mathbb{C}; n)$, and therefore convergent.

Let $\varepsilon > 0$. Define $B := \sup_{m\in\mathbb{N}} \|\phi_m\|_{PW,n-1}$, and choose $R > 0$ so large that $\frac{1}{1+R} < \frac{\varepsilon}{4B}$. Since (ϕ_m) is a Cauchy sequence in $H(\mathbb{C})$, there exists an $m_0 \in \mathbb{N}$ such that for all $m, p > m_0$

$$\sup_{|z|\leq R} |\phi_m(z) - \phi_p(z)| < \frac{\varepsilon}{2}.$$

Then for all $m > m_0$

$$\begin{aligned}
\|\phi_m - \phi_p\|_{PW,n} &= \sup_{z\in\mathbb{C}} \left(A(z)^{-n}|\phi_m(z) - \phi_p(z)|\right) \leq \\
&\leq \sup_{|z|\leq R} |\phi_m(z) - \phi_p(z)| + \frac{1}{1+R}\|\phi_m - \phi_p\|_{PW,n-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4B}2B = \varepsilon.
\end{aligned}$$

(iv) $\Longrightarrow$ *(i)*: By definition, the canonical injection from $PW(\mathbb{C}; n)$ into $PW(\mathbb{C})$ is continuous. ∎

Corollary 7.5 *The (LB)-space $PW(\mathbb{C})$ is sequentially retractive.* ∎

Corollary 7.6 *The (LB)-space $PW(\mathbb{C})$ is sequentially complete.* ∎

Note that the arguments of Theorem 7.4 cannot be used to prove that $PW(\mathbb{C})$ is complete, because a Cauchy net in $PW(\mathbb{C})$ is not necessarily bounded.

Finally, we want to show that $PW(\mathbb{C})$ is a Montel space, i.e. it satisfies the Heine-Borel property. In the proof, we need the following preliminary result:

Lemma 7.7 *Let $(\phi_m)_{m\in\mathbb{N}}$ be a bounded sequence in $H(\mathbb{C})$. Then the sequence has a convergent subsequence.*

Proof: Let $C(\mathbb{C})$ denote the vector space of all continuous functions on $\mathbb{C}$ with Fréchet topology. $H(\mathbb{C})$ is a closed subspace of $C(\mathbb{C})$ because the Fréchet topologies on both spaces are brought about by the same seminorms (11). Therefore the sequence (ϕ_m) is also bounded in $C(\mathbb{C})$. We will show that the sequence (ϕ_m) is locally equicontinuous.

Let K be a compact subset of $\mathbb{C}$, and define $R := 1 + \max\{|z| \mid z \in K\}$. Then the circle $\Gamma = \{z \in \mathbb{C} \mid |z| = R\}$ is a Jordan curve in $\mathbb{C}$, encircling K at a distance of at least 1. Let $m \in \mathbb{N}$ and $z, w \in K$. Using Cauchy's integral formula, we obtain

$$|\phi_m(z) - \phi_m(w)| \leq \frac{1}{2\pi} \oint_\Gamma \frac{|\phi_m(\zeta)||z-w|}{|(\zeta - z)(\zeta - w)|} |d\zeta| \leq \frac{1}{2\pi} \sup_{\zeta\in\Gamma} (|\phi_m(\zeta)|) \cdot 2\pi R \cdot |z-w|.$$

Since $\{\phi_m \mid m \in \mathbb{N}\}$ is bounded in $H(\mathbb{C})$,

$$C := \sup_{m\in\mathbb{N}} \sup_{\zeta\in\Gamma} (|\phi_m(\zeta)|),$$

is well-defined and finite. So for all $m \in \mathbb{N}$ and $z, w \in K$,

$$|\phi_m(z) - \phi_m(w)| \leq CR \cdot |z - w|,$$

and the sequence (ϕ_m) is locally equicontinuous.

Eventually, application of the generalized version of the Arzela-Ascoli theorem, as stated in Theorem 1.2 (with $\mathbb{R}$ replaced by $\mathbb{C}$), yields that (ϕ_m) has a convergent subsequence in $C(\mathbb{C})$. Since $H(\mathbb{C})$ is a closed subspace of $C(\mathbb{C})$, this subsequence is also convergent in $H(\mathbb{C})$.

■

Corollary 7.8 *The Fréchet space $H(\mathbb{C})$ is a Montel space.* ■

Theorem 7.9 *The (LB)-space $PW(\mathbb{C})$ is a Montel space.*

Proof: Let K be a closed and bounded subset of $PW(\mathbb{C})$. Then there is an $n \geq 1$, such that $K \subset PW(\mathbb{C}; n-1)$. We prove that K is a compact subset of $PW(\mathbb{C}; n)$.

So, let (ϕ_m) be a sequence in K. Then (ϕ_m) is a bounded sequence in $H(\mathbb{C})$, and according to Lemma 7.7 there is a convergent subsequence (ϕ_{m_ℓ}), with limit $\phi \in H(\mathbb{C})$. It follows as in the proof of Theorem 7.4, that (ϕ_{m_ℓ}) is a convergent sequence in the Banach space $PW(\mathbb{C}; n)$.

■

8 Fourier transformation on $\mathcal{E}'(\mathbb{R})$ as a homeomorphism

In this final section, we relate the commutative convolution algebra $\mathcal{E}'(\mathbb{R})$ and the function algebra $PW(\mathbb{C})$ topologically.

Theorem 8.1 *The Fourier transformation $\mathcal{F}$ is an algebra homeomorphism from the (LB)-space $\mathcal{E}'(\mathbb{R})$ onto the (LB)-space $PW(\mathbb{C})$.*

In the proof of this theorem, we need the following result, stated in [5, pp 12—13]:

Proposition 8.2 *Let* $\psi \in H(\mathbb{C})$, *and assume that there exist* $a > 0$ *and* $C > 0$ *such that for all* $z \in \mathbb{C}$:

$$|\psi(z)| \leq C(1+|z|)^{-2}e^{a|\operatorname{Im} z|}.$$

Then there is a continuous functional F *on the Fréchet space* $C(\mathbb{R})$, *with support contained in* $[-a, a]$, *such that*

$$F(e_z) = \psi(z), \qquad z \in \mathbb{C}.$$

■

Proof of Theorem 8.1: Let $G \in \mathcal{E}'(\mathbb{R}; n)$. Then for all $z \in \mathbb{C}$:

$$\begin{aligned} |(\mathcal{F}G)(z)| &= |G(e_z)| \leq \|G\|'_n \cdot p_n(e_z) = \|G\|'_n \cdot \max_{t\in[-n,n]} \sum_{j=0}^{n} |z|^j e^{t \operatorname{Im} z} \leq \\ &\leq \|G\|'_n A(z)^n, \end{aligned}$$

so that

$$\|\mathcal{F}G\|_{PW,n} \leq \|G\|'_n.$$

Hence $\mathcal{F}$ is a continuous operator from $\mathcal{E}'(\mathbb{R}; n)$ into $PW(\mathbb{C}; n)$, and $\mathcal{F} : \mathcal{E}'(\mathbb{R}) \longrightarrow PW(\mathbb{C})$ is continuous.

For the converse we apply Proposition 8.2. Let $\phi \in PW(\mathbb{C}; n)$, with $n \in \mathbb{N}_0$. Then there exists a $C > 0$ such that

$$|\phi(z)| \leq C(1+|z|)^n e^{n|\operatorname{Im} z|}, \qquad z \in \mathbb{C}.$$

If ϕ has less than $n+2$ zeros, then

$$\phi(z) = p(z)e^{iaz},$$

for some polynomial p of degree smaller than $n+2$, and some $|a| \leq n$. Hence $\phi = \mathcal{F}G$ with

$$G(f) = (p(i\frac{d}{dt})f)(a),$$

i.e. $G \in \mathcal{E}'(\mathbb{R}; n+1)$. In the other case, there is a polynomial p of degree $n+2$ such that $\psi(z) := \frac{\phi(z)}{p(z)}$ is holomorphic. Then

$$|\psi(z)| \leq \tilde{C}(1+|z|)^{-2}e^{n|\operatorname{Im} z|}, \qquad z \in \mathbb{C},$$

and so there is a continuous functional F on $C(\mathbb{R})$ with support in $[-n, n]$ such that $F(e_z) = \psi(z)$. So, with $G = F \circ p(i\frac{d}{dt})$ we define an element of $\mathcal{E}'(\mathbb{R}; n+2)$, satisfying $G(e_z) = \phi(z)$, $z \in \mathbb{C}$. We see that the inverse Fourier transformation maps $PW(\mathbb{C}; n)$ into $\mathcal{E}'(\mathbb{R}; n+2)$.

Since $\mathcal{F}$ is continuous from $\mathcal{E}'(\mathbb{R})$ into $PW(\mathbb{C})$, $\mathcal{F}^{-1}$ is closed, and so $\mathcal{F}^{-1}$ as an operator from $PW(\mathbb{C}; n)$ into $\mathcal{E}'(\mathbb{R}; n+2)$ is closed, and therefore continuous. We conclude that $\mathcal{F}^{-1}$ from $PW(\mathbb{C})$ onto $\mathcal{E}'(\mathbb{R})$ is continuous. ■

Since the Fourier transformation $\mathcal{F} : \mathcal{E}'(\mathbb{R}) \longrightarrow PW(\mathbb{C})$ is an algebra homeomorphism, all topological properties proved in Section 7 for the (LB)-space $PW(\mathbb{C})$ carry over to $\mathcal{E}'(\mathbb{R})$.

Theorem 8.3 *The (LB)-space $\mathcal{E}'(\mathbb{R})$ has the following properties:*

(i) *$\mathcal{E}'(\mathbb{R})$ is regular, i.e. every bounded set in $\mathcal{E}'(\mathbb{R})$ is bounded in some $\mathcal{E}'(\mathbb{R}; n)$;*

(ii) *$\mathcal{E}'(\mathbb{R})$ is sequentially retractive, i.e. every convergent sequence in $\mathcal{E}'(\mathbb{R})$ is contained in $\mathcal{E}'(\mathbb{R}; n)$ for some $n \in \mathbb{N}_0$, and converges with respect to the topology on $\mathcal{E}'(\mathbb{R}; n)$;*

(iii) *$\mathcal{E}'(\mathbb{R})$ is sequentially complete;*

(iv) *$\mathcal{E}'(\mathbb{R})$ is a Montel space.* ■

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