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# SPACES OF HARMONIC FUNCTIONS AND EVOLUTION EQUATIONS IN THEM

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# Contents

# 1. Some properties of harmonic functions

**Definition.** A complex valued function on an open set $\Omega \subset \mathbb{R}^q$ is said to be harmonic on $\Omega$ if $f \in C^\infty(\Omega)$ and $\dfrac{\partial^2 f}{\partial x_1^2} + \ldots + \dfrac{\partial^2 f}{\partial x_q^2} = \Delta f = 0$ at each point $\underline{x} \in \Omega$.

**Theorem.** Let $f$ be harmonic on $\Omega \subset \mathbb{R}^q$. The following functions are all harmonic on suitable open sets:

$$
\begin{aligned}
(T_{\underline{a}} f)(\underline{x}) &= f(\underline{x} + \underline{a}) &&, \quad \underline{a} \in \mathbb{R}^q \\
(L_A f)(\underline{x}) &= f(A\underline{x}) &&, \quad A \in O(q) \\
(Z_\lambda f)(\underline{x}) &= f(\lambda \underline{x}) &&, \quad \lambda \in \mathbb{R} \\
(Kf)(\underline{x}) &= |x|^{2-q} f\Big(\frac{x}{|x|^2}\Big) &&, \quad q > 2 \\
(\mathcal{P}_i f)(\underline{x}) &= \frac{\partial f}{\partial x_i}(\underline{x}) && \\
(\mathcal{L}_B f)(\underline{x}) &= (B\underline{x}, \nabla f(\underline{x})) &&, \quad B^T = -B \\
(Nf)(\underline{x}) &= (\underline{x}^T \nabla f)(\underline{x}) = x_1 \frac{\partial f}{\partial x_1} + \ldots + x_q \frac{\partial f}{\partial x_q} \,. &&
\end{aligned}
$$

**Proof.** All straightforward calculations. Note that for general $A \in \mathbb{R}^{q \times q}$ one has

$$\Delta f(A\underline{x}) = \operatorname{trace} AD^2 f A^T \;, \qquad \text{evaluated at } A\underline{x} \,.$$

(In index notation $\partial_{j'} \partial_{i'} f = A^i_{i'} A^j_{i'} \partial_j \partial_i f$.)

**Notation.** The complex vector space of all harmonic functions on $\Omega$ is denoted by $HA(\Omega)$.

**Mean Value Theorem.** Let $B \subset \Omega$ be a ball with centre $\underline{a}$ and radius $R$. If $u \in HA(\Omega)$ then

$$u(\underline{a}) = \frac{1}{\omega_q R^{q-1}} \int_{\partial B} u(\underline{x})\, d\sigma$$

or, equivalently,

$$u(\underline{a}) = \frac{q}{\omega_q R^q} \int_B u(\underline{x})\, d\underline{x} \,.$$

Here $\omega_q R^{q-1}$ denotes the total surface measure of the boundary $\partial B$ of $B$ ($\omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$).

**Proof.** Let $B_0 < B$ be a ball with centre $\underline{a}$ and radius $R_0 < R$. Then with Green II

$$\int_{\partial(B\backslash B_0)} \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) d\sigma = \int_{B\backslash B_0} (u\Delta v - v\Delta u)\, d\underline{x} \;.$$

We take $u \in HA(\Omega)$ and

$$v(\underline{x}) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{|\underline{x}-\underline{a}|} & , \quad q = 2 \\[2ex] \dfrac{1}{\omega_q(q-2)\,|\underline{x}-\underline{a}|^{q-2}} & , \quad q > 2 \;. \end{cases}$$

Since $v \in HA(\Omega\backslash\{\underline{a}\})$, we find

$$\frac{1}{\omega_q\, R^{q-1}} \int_{\partial B} u\, d\sigma = \frac{1}{\omega_q\, R_0^{q-1}} \int_{\partial B_0} u\, d\sigma \;.$$

Now let $R_0 \to 0$. This yields the first formula. The second one follows by integrating over spheres. □

**Theorem** (Maximum-Minimum-Principle). If $f \in HA(\Omega)$, $\Omega$ open, attains its maximum at $\underline{a} \in \Omega$, then $f$ is a constant.
If, in addition, $f \in C(\bar{\Omega})$ and $\Omega$ bounded then $f$ takes its extreme values on $\partial\Omega$.

**Proof.** Let $M$ denote the set of maximum points in $\Omega$. Then

- $M$ is closed in $\Omega$ because $f$ is continuous.
- $M$ is open in $\Omega$. To see this, let $\underline{a} \in \Omega$ and $B \subset \Omega$ a ball, centered at $\underline{a}$. Then, because of the continuity of $f$, the mean-value theorem and $\underline{a}$ being a maximum point, $f$ must be constant on $B$. So $M = \Omega$.[1] □

**Theorem** (Uniqueness Dirichlet problem). The boundary value problem

$$\left.\begin{array}{l} \Delta u = 0 \text{ in } \Omega, \text{ bdd.} \\ u \in C(\bar{\Omega}) \\ u(x) = \varphi(x) \text{ on } \partial\Omega \end{array}\right\}$$

has, at most, one solution.

**Proof.** Let $v$ be a second solution, then $w = u - v$ satisfies $\Delta w = 0$, $w \in C(\bar{\Omega})$, $w = 0$ on $\partial\Omega$. According to the max-min-principle $w$ must be zero all over $\Omega$. □

[1] Note that we only used the continuity and the mean value property in the proof!!

**Theorem** (Solution Dirichlet problem on the ball). The boundary value problem

$$\begin{aligned} &\Delta u = 0 \text{ in } B_R = \{\underline{x} \mid |\underline{x}| < R\} \\ &u \in C(\bar{B}_R) \\ &u(\underline{x}) = \varphi(\underline{x}) \ , \qquad \varphi \in C(\partial \bar{B}_R) \end{aligned}$$

is solved by

$$u(\underline{x}) = \frac{R^2 - |\underline{x}|^2}{\omega_q R} \int\limits_{\partial \bar{B}_R} \frac{\varphi(\underline{y})}{|\underline{x} - \underline{y}|^q} \, d\sigma \ .$$

**Sketch of the proof.** The harmonicity of $u$ in $B_R$ follows by straightforward differentiation. The proof of the continuity up to the boundary involves more subtle $\varepsilon/\delta$ arguments. See [A]. □

Next we show that harmonic functions are characterised by the mean value property.

**Theorem.** Let $u \in C(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\Omega$ open and suppose that for each $\underline{a} \in \Omega$ and each ball $B(R\,;\,\underline{a}) = \{\underline{x} \mid |\underline{x} - \underline{a}| < R\}$ such that $N(R\,;\,\underline{a}) \subset \Omega$, $u(\underline{a}) = \dfrac{q}{\omega_q E^q} \displaystyle\int\limits_{B(R;\underline{a})} u(\underline{x})\, d\underline{x}$, then $u \in HA(\Omega)$.

**Proof.** Consider an arbitrary ball $B \subset \Omega$. Let $w$ denote the solution of the Dirichlet problem inside of $B$ with $w|_{\partial B} = u|_{\partial B}$. Then $w - u$ has the mean value property in $B$ and $w - u = 0$ on $\partial B$. Therefore $w - u = 0$ on the whole of $B$. □

**Remark.** The condition $u \in C(\Omega)$ can be weakened to $u \in L_{1,\text{loc}}(\Omega)$.

**Harnack's Theorem.** Let $\{u_n\}$ be a sequence of harmonic functions in $HA(\Omega)$. Suppose that $u_n(\underline{x}) \to u(\underline{x})$ uniformly on compact sets in $\Omega$. Then also $u \in HA(\Omega)$.

**Proof.** The equality $u_n(\underline{a}) = \dfrac{q}{w_q R^q} \displaystyle\int\limits_B u_n(\underline{x})\, dx$ persists if $n \to \infty$. □

**Remark.** It is not difficult to show that also the sequence $\left\{\dfrac{\partial u_n}{\partial x_i}\right\}$ converges uniformly on compact sets in $\Omega$ to $\dfrac{\partial u}{\partial x_i}$ .

# 2. Some properties of harmonic polynomials

In the space $P(\mathbb{R}^q)$ of all polynomials on $\mathbb{R}^q$ we introduce the inner product

$$(p,q) = p(\nabla)\, q(\underline{x})|_{\underline{x}=\underline{0}} \ .$$

Note that the monomials $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_q^{\alpha_q} = \underline{x}^{\underline{\alpha}}$ and $\underline{x}^{\underline{\beta}}$ are orthogonal if $\underline{\alpha} \neq \underline{\beta}$ and

$$(\underline{x}^{\underline{\alpha}}, \underline{x}^{\underline{\alpha}}) = \underline{\alpha}! = \alpha_1!\alpha_2! \cdot \ldots \cdot \alpha_q! \ .$$

Let $P_m(\mathbb{R}^q)$ denote the subspace of all homogeneous polynomials of degree $m$. The harmonic polynomials or, spherical harmonics, are denoted by $HP(\mathbb{R}^q)$ and $HP_m(\mathbb{R}^q)$.
Note that $\dim P_m(\mathbb{R}^q) = \binom{q+m-1}{m}$.

**Theorem.**

- In $P_m(\mathbb{R}^q)$ we have $HP_m(\mathbb{R}^q)^\perp = |\underline{x}|^2 \cdot P_{m-2}(\mathbb{R}^q)$.
- $\dim HP_m(\mathbb{R}^q) = \binom{q+m-1}{m} - \binom{q+m-3}{m-2} = d_m^q \leq K_q\, m^{q-2}$.

**Proof.** Define the polynomial $s(\underline{x}) = |\underline{x}|^2$.
If $f \in HP_m(\mathbb{R}^q)$ then for all $q \in P_{m-2}(\mathbb{R}^q)$

$$(qs, f) = q(\nabla)\, s(\nabla)\, f = q(\nabla)\, \Delta f = 0 \ .$$

Conversely, let $f \perp s \cdot P_{m-2}(\mathbb{R}^q)$. This means $\forall q \in P_{m-2}(\mathbb{R}^q) \quad q(\nabla)\, s(\nabla)\, r = 0$. Take $q = s(\nabla)\, r = \Delta r$ then $(\Delta r, \Delta r) = 0$ so $\Delta r = 0$. □

Next we consider restrictions of harmonic polynomials to the spheres $S_R^{q-1} = \{\underline{x} \,|\, \underline{x} \in \mathbb{R}^q,\ |\underline{x}| = R > 0\}$. If $R = 1$ we write $S_1^{q-1} = S^{q-1}$.

**Theorem.** Consider $f \in P_m(\mathbb{R}^q)$ and $g \in HP_k(\mathbb{R}^q)$.

(a) If $k > m$, then

$$\int\limits_{|\underline{x}|=R} fg\, d\sigma = 0 \ .$$

(b) If $m = k$, then

$$\int_{|\underline{x}|=R} fg\, d\sigma = \frac{\omega_n\, R^{2k+q-1}}{q \cdot (q+2) \cdot \ldots \cdot (q+2k-2)} (f(\nabla)g)(\underline{0}) \; .$$

**Proof.**

(a) According to the preceding theorem $f$ can be splitted

$$f = f_m + |\underline{x}|^2 f_{m-2} + |\underline{x}|^4 f_{m-4} + \ldots \; ,$$

with all $f_j \in HP_j(\mathbb{R}^q)$. So it suffices to prove the result for $f \in HP_m(\mathbb{R}^q)$, $m < k$. By Green II we have

$$0 = \int_{|\underline{x}|\leq R} (f\nabla g - g\nabla f) d\underline{x} = \int_{|\underline{x}|=R} \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\right) d\sigma =$$

$$= (k-m) \int_{|\underline{x}|=R} fg\, d\sigma \; .$$

(b) See [B]. □

**Theorem.**

(a) Let $f \in HP_k(\mathbb{R}^q)$. For each $\underline{a} \in \mathbb{R}^q$ and each $\varepsilon$, $0 < \varepsilon < 1$, we have the estimation

$$|f(\underline{a})|^2 \leq \left[\frac{q}{w_q(2k+q)} \left(1 + \frac{1}{\varepsilon}\right)^q\right] (1+\varepsilon)^{2k} |\underline{a}|^{2k} \int_{|\underline{x}|=1} |f(\underline{x})|^2\, d\sigma \; .$$

(b) And even, more subtle,

$$|f(\underline{a})|^2 \leq \frac{d_k^q}{w_q} \int_{|\underline{x}|=1} |f(\underline{x})|^2\, d\sigma \; , \qquad \text{if } |\underline{a}| = 1 \; .$$

**Proof.** First note that

$$\int_{|\underline{x}|\leq R} |f(\underline{x})|^2\, d\underline{x} = \frac{R^{2k+q}}{2k+q} \int_{|\underline{x}|=1} |f(\underline{x})|^2\, d\sigma \; .$$

(a) Since $f$ is homogeneous it suffices to prove the result for $\underline{a} \in \mathbb{R}^q$ with $|\underline{a}| = 1$. From the mean-value-theorem

$$f(\underline{a}) = \frac{q}{\omega_q\, \varepsilon^q} \int\limits_{|\underline{x}-\underline{a}|<\varepsilon} f(\underline{x})\, dx$$

we find, applying Cauchy-Schwarz,

$$|f(\underline{a})|^2 \leq \frac{q}{\omega_q \varepsilon^2} \int\limits_{|\underline{x}-\underline{a}|<\varepsilon} |f(\underline{x})|^2\, dx \leq$$

$$< \frac{q}{\omega_q \varepsilon^2} \int\limits_{|\underline{x}|\leq 1+\varepsilon} |f(\underline{x})|^2\, d\underline{x} = \frac{q}{\omega_q \varepsilon^q} \frac{(1+\varepsilon)^{2k+q}}{2k+q} \int\limits_{|\underline{x}|=1} |f(\underline{x})|^2\, d\sigma\ .$$

(b) See e.g. [M]. □

Because of the Stone-Weierstrass theorem any continuous function on the sphere. $S_R^{q-1} \subset \mathbb{R}^q$ can be uniformly approximated by a sequence of (restrictions to $S_R^{q-1}$ of) polynomials. From the foregoing it is clear that such an approximating sequence can be replaced by a sequence of *harmonic* polynomials. As a consequence, the restrictions of the elements of $HP_k(\mathbb{R}^q)$ to $S_R^{q-1}$ establish a dense linear subspaqce of $L_2(S_R^{q-1})$. From now on we suppose that $HP_k(\mathbb{R}^\varepsilon)$ carries the $L_2(S^{q-1})$-inner product $(f,g) = \int\limits_{|\underline{x}|=1} f(\underline{x})\,\overline{g(\underline{x})}\, d\sigma$. Gathering our results we find part (a) of the following

**Theorem.**

(a) $L_2(S^{q-1}) = \oplus_{k=0}^{\infty} HP_k(\mathbb{R}^q)$.

(b) The projection $\Pi_k$ of $L_2(S^{q-1})$ on $HP_k(\mathbb{R}^q)$ is realised by

$$(\Pi_k f)(\underline{\xi}) = \int\limits_{S^{q-1}} k(\underline{\xi},\underline{\eta})\, f(\underline{\eta})\, d\sigma\ ,$$

with

$$k(\xi,\eta) = \frac{\Gamma\left(\frac{q}{2}\right) d_k^q}{2^k\, \Gamma\left(k+\frac{q}{2}\right)} P_k^q\left((\underline{\xi},\underline{\eta})\right)\ .$$

The $P_k^q$ are Gegenbauer polynomials.

For the proof of (b) see [M].

We now come to a central expansion result for harmonic functions.

**Theorem.** Let $f \in HA(B_R)$, $R > 0$. Then there is a unique expansion

$$f = \sum_{k=0}^{\infty} f_k \qquad \text{with } f_k \in HP_k(\mathbb{R}^q)$$

which converges uniformly on compact sets in $B_R$.
The $L_2(S^{q-1})$-norms $\|f_k\|$ of $f_k$ satisfy

$$\forall r < R \quad \sum_{k=0}^{\infty} r^{2k} \|f_k\|^2 < \infty$$

or, equivalently

$$\forall r < R \quad \sup_k r^k \|f_k\| < \infty \ .$$

Conversely, if a sequence $\{f_k\}$ of spherical harmonics satisfies this condition then the sum $g(\underline{x}) = \sum_{k=0}^{\infty} f_k(\underline{x})$ exists at each $\underline{x} \in B_R$ and $g \in HA(B_R)$.

**Proof.** Because of the scaling properties of harmonic functions and spherical harmonics it is sufficient to show that, given $f \in HA(B_R)$, $R > 1$, the expansion result is valid on the closed unit ball $\bar{B}$.

First, note that $f|_{S^{q-1}} = \sum_{k=0}^{\infty} f_k|_{S^{q-1}}$ in $L_2(S^{q-1})$-sense; with $f_k \in HP_k(\mathbb{R}^q)$.

Further, take $R_1$, $1 < R_1 < R$. Then

$$R_1^{q-1} \sum_{k=0}^{\infty} R_1^{2k} \|f_k\|^2 = \sum_{k=0}^{\infty} R_1^{q+2k-1} \int_{|\underline{x}|=1} |f_k(\underline{x})|^2 \, d\sigma =$$

$$= \sum_{k=0}^{\infty} \int_{|\underline{x}|=R_1} |f_k(\underline{x})|^2 \, d\sigma = \int_{|\underline{x}|=R_1} |f(\underline{x})|^2 \, d\sigma < \infty \ .$$

Then, from our point evaluation result for spherical harmonics, take $\varepsilon = \frac{1}{2}(R_1 - 1)$, it follows that $\sum_{k=0}^{\infty} f_k(\underline{x})$ converges in $L_\infty(S^q)$-sense to $f(\underline{x})$ on $S^{q-1}$. But then, because of the maximum principle $\left[f(\underline{x}) - \sum_{k=0}^{N} f_k(\underline{x})\right] \to 0$, uniformly on $B$, as $N \to \infty$.

Conversely, $\sum_{k=0}^{N} f_k$ is a Cauchy sequence in $L_\infty(B_{R_1})$ on each ball $B_{R_1}$ with $R_1 < R$. The estimation is as follows, take $R_1 < R_2 < R$;

$$\Big| \sum_{k=N+1}^{M} f_k(\underline{x})| \leq C_{q,\varepsilon} \sum_{k=N+1}^{M} (1+\varepsilon)^k R_1^k \|f_k\| =$$

$$= C_{q,\varepsilon} \sum_{k=N+1}^{M} \frac{((1+\varepsilon) R_1)^k}{R_2^k} R_2^k \|f_k\| \leq$$

$$\leq C_{q,\varepsilon} \left\{ \sum_{k=N+1}^{M} \frac{((1+\varepsilon)\,R_1)^{2k}}{R_2^{2k}} \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{\infty} R_2^{2k} \, \|f_k\|^2 \right\}^{\frac{1}{2}} .$$

If $\varepsilon$ is such that $\dfrac{(1+\varepsilon)\,R_1}{R_2} < 1$, the wanted result follows. □

# 3. Spaces of harmonic functions. Duality

We start with the introduction of a pairing between two spaces of harmonic functions on open balls.

**Theorem.** Let $R > r > 0$ and $Rr > 1$. Let $f \in HA(B_R)$ and $g \in HA(B_r)$. We define

$$\langle f,g\rangle_\alpha = \int\limits_{|\underline{x}|=1} f(\alpha\underline{x})\,\overline{g\Big(\frac{1}{\alpha}\,\underline{x}\Big)}\,d\sigma\ , \qquad \alpha \text{ such that } \alpha r > 1\,,\ \ \frac{R}{\alpha} > 1\ .$$

$$\langle f,g\rangle_\rho = \frac{1}{\rho}\int\limits_{|\underline{x}|=\rho} f(\underline{x})\,\overline{(Kg)(\underline{x})}\,d\sigma\ , \qquad \rho \text{ such that } \frac{1}{r} < \rho < R\ .$$

These pairings do not depend on the choice of $\alpha$ and $\rho$. In the sequel we omit those indices.

**Proof.** Follows easily from expansion in homogeneous harmonic polynomials $f = \sum_{k=0}^{\infty} b_k$.
Note that $f(\alpha\underline{x}) = \sum_{k=0}^{\infty} f_k(\alpha\underline{x}) = \sum_{k=0}^{\infty} \alpha^k\, f_k(\underline{x})$. Note also that $(K\,g_m)(\underline{x}) = |\underline{x}|^{2-q}\, g_m\Big(\frac{\underline{x}}{|\underline{x}|^2}\Big) = |x|^{2-q-2m}\, g_m(\underline{x})$. So

$$\frac{1}{\rho}\int\limits_{|\underline{x}|=\rho} f(\underline{x})\,\overline{(Kg)(\underline{x})}\,d\sigma = \sum_{m=0}^{\infty}\frac{1}{\rho}\int\limits_{|\underline{x}|=\rho} \rho^{2-q-2m}\, f_m(\underline{x})\,\overline{g_m(\underline{x})}\,d\sigma =$$

$$= \sum_{m=0}^{\infty}\frac{1}{\rho}\int\limits_{|\underline{x}|=1} \rho^{2-q-2m}\, f_m(\rho\underline{\xi})\,\overline{g_m(\rho\underline{\xi})}\,\rho^{q-1}\,d\sigma =$$

$$= \sum_{m=0}^{\infty}\int\limits_{|\underline{x}|=1} b_m(\underline{\xi})\,\overline{g_m(\underline{\xi})}\,d\sigma\ . \qquad \square$$

**Remark.** Let $HA(\bar{B})$ denote the space of functions which are harmonic on a neighbourhood of the *closed* unit ball, i.e. $HA(\bar{B}) = \bigcup_{R>1} HA(B_R)$.
The spaces $HA(B)$ and $HA(\bar{B})$ are in duality by means of the pairing $\langle\cdot,\cdot\rangle$.

Next we introduce weighted $L_2$-spaces of harmonic functions. We always consider radial weight functions $\mu$ which satisfy the conditons: $\mu \geq 0$, and

$$(r \mapsto \mu(r)) \in L_1(0,R) \qquad \text{if } R < \infty$$

$$\forall m \in \mathbb{N}\cup\{0\}\ (r \mapsto r^m\,\mu(r)) \in L_1(0,\infty) \qquad \text{if } R = \infty\ .$$

**Definition.**

$$HA(B_R\,;\,\mu) = \{f \mid f \in L_2(B_R\,;\,\mu),\ f \in HA(B_R)\}\ , \qquad 0 < R < \infty$$
$$HA_2(\mathbb{R}^q\,;\,\mu) = \{f \mid f \in L_2(\mathbb{R}^q\,;\,\mu),\ f \in HA(\mathbb{R}^q)\}\ .$$

As a consequence of the mean value theorem both these subspaces are closed. Hence they are Hilbert spaces themselves. The inner products are given by

$$(f,g) = \int\limits_{|\underline{x}|\le R} f(\underline{x})\,\overline{g(\underline{x})}\,\mu(|\underline{x}|)\,d\underline{x}\ .$$

Further, because of the conditons on the weight function, the spaces $HP_m(B_R)$ are mutually orthogonal subspaces in $HA_2(B_R\,;\,\mu)$ for all $R$, $0 < R \le \infty$, and all weights $\mu$.

**Theorem.**

- $HA_2(B_R\,;\,\mu) = \oplus_{m=0}^{\infty} HP_m(B_R)$.
- Let $f = \sum\limits_{m=0}^{\infty} f_m \in HA(B_R)$, $f_m \in HP_m$, then

$$f \in HA_2(B_R\,;\,\mu) \qquad \text{iff } (\alpha_m\,\|f_m\|)_{m=0}^{\infty} \subset l_2\ .$$

Here $\|f_m\| = \left\{ \int\limits_{|\underline{x}|=1} |f_m(\underline{x})|^2\,d\sigma \right\}^{\frac{1}{2}}$ and the sequence $(\alpha_m)_{m=0}^{\infty}$ is defined by

$$\alpha_m = \left\{ \int\limits_0^R r^{2m+q-1}\,\mu(r)\,dr \right\}^{\frac{1}{2}}\ .$$

**Proof.** Any $f \in HA(B_R)$ can be written $f = \sum\limits_{m=0}^{\infty} f_m$ with convergence uniformly on compact sets in $B_R$. The remaining statements follow from

$$\int\limits_{|\underline{x}|\le R} |f(\underline{x})|^2\,\mu(|\underline{x}|)\,d\underline{x} = \int\limits_0^R r^{q-1}\,\mu(r)\,dr \int\limits_{|\underline{\xi}|=1} f(r\underline{\xi})|^2\,d\sigma =$$

$$= \int\limits_0^R r^{q-1}\,\mu(r)\,dr \cdot \sum_{m=0}^{\infty} \int\limits_{|\underline{\xi}|=1} |b_m(r\underline{\xi})|^2\,d\sigma =$$

$$= \sum_{m=0}^{\infty} \int_0^R r^{2m+q-1} \mu(r)\, dr \cdot \|f_m\|^2_{L_2(S^{q-1})} = \sum_{m=0}^{\infty} \|f_m\|^2_{HA_2(B_R;\mu)} \,. \qquad \square$$

We introduce a special two-parameter class of Hilbert spaces on the open unit ball $B = B_1$. The two parameters are $\beta > 0$ and $r \in \mathbb{N} \cup \{0\}$.

**Definition.** $HA_2(B\,;r,\beta) = \{f \mid f \in HA(B),\ \|f\|_{r,\beta} < \infty\}$ with

$$(\|f\|_{r,\beta})^2 = \int_{|\underline{x}|\leq 1} (f(\underline{x})|^2 + |(N^r f)(\underline{x})|^2)(1 - |\underline{x}|)^{2\beta-1}\, d\underline{x}$$

and

$$\beta > 0\,, \quad r = 0, 1, 2, \dots\,.$$

**Theorem.**

- If $f \in HP_m$ then

$$\|f_m\|^2_{r,\beta} = (1 + m^{2r}) \cdot \frac{\Gamma(2m+q)\,\Gamma(2\beta)}{\Gamma(2m+q=2\beta)} \|f_m\|^2_{L_2(S^{q-1})} \,.$$

- $\lim\limits_{m\to\infty} [(1+m^2)^{r-\beta}]^{-1} (1+m)^{2r} \dfrac{\Gamma(2m+q)\,\Gamma(2\beta)}{\Gamma(2m+q+2\beta)} = 2^{-2\beta}\,\Gamma(2\beta) > 0$ .

- $F = \sum\limits_{m=0}^{\infty} F_m \in HA_2(B\,;r,\beta)$ iff

$$\left((1+m^2)^{\frac{r-\beta}{2}} \|F_m\|\right)_{m=0}^{\infty} \subset l_2 \,.$$

- As topological vector spaces the $HA_2(B\,;r,\beta)$ only depend on the difference $s = r - \beta$. Abusing the notation we write $HA_2(B\,;s)$ instead.

**Proof.** Calculate $\|f_m\|^2_{r,\beta} = (1+m^{2r}) \cdot \int_0^1 r^{2m+q-1}(1-r)^{2\beta-1}\, dr \cdot \|f_m\|^2_{L_2(S^{q-1})}$. Introduction of the Beta-function and of some asymptotics for the Gamma-function leads to the remaining results. $\square$

**Theorem.** For each $s \geq 0$ the Hilbert spaces $HA_2(B\,;-s)$ and $HA_2(B\,;s)$ are each others strong dual. For $F = \sum\limits_{m=0}^{\infty} F_m \in HA(B\,;-s)$ and $f = \sum\limits_{m=0}^{\infty} f_m \in HA_2(B\,;-s)$ the duality is given by

$$\langle F, f\rangle = \lim_{r\uparrow 1} \int_{|\underline{x}|=r} \overline{F(\underline{x})}\, g(\underline{x})\, d\sigma = \sum_{m=0}^{\infty} (f_m, F_m)_{L_2(S^{q-1})} \,,$$

where the sum is absolutely convergent.

**Proof.** Consider $\int\limits_{|\underline{x}|=r} \overline{F(\underline{x})}\, g(\underline{x})\, d\sigma = \sum\limits_{m=0}^{\infty} r^{2m+q-1}(f_m, F_m)_{L_2(S^{q-1})}$. Since $(m^{-s}\,\|F_m\|)$ and $(m^s\,\|f_m\|)$ are $l_2$-sequences, the sum is absolutely convergent for all $r \leq 1$. The remaining part of the proof follows by standard reflections on the Gel'fand Triple

$$\begin{array}{c} HA_2(B\,;\,s) \hookrightarrow HA_2B\,;\,0) \hookrightarrow HA_2(B\,;\,-s)\ . \\ \cong \\ L_2(S^{q-1}) \end{array}$$

□

**Remark.** The Schwartz space of test functions $C^\infty(S^{q-1})$ on the unit sphere $S^{q-1}$ consists precisely of the restrictions to $S^{q-1}$ of the harmonic functions in the projective limit

$$HA_2(B\,;\,\infty) = \bigcap_{s\in\mathbb{R}} HA_2(B\,;\,s)\ .$$

The Schwartz space $(C^\infty(S^{q-1}))'$ of distributions on the unit sphere consists precisely of the "boundary values" on $S^{q-1}$ of the harmonic functions in the inductive limit

$$HA_2(B\,;\,-\infty) = \bigcup_{s\in\mathbb{R}} HA_2(B\,;\,s)\ .$$

All classical properties of Schwartz' distributions on the unit sphere can easily be derived from the above representation.

**Remark.** The harmonic function $\underline{x} \mapsto \dfrac{1-|\underline{x}|^2}{\omega_q}\,\dfrac{1}{|\underline{x}-\underline{a}|^q}$, $|\underline{a}| = 1$, belongs to $HA_2(B\,;\,-\beta)$ for each $\beta > \frac{1}{2}\,q$ because

$$\Big(\frac{1-|\underline{x}|}{|\underline{x}-\underline{a}|^q}\Big)^2 (a-|x|)^{2\beta-1} \leq \frac{1}{|\underline{x}-\underline{a}|^{2q-\beta-1}}\ .$$

The latter function belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^q)$ if $2q-2\beta-1 < q-1$.
This means that the point evaluation $f \mapsto f(\underline{a})$, $|\underline{a}| = 1$, is a continuous linear functional on $HA_2(B\,;\,\beta)$ for each $\beta > \frac{1}{2}\,q$. In other words

$$|f(\underline{a})| \leq c\,\|f\|_{HA_2(B;\beta)}\ .$$

Specializing to $f = f_m \in HP_m$ yields

$$|f_m(\underline{a})| \leq c_\beta\, M^\beta\, \|f_m\|_{L_2(S^{q-1})} \qquad \text{for any } \beta > \tfrac{1}{2}\,q\ .$$

**Question.** Why don't I get the best estimate for point-evaluations of harmonic polynomials in this way?

# 4. Operators in $L_2(S^{q-1})$ which are also operators on $\boldsymbol{HA}(B_R)$

The Laplace operator $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \ldots + \dfrac{\partial^2}{\partial x_q^2}$ satisfies

$$|\underline{x}|^2\, \Delta u = N^2 u + (q-2)\, N u - \Delta_{LB} u$$

with

$$(Nu)(\underline{x}) = (\underline{x}, \nabla u(\underline{x}))$$

$$\Delta_{LB} = - \sum_{1 \leq j < k \leq q} \left( x\, ;\, \frac{\partial}{\partial x^k} - x_k \frac{\partial}{\partial x^j} \right)^2 .$$

(The Laplace Beltrami operator $\Delta_{LB}$ is the same as the moment of momenum operator $\vec{L}^2$ in quantum mechanics.)
If we take $g \in HP_k(\mathbb{R}^q)$ then

$$Ng = kg \quad \text{and} \quad \Delta_{LB}\, g = k(k+q-2)\, g\ .$$

In $L_2(S^{q-1})$ the operators $N$ and $\Delta_{LB}$ are essentially self adjoint on the (dense) subspace of spherical harmonics.

**Theorem.**

- $HA(\bar{B}) = S_{L_2(S^{q-1}),N} = S_{L_2(S^{q-1}),\Delta_{LB}^{1/2}}$
  (with $S_{X,A} = \bigcup_{t>0} e^{-tA}(X)$).
- $HA(\mathbb{R}^q) = \tau(L_2(S^{q-1}), N) = \tau(L_2(S^{q-1}), \Delta_{LB}^{1/2})$
  (with $\tau(X, A) = \bigcap_{t>0} e^{-tA}(X)$).

**Remark.** For the general theory of those topological vector spaces, the operator algebras on them and applications, see [EG1], [EG2].

Note that

$$\forall t > 0 \lim_{k \to \infty} \frac{e^{-tk}}{e^{-t\sqrt{k(k+q-2)}}} = e^{-\frac{1}{2}t(q-2)} > 0\ .$$

**Problems.** Show that

$$e^{-tN}\,(L_2(S^{q-1})) \hookrightarrow HA(B_R) \qquad \text{with } R = e^t\ .$$

Find a suitable weight function $\mu$ such that

$$e^{-tN}\,(L_2(S^{q-1})) = HA(B_R\,;\,\mu)\ , \qquad \text{with } R = e^t$$

as sets and as topological vector spaces.

We now turn to the resolvent of the operator $N$.

**Theorem.**

(i) For all $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, and all $R > 0$ the operator $N + \alpha I$ is a bijection on $C^\infty(B_R)$ and also on $HA(B_R)$.

(ii) For all $R > 0$ and all $\alpha \in \mathbb{C}$, $\alpha \notin \{0, -1, -2, ...\}$, $N + \alpha$ is a bijection on $HA(B_R)$.

**Proof.**

(i) Straightforward calculation shows that the operator $M_\alpha$ defined by $(M_\alpha u)(\underline{x}) = \int_0^1 s^{\alpha-1}\, u(s\underline{x})\, ds$ is both a left and a right inverse of $(N + \alpha)$.
Moreover, $M_\alpha$ and $N + \alpha$ save harmonicity.

(ii) Expand in spherical harmonics and apply the characterization theorem on page 6. □

**Remark.** The null space of $(N + \alpha)$ contains singular functions of the form

$$|\underline{x}|^{-\alpha}\, \psi\Big(\frac{\underline{x}}{|\underline{x}|}\Big)\ , \qquad \psi \text{ arbitrary}\ .$$

**Theorem** (harmonic multiplication). Let $f \in HA(B_R)$, $g \in HA(B_R)$ with $R > 1$. Define for $\underline{\xi} \in S^{q-1}$ $(f \cdot g)(\underline{\xi}) = f(\underline{\xi}) \cdot g(\underline{\xi})$. Then $fg$ can be extended to $(fg)_e \in HA(B_R)$. In this way $HA(B_R)$ becomes a commutative algebra (with no zero divisors).

**Proof.** Write $f = \sum_{n=0}^{\infty} f_n$, $g = \sum_{m=0}^{\infty} g_m$, $f_n, g_n \in HP_n(\mathbb{R}^q)$. In case of absolute convergence we can write

$$f(\underline{\xi})\, g(\underline{\xi}) = \sum_{l=0}^{\infty} \sum_{m+n=l} f_n(\underline{\xi})\, g_m(\underline{\xi})\ . \tag{$*$}$$

Let $1 < R_1 < R$. Uniform convergence of $(*)$ on $S^{q-1}$ follows from the estimate

$$|f_0(\underline{\xi})\, g_l(\underline{\xi})| + |f_1(\underline{\xi})\, g_{l-1}(\underline{\xi})| + ... + |f_l(\underline{\xi})\, g_0(\underline{\xi})| \le$$

$$\le C_q\, l^q\, R_1^{-l} \Big\{ \sum_{k=0}^{l} R_1^{2k}\, \|f_k\|^2 + \sum_{k=0}^{l} R_1^{2k}\, \|g_k\|^2 \Big\} \le C_{fg}\, l^q\, R_1^{-l}\ .$$

Here $C_{fg}$ is a constant which only depends on $f$ and $g$. From the last inequality it also follows that

$$\Big\| \sum_{m+n=l} f_n(\underline{\xi})\, g_m(\underline{\xi}) \Big\| \leq \omega_q^{\frac{1}{2}}\, C_{fg}\, l^q\, R_1^{-l} \ .$$

Next we estimate the norm of the $L_2(S^{q-1})$-projection $(f \cdot g)_k$ of $f \cdot g$ on $HP_k$.
In $L_2(S^{q-1})$ we write

$$(f \cdot g)_k = \sum_{l=k}^{\infty} \sum_{m+n=l} f_n\, g_m \ .$$

(Note: there is no contribution for $l < k$ !)

$$\|(f \cdot g)_k\| \leq \sum_{l=k}^{\infty} \Big\| \sum_{m+n=l} f_n\, g_m \Big\| \leq$$

$$\leq \omega_q^{\frac{1}{2}}\, C_{fg} \sum_{l=k}^{\infty} R_1^{-l}\, l^q \leq R_1^{-k}\, \omega_q^{\frac{1}{2}}\, C_{fg} \sum_{l=k}^{\infty} R_1^{-(l-k)}\, l^q \leq C_1\, R_1^{-k} \ .$$

Finally, for all $R_2$, $1 < R_2 < R_1 < R$,

$$\sum_{k=0}^{\infty} R_2^{2k}\, \|(f \cdot g)_k\|^2 < \infty \ . \qquad \square$$

**Corollary.** The space $HA(\bar{B})$ becomes a commutative algebra by harmonic multiplication. The elements in $HA(\bar{B})$ are *multipliers* for the hyperfunctions on $S^{q-1}$. This means that the operator $M_\varphi$, $\varphi \in HA(\bar{B})$, defined by $M_\varphi\, f = (\varphi f)_\epsilon$, extends to hyperfunctions by $\langle M_\varphi\, F, f \rangle = \langle F, \bar{\varphi} f \rangle$.

**Further remarks.** It is an interesting problem to find an explicit expression for the harmonic multiplication. One could proceed in the following way:
Let $f, g \in HA(B_R)$, $R > 0$ (!). Try to find $h \in C^\infty(B_R)$ such that

$$fg + (1 - |\underline{x}|^2)\, h \in HA(B_R) \ .$$

Calculation of the Laplacian leads to the condition

$$-\tfrac{1}{2}(1 - |\underline{x}|^2)\, \Delta h + (2N + q)\, h =$$
$$= -\tfrac{1}{2}\Delta\, \{(1 - |x|^2)\, h\} + Nh = (\nabla f)^T\, \nabla g \ .$$

Perhaps there is a $C^\infty$ solution to this equation on $B_R$ !?

- If $(\nabla f)^T \nabla g$ happens to be harmonic, then $h = (2N+q)^{-1} (\nabla f)^T \nabla g$ solves the problem.
- This is true if $f(\underline{x}) = x_k$, then $(\nabla f)^T \nabla g = \dfrac{\partial f}{\partial x_k}$. So $x_k g + (1 - |\underline{x}|^2)(2N+q)^{-1} \dfrac{\partial g}{\partial x^k} \in HA(B_R)$.
- In particular if $g \in HP_m(\mathbb{R}^q)$, this leads to

$$x_k g = \underbrace{x_k g - \frac{1}{2k-2+q} |\underline{x}|^2 \frac{\partial g}{\partial x_k}}_{\in\ HP_{m+1}(\mathbb{R}^q)} + \underbrace{\frac{1}{2k-2+q} \frac{\partial g}{\partial x_k}}_{\in\ HP_{m-1}(\mathbb{R}^q)} .$$

- If we take $R = 1$ the above procedure leads to a product for special pairs of hyperfunctions on $S^{q-1}$, viz. $F$ and $G$ with $(\nabla F)^T (\nabla G) \in HA(B)$.
- Looking at the expression for $x_k g$ it seems reasonable that there exists a relation between the harmonic product and a product of type $f \bullet g = f(\nabla) g$.

The operators $T_{\underline{a}}$, $\underline{a} \in \mathbb{R}^q$; $L_A$, $A \in O(\mathbb{R}, q)$; $Z_\lambda$, $\lambda \in \mathbb{R}$; $\mathcal{P}_i$; $\mathcal{L}_B$, $B \in \mathbb{R}^{q\times q}$, $B^T = -B$ and $N$ of Chapter 1 can all be considered as densely defined linear operators in $L_2(S^{q-1})$. Their domains are restrictions of functions which are harmonic on a sufficiently large neighbourhood of the closed unit ball. Of course, lots of questions on the closed extendibility in $L_2(S^{q-1})$ arise!
We now present a wider class of operators for which the relation to harmonic functions is not so obvious.

**Theorem.** Let $f \in HA(B_R)$, $R > 1$. let $A \in \mathbb{R}^{q\times q}$. Suppose $||A|| = R_1 < R$. Define $g \in L_2(S^{q-1})$ by $g(\underline{\xi}) = f(A\underline{\xi})$, $|\underline{\xi}| = 1$. Then: $g$ can be extended to $g_e \in HA(B_{R_2})$ with $R_2 = \dfrac{R}{R_1}$.

**Proof.** Again, write $f = \sum_{m=0}^{\infty} f_m$, $g(\underline{\xi}) = \sum_{m=0}^{\infty} f_m(A\underline{\xi})$, $f_m(A\underline{x})$ is a homogeneous polynomial of degree $m$. We estimate

$$|f_m(A\underline{\xi})| \leq C_q\, m^q\, ||A||^m\, ||f_m|| .$$

Hence, in $L_2(S^{q-1})$,

$$||f_m(A\cdot)|| \leq D_q\, m^q\, ||A||^m\, ||f_m|| .$$

Next we estimate the norm of the $L_2(S^{q-1})$-projection $g_k$ of $g$ on $HP_k$. We have

$$g_k(\underline{\xi}) = \sum_{m=k}^{\infty} f_m(A\underline{\xi}) .$$

So,

$$\|g_k\| \le D_q \sum_{m=k}^{\infty} m^q \, R_1^m \, \|f_m\| \ .$$

Let $R_1 < R_2 < R$. Let $1 \le L < \dfrac{R_2}{R_1}$, then

$$L^{2k} \, \|g_k\|^2 \le D_q^2 \Big( \sum_{m=k}^{\infty} L^k \, m^q \, R_1^m \, \|f_m\| \Big)^2 \le$$

$$\le D_q^2 \Big( \sum_{m=k}^{\infty} \Big(\frac{L\,R_1}{R_2}\Big)^m m^q \, R_2^m \, \|f_m\| \Big)^2 \le$$

$$\le \tfrac{1}{2} \, D_q^2 \Big( \sum_{m=0}^{\infty} \Big(\frac{L\,R_1}{R_2}\Big)^{2m} m^{2q} + \sum_{m=0}^{\infty} R_2^{2m} \, \|f_m\|^2 \Big) < \infty \ .$$

Since this is true for all $L < \dfrac{R_2}{R_1} < \dfrac{R}{R_1}$ we conclude

$$\sum_{k=0}^{\infty} L^{2k} \, \|g_k\|^2 < \infty \qquad \text{for all } L < \frac{R}{R_1} \ . \qquad \Box$$

**Corollary.**

- In $HA(\bar{B})$ the operators $L_A$, $A \in \mathbb{R}^{q \times q}$, $\|A\| \le 1$, defined by $(L_A\, f)(\underline{\xi}) = f(A\underline{\xi})$, $|\underline{\xi}| = 1$, and then followed harmonic extension, make sense.
- In $HA(\mathbb{R}^q)$, all operators $L_A$, $A \in \mathbb{R}^{q \times q}$, defined by $(L_A\, f)(\underline{\xi}) = f(A\underline{\xi})$, $|\underline{\xi}| = 1$, and then followed by harmonic extension, make sense.

**Some remarks.**

- In case $A \in O(\mathbb{R}, q)$, $L_A$ coincides with the earlier definition.
- Related operators corresponding to matrices $A$ have been introduced by Van Eijndhoven and Martens. The latter has also investigated their mutual relations and their extendibility properties.

The following is quite remarkable.

**Theorem** [Ma]. Let $A \in \mathbb{R}^{q \times q}$ with $|\det A| = 1$. Define $\varphi_A : L_2(S^{q-1}) \to L_2(S^{q-1})$ by

$$(\varphi_A\, f)(\underline{\xi}) = |A^T \underline{\xi}|^{-\frac{q}{2}} \, f\Big( \frac{A^t\, \underline{\xi}}{|A^T \underline{\xi}|} \Big) \ .$$

(i) All $\varphi_A$ are unitary operators.

(ii) $A \mapsto \varphi_A$ is a faithful unitary representation of $SL(\mathbb{R}, q)$.

(iii) The representation $(A \mapsto \varphi_A)$ is irreducible on the odd subspace and on the even subspace in $L_2(S^{q-1})$.

(iv) The Lie algebra $sl(\mathbb{R}, q) = \{B \mid B \in \mathbb{R}^{q \times q},\ \operatorname{tr} B = 0\}$ is represented by the skew adjoint operators $B$:

$$(Bf)(\underline{\xi}) = (B\underline{\xi})^T \nabla f - (B\underline{\xi})^T \underline{\xi}\, \underline{\xi}^T \nabla f - q(B\underline{\xi})^T \underline{\xi} f .$$

(Note that the first two terms constitute a tangent vector field on $S^{q-1}$.

**Remarks.**

- If $A \in O(\mathbb{R}, q)$, which is a compact subgroup of $SL(\mathbb{R}, q)$, then $\varphi_A = L_A$ (again).
- The mapping $\underline{\xi} \mapsto |A\underline{\xi}|^{-1} A\underline{\xi}$ carries subspheres of radius 1 into subspheres of radius 1. So this mapping corresponds to projective mappings in the $(q-1)$-dimensional projective space $P\mathbb{R}^{q-1}$.
- The representation on the even subspaces is an irreducible representation of the projective group.

**Questions.**

- The analytic vectors of the representation $A \mapsto \varphi_A$ with $A \in O(\mathbb{R}, q)$ are precisely the harmonic functions in $HA(\bar{B})$. Is it easy to see that $HA(\bar{B})$ contains precisely the analytic vectors of the whole of $SL(\mathbb{R}, q)$?
- Can the Gevrey spaces for the generators of $O(\mathbb{R}, q)$ and $SL(\mathbb{R}, q)$ be characterized by means of harmonic functions?
- What is the relation between the "odd representation" and the projective group?
- The whole linear group $GL(\mathbb{R}, q)$ can be represented unitarily on $L_2(S^q)$ by means of

  $$L_{\tilde{A}} \qquad \text{with } \operatorname{diag}[A, (\det A)^{-1}] .$$

  What about the irreducibility properties for these operators?

**Theorem.** Consider the diffusion equation

$$\frac{\partial u}{\partial t} = -\Delta_{LB}\, u \text{ on } S^{q-1} \qquad (\Delta_{LB} > 0)$$

with initial condition

$$u(\cdot,0) = f \in L_2(S^{q-1}) \ .$$

The solution $u(\cdot,t)$ at time $t$ can be extended to an entire harmonic function in $HA_2(\mathbb{R}^q,\mu_t)$ with

$$\mu_t(r) = \frac{1}{r^2}\, e^{-\frac{(\log r)^2}{4t}} \ .$$

In fast there is a continuous bijection between the Hilbert spaces $w^{-t\Delta_{LB}}\,(L_2(S^{q-1}))$, with graph norm, and $HA_2(\mathbb{R}^q,\mu_t)$.

**Sketch of the proof.** The theorem on page 10 is applied. We have to guess a measure $\mu_t$ such that

$$\lim_{m\to\infty} e^{-2tm(m+q-2)} \int_0^\infty r^{2m+q-1}\,\mu_t(r)\,dr = c > 0 \ .$$

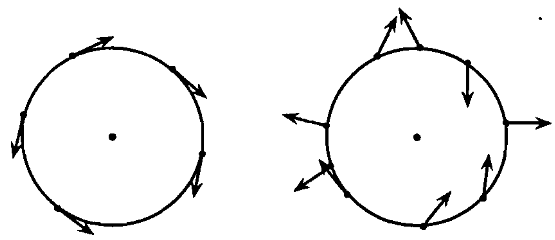
The trick is to calculate the asymptotic behaviour of the integral with

$$\mu(r) = (1+|\log r|)^\gamma\, r^\beta\, e^{-\frac{1}{\Omega}|\log r|^{\frac{1}{\alpha}}}$$

and then to "adjust" the parameters $\gamma$, $\beta$, $\alpha$, $\Omega$. The same trick works well for $\Delta_{LB}$ replaced by $(\Delta_{LB})^\nu$ with $\frac{1}{2} < \nu \le 1$. For greater $\nu$ creative suggestions are needed.

**Corollary.** $e^{t\Delta_{LB}}$ acts as a one-parameter group on the space of harmonic functions $\bigcap\limits_{t>0} HA_2(\mathbb{R}^q,\mu_t)$.

## 5. Lie-algebras of vector fields on $S^{q-1}$ and their skew-adjoint representations



In this section the emphasis is on algebra. Therefore, sometimes, we will be sloppy about the distinction between the concepts 'skew-symmetric' and 'skew-adjoint'.

In $L_2(S^{q-1})$ on the domain $HA(\bar{B})$ we introduce the operators $\mathcal{P}_j$, $\mathcal{N}$, $Q_j$, $R_j$, $1 \le j \le q$, by

$$\mathcal{P}_j\, u = \frac{\partial u}{\partial x_j}\ ,$$

$$\mathcal{N}\, u = \sum_{j=1}^{q} x_j \frac{\partial u}{\partial x_j}\ ,$$

$$Q_j\, u = x_j\, u \text{ on } S^{q-1} \text{ (with harmonic extension if necessary)}\ ,$$

$$R_j = \mathcal{P}_j - Q_j \mathcal{N}\ .$$

Note that the differential operators $R_j$ correspond to the tangent vector fields

$$\underline{r}_j(\underline{x}) = \underline{e}_j - x_j\, \underline{x} = \underline{e}_j - x_j\, \underline{n}\ , \qquad \text{on } S^{q-1}\ .$$

As a set, our Lie algebra of tangent vector fields will be (a submodule of) a module over $HA(\bar{B})$ generated by the $\underline{r}_j(\underline{x})$, $1 \le j \le q$. First, we calculate the adjoint $\mathcal{P}_j^*$ of $\mathcal{P}_j$ in $L_2(S^{q-1})$.

**Lemma** (Martens). Let $f$ and $g$ be continuously differentiable on a neighbourhood of the unit ball $\bar{B}$. Then, we have the identity

$$\int_{S^{q-1}} \frac{\partial f}{\partial x_j}\, \bar{g}\, d\sigma = -\int_{S^{q-1}} f \frac{\partial \bar{g}}{\partial x_j}\, d\sigma + \int_{S^{q-1}} x_j \left[\frac{\partial f}{\partial n}\, \bar{g} + f\, \frac{\partial \bar{g}}{\partial n}\right] d\sigma + $$

$$+ (q-1) \int_{S^{q-1}} x_j\, f \bar{g}\, d\sigma\ .$$

**Proof.** Define $M$ by

$$M = \int\limits_{S^{q-1}} \left[\frac{\partial f}{\partial x_j}\right](\underline{\xi})\, \overline{g(\underline{\xi})}\, d\sigma + \int\limits_{S^{q-1}} f(\underline{\xi})\, \overline{\left[\frac{\partial g}{\partial x_j}\right](\xi)}\, d\sigma \ .$$

Straightforward calculation yields

$$\begin{aligned} M &= \int\limits_{S^{q-1}} \left[\frac{\partial}{\partial x_j}(f\bar{g})\right] d\sigma = \sum_{k=1}^{q} \int\limits_{S^{q-1}} \frac{\partial f\bar{g}}{\partial x_j}\, \xi_k^2\, d\sigma = \qquad \text{(Gauss)} \\ &= \sum_{k=1}^{q} \int\limits_{B} \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j}(f\bar{g})\right] x_k\, d\underline{x} + q \int\limits_{B} \frac{\partial}{\partial x_j}(f\bar{g})\, d\underline{x} = \\ &= \sum_{k=1}^{q} \int\limits_{B} \frac{\partial}{\partial x_j}\left[\frac{\partial f\bar{g}}{\partial x_k} x_k\right] d\underline{x} + (q-1) \int\limits_{B} \frac{\partial f\bar{g}}{\partial x_j}\, d\underline{x} = \\ &= \int\limits_{S^{q-1}} x_j \left[\frac{\partial f}{\partial n}\bar{g} + f\frac{\partial \bar{g}}{\partial n}\right] d\sigma + (q-1) \int\limits_{S^{q-1}} x_j\, f\bar{g}\, d\sigma \ . \end{aligned}$$

□

**Corollary.**

$$\left(\left(\mathcal{P}_j - Q_j\mathcal{N} - \frac{q-1}{2} Q_j\right) f, g\right) = -\left(f, \left(\mathcal{P}_j - Q_j\mathcal{N} - \frac{q-1}{2} Q_j\right) g\right) .$$

So, the operators

$$R_j + \alpha\, Q_j \ , \qquad \alpha = -\frac{q-1}{2} \ , \qquad 1 \le j \le q \ ,$$

are skew symmetric.

**Theorem.** We have the following algebraic relatioins between the mentioned operators on $HA(\bar{B})$:

- $\mathcal{P}_k\, Q_l = \delta_{kl}\, J + Q_l\, \mathcal{P}_k - 2Q_k(2\mathcal{N} + q)^{-1}\, \mathcal{P}_l + (1 - |x|)^2 \dots$ .
- $\mathcal{P}_k\, \mathcal{N} = \mathcal{P}_k + \mathcal{N}\, \mathcal{P}_k$ .
- $\mathcal{N}\, Q_k = Q_k + Q_k\, \mathcal{N} - 2(2\mathcal{N} + q)^{-1}\, \mathcal{P}_k + (1 - |x|^2) \dots$ .
- $R_k\, R_l = R_l\, R_k = Q_k\, \mathcal{P}_l - Q_l\, \mathcal{P}_k = Q_k\, R_l - Q_l\, R_k$ .

**Proof.** The proof is tedious, but straightforward, calculation using the explicit expression for harmonic multiplication by $x_k$

$$Q_k f = x_k f + (1 - |\underline{x}|^2)(2\mathcal{N} + q)^{-1} \delta_k f \ . \qquad \square$$

If we put a non-constant coefficient $a$ in front of $R_k$, the operator $a\, R_k$ will no longer be skew symmetric. However,

**Theorem.** Let $a \in HA(\bar{B})$. The operator $a\, R_k + \alpha a\, Q_k + \frac{1}{2}(R_k\, a)$ with $1 \le k \le q$, $\alpha = -\dfrac{q-1}{2}$ and $(R_k\, a) = \dfrac{\partial a}{\partial x_k} + x_k \dfrac{\partial a}{\partial n}$ is skew-symmetric. The domain $HA(\bar{B})$ is invariant. Further, for the commutator of two such operators, we have

$$\begin{aligned}
&[a\, R_k + \alpha a\, Q_k + \tfrac{1}{2}(R_k\, a),\ b\, R_l + \alpha b\, Q_l - \tfrac{1}{2}(R_l\, b)] = \\
&= -\Big(c\, R_k + \alpha c\, Q_k + \tfrac{1}{2}(R_k\, c)\Big) + \Big(d\, R_l + \alpha d\, Q_l - \tfrac{1}{2}(R_l\, d)\Big)
\end{aligned}$$

with $c = b(R_l\, a) + x_l\, a$ and $d = a(R_k\, c - x_k\, b)$. $\square$

**Proof.** For the first part of the theorem we apply the general trick: Let $\mathcal{A}$, $B$ and $\mathcal{S}$ be operators in a Hilbert space $\mathcal{H}$ with common invariant domain $W$, and suppose $\mathcal{A}^* = \mathcal{A}$, $B^* = -B$, $\mathcal{S}^* = \mathcal{S}$, $\mathcal{A}B = \mathcal{S} + B\mathcal{A}$, then

$$(\mathcal{A}B - \tfrac{1}{2}\mathcal{S})^* + (\mathcal{A}B - \tfrac{1}{2}\mathcal{S}) = 0 \ .$$

In our case, take $\mathcal{A}$ = multiplication by $a$ in $L_2(S^{q-1})$, $B = R_k + \alpha\, Q_k$. Let $u \in HA(\bar{B}) \subset L_2(S^{q-1})$. Then

$$\begin{aligned}
&a(\mathcal{P}_k - Q_k \mathcal{N} + \alpha\, Q_k)\, u - (\mathcal{P}_k - Q_k \mathcal{N} + \alpha\, Q_k)\, au = \\
&= -(\mathcal{P}_k\, a - Q_k \mathcal{N} a) = -(R_k\, a) \ .
\end{aligned}$$

By taking $\mathcal{S}$ = multiplication by $-(R_k\, a)$, the first result follows.
Since $a \in HA(\bar{B})$ and $HA(\bar{B})$ is an invariant domain for all occurring operators, it is also an invariant domain for the operator as a whole. Finally, calculating the commutator is a matter of bookkeeping. $\square$

The general type of operator that we want to consider is the following:
Let $\underline{a} = a^k\, \underline{e}_k$, $a^k \in HA(\bar{B})$, be a vector field (not necessarily tangent) on $S^{q-1}$. With $\underline{a}$ we associate the operator ($\alpha = -\dfrac{q-1}{2}$)

$$G(\underline{a}) = a^k\, R_k + \alpha\, a^k\, Q_k + \tfrac{1}{2}(R_k\, a^k) = \underline{a}^T\, \underline{R} + \alpha\, \underline{a}^T\, \underline{x} + \tfrac{1}{2}(\underline{R}^T\, \underline{a}) \ .$$

The commutator of $G(\underline{a})$ and $G(\underline{b})$ is $G(\underline{c})$ with $\underline{c} = (R\,\underline{b}^T)^T\, \underline{a} - (R\,\underline{a}^T)^T\, \underline{b} + (\underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T)\, \underline{x}$.
Note that $\underline{a}^T\, \underline{R}$ corresponds to the vector field

$$a^k(e_k - x_k\, \underline{x}) = \underline{a} - (\underline{a}^T\, \underline{n})\, \underline{n}$$

which is tangent to $S^{q-1}$. So, $G(\underline{a})$ is the same operator as $G(\underline{a} - (\underline{a}, \underline{n})\, \underline{n})$.
Let us look at some special cases:

I. $\underline{a} = K\,\underline{x}$, $\underline{b} = L\,\underline{x}$, $K, L \in \mathbb{R}^{q\times q}$, $K^T = -K$, $L^T = -L$.
Substitution leads to $G(\underline{a}) = \underline{x}^T\, K\, \underline{R} = \underline{x}^T\, K^T\, \underline{P}$ which is a "moment of momentum" operator. (We used $R\,\underline{x}^T = \underline{T} - \underline{x}\,\underline{x}^T$ and $\underline{x}^T\, K\, \underline{x} = 0$.)
For the commutator of $G(\underline{a})$ and $G(\underline{b})$ we find $G(\underline{c})$ with $\underline{c} = (KL - LK)\,\underline{x}$.
Operators of this type establish a Lie-algebra which corresponds to $so(\mathbb{R}, q)$.

II. $\underline{a} = \underline{\text{constant}}$, $\underline{b} = \underline{\text{constant}}$.

$$[G(\underline{a}), G(\underline{b})] = G(\underline{c}) \qquad \text{with } \underline{c} = (\underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T)\,\underline{x}\ .$$

Note that $(\underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T \in so(\mathbb{R}, q)$.
Operators of the form $G(\underline{a} + K\,\underline{x})$, $\underline{a} \in \mathbb{R}^q$, $K \in so(\mathbb{R}, q)$ establish a Lie algebra.
A $(q+1) \times (q+1)$-matrix representation is $\begin{bmatrix} K & \underline{a} \\ \underline{a}^T & 0 \end{bmatrix}$.
This is an extension of $so(\mathbb{R}, q)$. Cf. also III.

III.

$$\underline{a} = K\,\underline{x} \qquad \underline{b} = L\,\underline{x} \qquad K, L \in \mathbb{R}^{q\times q}$$

$$G(\underline{a}) = \underline{x}^T\, K^T(\nabla - \underline{x}\,\underline{x}^T\,\nabla) + \alpha\,\underline{x}^T\, K^T\,\underline{x} + \tfrac{1}{2}\,(\nabla - \underline{x}\,\underline{x}^T\,\nabla)^T\, K\,\underline{x} =$$

$$= \underline{x}^T\, K^T(I - \underline{x}\,\underline{x}^T)\,\nabla + \frac{q-1}{2}\,\underline{x}^T\, K\,\underline{x} - \tfrac{1}{2}\,\underline{x}^T\, K\,\underline{x}$$

$$G(\underline{a}) = \underline{x}^T\, K^T\,\nabla u - (\underline{x}^T\, K\,\underline{x})\,\underline{x}^T\,\nabla u - \frac{q}{2}\,(\underline{x}^T\, K\,\underline{x}) + \tfrac{1}{2}\,\mathrm{tr}\, K\ .$$

So, we have another extension of $so(\mathbb{R}, q)$ which corresponds both to $sl(\mathbb{R}, q)$ and to Martens' representation.
The extensions of II and III cannot be combined into one single small Lie-algebra in a *simple* way. Note that the matrices $\left\{ \begin{pmatrix} A & \underline{a} \\ \underline{a}^T & 0 \end{pmatrix} \right\}$ establish a Lie algebra only if $A^T = -A$.
Note also that, taking $\underline{a}$ = constant and $\underline{b} = L\,\underline{x}$ we find a commutator with

$$\underline{c} = \left((\nabla - \underline{x}\,\underline{x}^T\,\nabla)\,\underline{x}^T\, L^T\right)^T\, \underline{a} + L\,\underline{x}\,\underline{a}^T\,\underline{x} - \underline{a}\,\underline{x}^T\, L^T\,\underline{x} =$$

$$= L\underline{a} - \underbrace{\underline{x}^T\,\underline{a}\, L\,\underline{x} + \underline{x}^T\,\underline{a}\, L\,\underline{x}}_{=\ 0} - \underline{a}\,\underline{x}^T\, L^T\,\underline{x}\ .$$

Only if $L^T = -L$ this reduces to $L\,\underline{a}$.

IV. $\underline{a} = f(\underline{x})\,\underline{x}$. Then $G(\underline{a}) = 0$.

For more details, see [G].

# Appendix A

**Some spectral properties of** $(\mathcal{P}_j - Q_j\mathcal{N} - \frac{1}{2}(q-1)\,Q_j)$
Take $j = 1$. After a change of variables

$$x_1 = \cos\vartheta \qquad x_\alpha = \sin\vartheta\, y_\alpha\ , \qquad 2 \leq \alpha \leq q\,,\ \ 0 \leq \vartheta \leq \pi\ ,$$

the operator becomes

$$-\sin\vartheta\,\frac{\partial}{\partial\vartheta} - \tfrac{1}{2}(q-1)\cos\vartheta$$

(skew hermitean in $L_2([0,\pi],(\sin\vartheta)^{q-2}\,d\vartheta)$).
The improper eigenfunctions are

$$u = \Big(\frac{1+\cos\vartheta}{\sin\vartheta}\Big)^{i\lambda}(\sin\vartheta)^{-\frac{1}{2}(q-1)}\cdot\psi(\underline{y})$$

with eigenvalue $i\lambda$, $\lambda \in \mathbb{R}$. The function $\psi$ can be taken arbitrarily.
Note that

$$L_2(S^{q-1}) = L_2(S^{q-2}) \otimes L_2([0,\pi],(\sin\vartheta)^{q-2}\,d\vartheta)\ .$$

With the transformation

$$\xi = \ln\tan\frac{\vartheta}{2}\ , \qquad \vartheta = 2\arctan e^{\xi}$$

the differential operation becomes

$$-\frac{d}{d\xi} + \tfrac{1}{2}(q-1)\tanh\xi$$

with eigenfunctions $e^{-i\lambda\xi}(\cosh\xi)^{\frac{1}{2}(q-1)}$.
Using the natural unitary bijection

$$L_2(\mathbb{R},(\cosh\xi)^{-(q-1)}\,d\xi) \to L_2(\mathbb{R},d\xi)$$

we conclude that our operator, acting in the subspace of rotation symmetric functions in $L_2(S^{q-1})$ is unitarily equivalent to

$$\frac{d}{dx}\ \text{ in }\ L_2(\mathbb{R},dx)\ .$$

**Solution of the evolution equation**

$$\frac{\partial u}{\partial t} = -\sin\vartheta\,\frac{\partial u}{\partial\vartheta} - \tfrac{1}{2}(q-1)\cos\vartheta\, u$$

$$u(0\,;\,\vartheta,\underline{y}) = \psi(\vartheta,\underline{y})\ , \qquad \underline{y} = y_2,\dots,y_q$$

$$u(t\,;\,\vartheta,y) = \psi\Big(2\arctan\Big(e^{-t}\tan\frac{\vartheta}{2}\Big),\underline{y}\Big)\cdot(\cosh t + \cos\vartheta\sinh t)^{-\frac{1}{2}(q-1)}\ .$$

Note that the, unitarily equivalent, solution of

$$\frac{\partial w}{\partial t} = -\frac{\partial w}{\partial x} + \tfrac{1}{2}(q-1)\cdot\tanh x\cdot w\ , \qquad w(t,0) = \varphi(x)$$

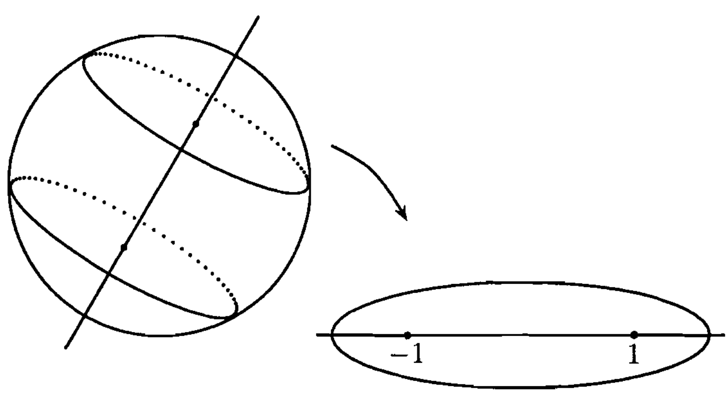
is easily found to be

$$w(x,t) = \varphi(x-t)\Big(\frac{\cosh x}{\cosh(x-t)}\Big)^{\frac{1}{2}(q-1)}\ .$$

If we start with $\psi(\vartheta,\underline{y}) = \psi(\vartheta)$, then $\psi\in HA(\mathbb{R}^q)$ iff $\psi$ is even, $2\pi$-periodic and entire analytic.

For $t>0$ $\quad u(t\,;\,\cdot)\notin HA(\mathbb{R}^q)$

$\qquad\qquad u(t\,;\,\cdot)\in HA(\bar{B})$ .




$$\frac{\partial u}{\partial t} = (1-z^2)\frac{\partial u}{\partial z} - \tfrac{1}{2}(q-1)\,z\,u$$

$$u(z,t) = \varphi\Big(\frac{z+\tanh t}{1+z\tanh t}\Big)(\cosh t + z\sinh t)^{-\frac{1}{2}(q-1)}\ .$$

# Appendix B
# (Harmonic) Polynomials in $q$ variables

**Some notations.**

- $\underline{x} = \mathrm{col}(x_1, ..., x_q) = \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \in \mathbb{R}^q$.
- $\underline{a} = \mathrm{col}(a_1, ..., a_1) \in \mathbb{C}^q$.
  $\Omega \subset \mathbb{R}^q$, $\Omega$ open.
- $f \in C^\infty(\Omega)$, $\mathrm{grad}\, f = \nabla f = \mathrm{col}\Big(\dfrac{\partial f}{\partial x^1}, ..., \dfrac{\partial f}{\partial x^q}\Big)$.
- $v_j \in C^\infty(\Omega)$, $1 \leq j \leq q$, $\mathrm{div}\, \underline{v} = \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + ... + \dfrac{\partial v_q}{\partial x_q}$.
- $\Delta f = \mathrm{div}\,\mathrm{grad}\, f = \dfrac{\partial^2 f}{\partial x_1^2} + \dfrac{\partial^2 f}{\partial x_2^2} + ... + \dfrac{\partial^2 f}{\partial x_q^2}$.
- $\mathcal{N} f = \underline{x}^T \nabla f = x_1 \dfrac{\partial f}{\partial x_1} + x_2 \dfrac{\partial f}{\partial x_2} + ... + x_q \dfrac{\partial f}{\partial x_q}$.

**Exercise.**

- Calculate $[\Delta, \mathcal{N}] = \Delta\mathcal{N} - \mathcal{N}\Delta$.
- Prove that $\Delta(f \cdot g) = f \cdot \Delta g + g \cdot \Delta f + 2(\nabla f)^T \nabla g$.

**Definition.** $f \in C^\infty(\Omega)$ is called *Harmonic*, $f \in \mathrm{Harm}(\Omega)$, if $\Delta f = 0$, i.e. if $f \in \mathrm{Null}(\Delta)$.

Case $q = 2$ Notation: $x_1 = x$, $x_2 = y$.

**Notations.**

- $\mathrm{Pol}_2$: Vector space of all polynomials in $x$ and $y$ with complex coefficients.
- $\mathrm{Pol}_2(m) = \left\{ \begin{array}{c} \text{polynomials of} \\ \text{degree at most } m \end{array} \right\} \subset \mathrm{Pol}_2$.
- $\mathrm{Hom\,Pol}_2(m) = \{a_m\, x^m + a_{m-1}\, x^{m-1}\, y + ... + a_1\, x\, y^{m-1} + a_0\, y^m \mid$ with $a_0, a_1, ..., a_m \in \mathbb{C}\}$.

  Note that

  - $\mathrm{Hom\,Pol}_2(m) \subset \mathrm{Pol}_2(m)$.

- $\dim \text{Hom Pol}_2(m) = m + 1$.
- $\text{Pol}_2(m) = \text{Hom Pol}_2(0) \oplus \text{Hom Pol}_2(1) \oplus \dots \oplus \text{Hom Pol}_2(m)$.
- $\dim \text{Pol}_2(m) = 1 + 2 + \dots + (m+1) = \frac{1}{2}(m+1)(m+2) = \binom{m+2}{2}$.

- $\text{Harm Hom Pol}_2(m) = \{f \mid f \in \text{Hom Pol}_2(m),\ \Delta f = 0\}$.

**Observation.**

$$\Delta\ :\ \text{Hom Pol}_2(m) \to \text{Hom Pol}_2(m-2)\ , \qquad m \geq 2$$

$$\mathcal{N}\ :\ \text{Hom Pol}_2(m) \to \text{Hom Pol}_2(m)$$

$$L_z\ :\ \text{Hom Pol}_2(m) \to \text{Hom Pol}_2(m)$$

$$\text{Here } L_z f = x\,\frac{\partial f}{\partial y} - y\,\frac{\partial f}{\partial x}\ .$$

**Theorem.** The set of polynomials

$$\{(x+iy)^m, (x+iy)^{m-1}(x-iy), \dots, (x+iy)(x-iy)^{m-1}, (x-iy)^m\}$$

is a basis for $\text{Hom Pol}_2(m)$.

**Proof.** Because there are $(m+1)$ of them we only have to show the independency. Put $x = r\cos\varphi$, $y = r\sin\varphi$ and assume constants $\alpha_m, \alpha_{m-2}, \dots, \alpha_{-m} \in \mathbb{C}$ such that

$$\alpha_m\, r^m\, e^{im\varphi} + \alpha_{m-2}\, r^m\, e^{i(m-2)\varphi} + \dots + \alpha_{-m}\, r^m\, e^{-im\varphi} = 0\ .$$

Multiply by $e^{-ik\varphi}$ and integrate over $\varphi$, $0 \leq \varphi \leq 2\pi$. Taking $k = m, m-2, \dots, -m$ successively we get that all $\alpha_j$ must be 0. □

**Exercise.**

- Show that $L_z(x+iy)^k(x-iy)^l = i(k-l)(x+iy)^k(x-iy)^l$.
- Show that $\Delta(x+iy)^k(x-iy)^l = 4kl(x+iy)^k(x-iy)^l$.

**Theorem.**

$$\Delta\,(\text{Hom Pol}_2(m)) = \text{Hom Pol}_2(m-2)\ .$$

**Proof.** Calculate

$$\Delta\,\frac{(x+iy)^{m-(l+1)}(x-iy)^{l+1}}{4\,(m-(l+1))(l+1)} = (x+iy)^{(m-2)-l}(x-iy)^l\ ,$$

$$m \geq 2\,,\quad 0 \leq l \leq m-2\ .$$

Together with the complex conjugate of this all the elements of a basis in Hom $\text{Pol}_2(m-2)$ can be reached. □

**Theorem.** Let $m \geq 1$. Then dim Harm Hom $\text{Pol}_2(m) = 2$. A basis in Harm Hom $\text{Pol}_2(m)$ is given by

$$\{(x+iy)^m, (x-iy)^m\} \ .$$

**Proof.** According to the 'Main Theorem of Linear Algebra'

$$\begin{array}{rcl} \dim \text{Null}(\Delta) + \ \dim \text{Range}(\Delta) & = & \dim \text{Hom}\,\text{Pol}_2(m) \ . \\ = & & = \\ (m-2)+1 & & m+1 \end{array}$$

So $\dim \text{Null}(\Delta) = 2$. Therefore

$$\text{Harm Hom Pol}_2(m) \text{ is 2-dimensional}.$$ □

**Note** that in Polar Coordinates $\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}$

$$(x \pm iy)^m = r^m e^{\pm im\varphi} \ .$$

A basis of real valued polynomials is given by $\{r^m \cos m\varphi, r^m \sin m\varphi\}$.

**Exercise.**

- Show that $(\underline{a}^T \underline{x})^m \in$ Harm Hom $\text{Pol}_2(m)$ iff $\underline{a}^T \underline{a} = 0$. $\underline{a} \in \mathbb{C}^2$.
- Show that for $\underline{a} \in \mathbb{C}^2$, $\underline{x} \in \mathbb{R}^q$ that

$$\Delta(\underline{a}^T \underline{x})^m = 0 \Leftrightarrow \underline{a}^T \underline{a} = 0 \ .$$

Case $q = 3$

**Notations.**

- $x_1 = x$, $x_2 = y$, $x_3 = z$.
- $\text{Pol}_3$: Vector space of all polynomials in $x$, $y$, $z$ with complex coefficients.
- $\text{Pol}_3(m) = \left\{ \begin{array}{c} \text{polynomials of degree} \\ \text{at most } m \end{array} \right\} \subset \text{Pol}_3$.

- Hom $\mathrm{Pol}_3(m) = \{p_0(x,y) + p_1(x,y)\,z + p_2(x,y)\,z^2 + \ldots +$
$+\, p_{m-1}(x,y)\,z^{m-1} + p_m(x,y)\,z^m \mid p_j \in$ Hom $\mathrm{Pol}_2(m-j)\}$.

  Note that

  - Hom $\mathrm{Pol}_3(m) \subset \mathrm{Pol}_3(m)$.
  - dim Hom $\mathrm{Pol}_3(m) = 1 + 2 + 3 + \ldots + (m+1) = \binom{m+2}{2}$.
  - $\mathrm{Pol}_3(m) =$ Hom $\mathrm{Pol}_3(0) \oplus$ Hom $\mathrm{Pol}_3(1) \oplus \ldots \oplus$ Hom $\mathrm{Pol}_3(m)$.
  - dim $\mathrm{Pol}_3(m) = \binom{2}{2} + \binom{3}{2} + \ldots + \binom{m+2}{2} = \binom{m+3}{3}$.

- Harm Hom $\mathrm{Pol}_3(m) = \{f \mid f \in$ Hom $\mathrm{Pol}_3(m),\ \Delta f = 0\}$.

**Theorem.**

- Let $p_0 \in$ Hom $\mathrm{Pol}_2(m)$, then the polynomial

$$p_0(x,y) - \frac{z^2}{2!}\,\Delta\, p_0(x,y) + \frac{z^4}{4!}\,\Delta^2\, p_0(x,y) - \ldots + \\ + (-1)^k \frac{z^{2k}}{(2k)!}\,\Delta^k\, p_0(x,y)$$

  with $m \leq 2k \leq m+1$ is harmonic.

- Let $p_1 \in$ Hom $\mathrm{Pol}_2(m-1)$, then the polynomial

$$z\,p_1(x,y) - \frac{z^3}{3!}\,\Delta\, p_1(x,y) + \frac{z^5}{5!}\,\Delta^2\, p_1(x,y) - \ldots + \\ + (-1)^k \frac{z^{2k+1}}{(2k+1)!}\,\Delta^k\, p_1(x,y)$$

  with $m-1 \leq 2k \leq m$ is harmonic.

**Remark.** We could write, respectively

$$\cos(z\,\Delta^{\frac{1}{2}})\,p_0(x,y)\,, \qquad \frac{\sin\, z\,\Delta^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}}\,p_1(x,y)$$

as 'formal' definitions.
Note that $\Delta^k\, p_0(x,y) = 0$ and $\Delta^k\, p_1(x,y) = 0$ in any case if $2k > m$.

**Proof.** Let $p \in$ Hom $\mathrm{Pol}_3(m)$. Then

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 ,$$

means

$$\Delta p_0 + z\,\Delta p_1 + z^2\,\Delta p_2 + \ldots + z^{m-2}\,\Delta p_{m-2} +$$
$$+\ 2.1.p_2 + 3.2z.p_3 + 4.3.z^2\,p_4 + \ldots + m(m-1)\,z^{m-2}\,p_m = 0 \ .$$

Equating equal powers of $z$ leads to the recursive relations

$$p_2 = -\frac{1}{2!}\,\Delta\,p_0$$
$$p_4 = -\frac{1}{4.3}\,\Delta\,p_2 = \frac{1}{4!}\,\Delta^2\,p_0 \ , \qquad \text{etc.}$$
$$p_3 = -\frac{1}{3.2}\,\Delta\,p_1$$
$$p_5 = -\frac{1}{5.4}\,\Delta\,p_3 = \frac{1}{5!}\,\Delta^2\,p_1 \ .$$

So a harmonic polynomial is completely determined by prescribing $p_0(x,y)$ and $p_1(x,y)$. Just apply $\Delta$ to the formulae of the theorem and find them to be 0. □

**Theorem.**

$$\dim \mathrm{Harm\,Hom\,Pol}_3(m) = 2m + 1 \ .$$

**Proof.** $p_0$ can be chosen freely out of $\mathrm{Hom\,Pol}_2(m)$ which is $m+1$ dimensional.
$p_1$ can be chosen freely out of $\mathrm{Hom\,Pol}_2(m-1)$ which is $m$ dimensional. That adds up to $2m+1$. □

In the next definition we construct a basis for $\mathrm{Harm\,Hom\,Pol}_3(m)$ by taking for $p_0$ and $p_1$ respectively the elements of the bases $\{(x+iy)^k\,(x-iy)^l\}$.

**Definition** of $Q_{m,k}$, $-m \le k \le m$. A special basis for $\mathrm{Harm\,Hom\,Pol}_3(m)$, "Spherical Harmonics".

- $Q_{m,m}(x,y,z) \quad = (x+iy)^m$
- $Q_{m,m-2}(x,y,z) \quad = (x+iy)^{m-1}\,(x-iy) - \dfrac{4.(m-1).1}{2!}\,z^2(x+iy)^{m-2}$
- $Q_{m,m-4}(x,y,z) \quad = (x+iy)^{m-2}\,(x-iy)^2 - \dfrac{4.(m-2).2}{2!}\,z^2(x+iy)^{m-3}\,(x-iy) + {} + \dfrac{4^2(m-2)(m-3).2.1}{4!}\,z^4.(x+iy)^{m-4}$

$\vdots$

- $Q_{m,-m+2}(x,y,z) = \overline{Q_{m,m-2}(x,y,z)}$
- $Q_{m,-m}(x,y,z) \quad = (x-iy)^m = \overline{Q_{m,m}(x,y,z)}$

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- $Q_{m,m-1}(x,y,z) \;= z(x+iy)^{m-1}$
- $Q_{m,m-3}(x,y,z) \;= z(x+iy)^{m-2}\,(x-iy) - \dfrac{4.(m-2).1}{3!}\,z^3(x+iy)^{m-3}$
- $Q_{m,m-5}(x,y,z) \;= z(x+iy)^{m-3}\,(x-iy)^2 - \dfrac{4.(m-3).2}{3!}\,z^3(x+iy)^{m-4}\,(x-iy) + {} + \dfrac{4^2(m-3)(m-4).2.1}{5!}\,z^5.(x+iy)^{m-5}$

$\vdots$

- $Q_{m,-m+3}(x,y,z) = \overline{Q_{m,m-3}(x,y,z)}$
- $Q_{m,-m+1}(x,y,z) = z(x-iy)^{m-1}$ .

**Remark.**

$$Q_{m,k}(\rho \sin \vartheta \cos \varphi, \rho \sin \vartheta \sin \varphi, \rho \cos \vartheta) = c_{m,k}\,\rho^m\,Y_{m,k}(\vartheta,\varphi)\ .$$

($c_{m,k}$: normalization constant; $Y_{m,k}$: surface harmonic, function on $S^2$)

**Verify**

$$\mathcal{N}\,Q_{m,k} = m\,Q_{m,k}$$
$$-i\,L_z\,Q_{m,k} = k\,Q_{m,k}$$
$$L_z\,\mathcal{N} = \mathcal{N}\,L_z$$

The proper values $m$, $k$ determine the 'state' $Q_{m,k}$ uniquely up to a constant. The $Q_{m,k}$ are common eigenvectors of the commuting 'observables' $\mathcal{N}$ and $L_z$. The integers $m$ and $k$ are 'good quantum numbers'.

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