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Inertial Algorithms for the Stationary Navier-Stokes Equations

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Several kind of new numerical methods for stationary Navier-Stokes equations based on the virtue of Inertial Manifold and Approximate Inertial Manifold, which we call them inertial algorithms in this paper, together with their error estimations are presented. All these algorithms are constructed under an uniform frame, that is to construct some kind of new projections for the Sobolev space in which the true solution is sought. It is shown that the proposed inertial algorithms can greatly improve the convergence rate of the standard Galerkin approximate solution. And we also give some numerical examples to verify our results.

1. Introduction

Even though the computing power improved rapidly in last decade, constructing higher accurate and more effective algorithms for numerically solving partial differential equations, especially for numerically solving the Navier-Stokes equations, still offer many challenges. Many authors derived new techniques and algorithms in the past several years, for example, Q. Lin[9], B. García-Achilla, J. Novo and E. Titi[5], W. Layton and W. Lenferink[8], J. Xu[14],[10] used superconvergent finite element methods, two-level finite element methods and nonlinear Galerkin methods, etc.

In this paper, we are interested in applying the original ideal of the inertial manifold [4] and approximate inertial manifold [3] to construct some kind of higher order finite spectral algorithms for the stationary Navier-Stokes equations, which we will call them inertial algorithms in the remainder of this paper. Although these algorithms derived here are designed for the stationary Navier-Stokes equations, they can be also applied to the nonstationary Navier-Stokes equations with or without modification which we will investigate elsewhere.

Assume H is a suitable Sobolev space, H_m a finite dimensional subspace of H and $\boldsymbol{u}$ and $\boldsymbol{u}_m$ the true solution and the standard spectral Galerkin approximation of the stationary Navier-Stokes equations in H and H_m respectively. Our main goal is to construct some kind of new projection Q_m which maps any function of H onto H_m. Then for any vector $\boldsymbol{w}$ in H, we have the following decomposition

$$\boldsymbol{w} = Q_m\boldsymbol{w} + \hat{\boldsymbol{w}}, \quad \forall \boldsymbol{w} \in H,$$

where $\hat{\boldsymbol{w}} \in \hat{H}$ and $\hat{H}$ is the orthogonal complement of H_m with respect to some scalar product. We shall identify $Q_m\boldsymbol{w}$ and $\hat{\boldsymbol{w}}$ as the lower frequency and higher frequency components of $\boldsymbol{w}$. Especially,

$$\boldsymbol{u} = Q_m\boldsymbol{u} + \hat{\boldsymbol{u}}.$$

The crucial point in the construction of Q_m is to ensure $Q_m\boldsymbol{u}$ to be closer to $\boldsymbol{u}_m$ than $\boldsymbol{u}$. For further discussion, we denote by P_m the standard L^2- orthogonal projection from H onto H_m further on. Based upon the idea of inertial manifold and approximate inertial manifold, there should

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exist interactive or at least some approximate interactive relations between the lower and higher frequency components of the true solution. Then we want to construct some finite dimensional mapping $\boldsymbol{\Phi} : H_m \to \hat{H}$, which in fact is a reflection of this kind of relations, such that $\boldsymbol{\Phi}(\boldsymbol{u}_m)$ can generate a suitable approximation of $\hat{\boldsymbol{u}}$ with approximate order at least equal to that of $Q_m\boldsymbol{u}-\boldsymbol{u}_m$. If it succeeds, $\boldsymbol{u}_m + \boldsymbol{\Phi}(\boldsymbol{u}_m)$ will be a higher order approximation of $\boldsymbol{u}$ than $\boldsymbol{u}_m$ with

$$||\boldsymbol{u} - (\boldsymbol{u}_m + \boldsymbol{\Phi}(\boldsymbol{u}_m))|| \leq ||Q_m\boldsymbol{u} - \boldsymbol{u}_m|| + ||\hat{\boldsymbol{u}} - \boldsymbol{\Phi}(\boldsymbol{u}_m)||.$$

The paper is arranged as follows. In section 2, we consider the stationary Navier-Stokes equations and its spectral Galerkin approximation. In section 3, three kind of projections from H onto H_m are constructed and studied. In section 4, we derive the estimations of $Q_m\boldsymbol{u}-\boldsymbol{u}_m$ respectively for various projections and then give the construction of corresponding finite dimensional mapping $\boldsymbol{\Phi}$. In section 5, three kinds of inertial algorithms based on the projections and the associated finite dimensional mappings are presented together with their convergence properties . Finally, some numerical examples are presented for the various algorithms and it is shown that such schemes can greatly improve the convergence rate of standard Galerkin method.

2. Preliminaries

In this paper, we will consider the following stationary Navier-Stokes equations defined on a bounded domain $\Omega \subset \mathcal{R}^d$, $d = 2, 3$

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{u} : \Omega \to \mathcal{R}^d$ denotes the velocity field, $p : \Omega \to \mathcal{R}$ the pressure, $\boldsymbol{F}$ the external force which drives the flow, $\nu > 0$ the kinematic viscosity. For simplicity, we consider $\partial\Omega$ to be class C^2.

We also define the following Hilbert space as usual

$$H := \{\boldsymbol{v} \in L^2(\Omega)^d : \ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \ \operatorname{div}\boldsymbol{v} = 0 \ \text{ in weak sense}\}.$$

Here $\boldsymbol{n}$ denotes the unit outer normal vector of $\partial\Omega$. By P we denote the orthogonal L^2- projection from $L^2(\Omega)^d$ onto H. Now projecting the above Navier-Stokes equations in velocity-pressure form onto H, we get the abstract Navier-Stokes equations in H

$$\nu A\boldsymbol{u} + B(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f}, \tag{2.1}$$

where $A = -P\Delta$, $B(\boldsymbol{u},\boldsymbol{u}) = P[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]$ and $\boldsymbol{f} = P\boldsymbol{F}$. Since $A : H \to H$ is a densely defined, unbounded, self-adjoint and positive operator with compact inverse, there exist eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty,$$

and associated eigenfunctions $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n, \cdots\}$ which form a complete orthogonal basis of H. In addition, for any $s \in \mathcal{R}$, we can define the power operator A^s, whose domain is the following linear space

$$D(A^s) := \{\boldsymbol{v} \in H : \ \boldsymbol{v} := \sum_{i=1}^{\infty} v_i\boldsymbol{\phi}_i, \ \sum_{i=1}^{\infty} \lambda_i^{2s}|v_i|^2 < \infty, \ v_i \in \mathcal{R}\}$$

with natural inner product and related norm defined as

$$(\boldsymbol{v}_1, \boldsymbol{v}_2)_s := (A^s\boldsymbol{v}_1, A^s\boldsymbol{v}_2), \quad |A^s\boldsymbol{v}|^2 = (\boldsymbol{v},\boldsymbol{v})_s.$$

Here $(\cdot,\cdot)$ and $|\cdot|$ stand for the standard L^2- inner product and norm respectively. At least, under the assumption of the smoothness of $\partial\Omega$ and for $s < \frac{5}{4}$, $|A^s\cdot|$ is an equivalent norm of $||\cdot||_{2s}$. In the following, for convenience, we will not distinguish between $|A^s\cdot|$ and $\|\cdot\|_{2s}$ for $s<\dfrac{5}{4}$. In the remainder of this paper, we often use the space

$$V := \{\boldsymbol{v}\in H_0^1(\Omega)^d:\ \mathrm{div}\boldsymbol{v}=0\}.$$

It is well known that $V=D(A^{\frac12})$.

The variational formulation of (2.1) is given by

$$\begin{cases}\text{find } \ \boldsymbol{u}\in V \ \text{ such that}\\ a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad \forall \boldsymbol{v}\in V,\end{cases}\tag{2.2}$$

where $a(\cdot,\cdot)$ is symmetric, positive definite and continuous in $V\times V$ defined as

$$a(\boldsymbol{u},\boldsymbol{v}) := \nu(A^{\frac12}\boldsymbol{u},A^{\frac12}\boldsymbol{v}),\quad \forall \boldsymbol{u},\boldsymbol{v}\in V,$$

and the trilinear form is defined as

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}),\quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H^1(\Omega)^d.$$

As well known, $b(\cdot,\cdot,\cdot)$ has the following properties[13]

$$\begin{cases} b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\quad \forall \boldsymbol{u}\in V,\boldsymbol{v},\boldsymbol{w}\in H^1(\Omega)^d,\\ b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\le c_b\|\boldsymbol{u}\|_{s_1}\|\boldsymbol{v}\|_{s_2+1}\|\boldsymbol{w}\|_{s_3},\quad \forall \boldsymbol{u}\in H^{s_1}(\Omega)^d,\boldsymbol{v}\in H^{s_2+1}(\Omega)^d,\boldsymbol{w}\in H^{s_3}(\Omega)^d,\end{cases}\tag{2.3}$$

where $s_1,s_2,s_3\ge 0$, $s_1+s_2+s_3\ge\dfrac{d}{2}$ and $(s_1,s_2,s_3)\ne(0,0,\dfrac{d}{2})$, $(0,\dfrac{d}{2},0)$, $(\dfrac{d}{2},0,0)$. Moreover, we will use the following Sobolev interpolation inequalities and Brezis-Gallouet inequality[2] later.

$$\begin{cases}\|\boldsymbol{v}\|_{\frac12}\le c_0\|\boldsymbol{v}\|^{\frac12}|\boldsymbol{v}|^{\frac12},\quad \forall\boldsymbol{v}\in D(A^{\frac12}),\\ |\boldsymbol{v}|_{L^\infty}\le c_0\|\boldsymbol{v}\|^{\frac12}\|\boldsymbol{v}\|_2^{\frac12},\quad d=3,\forall\boldsymbol{v}\in D(A),\\ |\boldsymbol{v}|_{L^\infty}\le c_0\|\boldsymbol{v}\|(1+\sqrt{\ln(1+\dfrac{\|\boldsymbol{v}\|_2}{\|\boldsymbol{v}\|})}),\quad d=2,\forall\boldsymbol{v}\in D(A).\end{cases}\tag{2.4}$$

For any given $m\in\mathcal{N}$, let

$$H_m=\{\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\cdots,\boldsymbol{\phi}_m\}=P_mH.$$

Then the standard spectral Galerkin approximation of (2.1) reads

$$\begin{cases}\text{find } \ \boldsymbol{u}_m\in H_m \ \text{ such that}\\ \nu A\boldsymbol{u}_m+P_mB(\boldsymbol{u}_m,\boldsymbol{u}_m)=P_m\boldsymbol{f}.\end{cases}\tag{2.5}$$

And its variational problem is

$$\begin{cases}\text{find } \ \boldsymbol{u}_m\in H_m \ \text{ such that}\\ a(\boldsymbol{u}_m,\boldsymbol{v})+b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad \forall\boldsymbol{v}\in H_m.\end{cases}\tag{2.6}$$

We will write $V_m=P_mV$ for short.

For the sake of convenience, we will use the symbol $\boldsymbol{\eta}$ to denote the distance between $\boldsymbol{u}$ and $\boldsymbol{u}_m$. As $\boldsymbol{u}_m\in H_m$, the error estimates of $\boldsymbol{\eta}$ can not be better than the norms of $\boldsymbol{u}-P_m\boldsymbol{u}$. Indeed,

$$|A^{-\frac12}\boldsymbol{\eta}|\le c_1\lambda_{m+1}^{-\frac32},\quad |\boldsymbol{\eta}|\le c_1\lambda_{m+1}^{-1},\quad \|\boldsymbol{\eta}\|\le c_1\lambda_{m+1}^{-\frac12},\tag{2.7}$$

where $c_1>0$ is a constant independent of m. The proof of the above results is classical.

3. Projections

In this section, we will construct three kinds of projections from H onto H_m. First of all, let us introduce the Navier-Stokes operator $\mathcal{F}$ from V to V^*: for any $\boldsymbol{w} \in V$, $\mathcal{F}(\boldsymbol{w}) \in V^*$ such that

$$< \mathcal{F}(\boldsymbol{w}), \boldsymbol{v} >= a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}; \boldsymbol{w}, \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V,$$

where V^* is the dual space of V and $< \cdot, \cdot >$ is the dual product. Thus, (2.1) is equivalent to

$$\mathcal{F}(\boldsymbol{u}) = 0.$$

We denote by $D\mathcal{F}(\boldsymbol{u})$ the Frechet derivative of $\mathcal{F}$ at $\boldsymbol{u}$: for any $\boldsymbol{w}, \boldsymbol{v} \in V$

$$< D\mathcal{F}(\boldsymbol{u})\boldsymbol{w}, \boldsymbol{v} >:= a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}). \tag{3.1}$$

And we set

$$\mathcal{L}_1(\boldsymbol{w}, \boldsymbol{v}) :=< D\mathcal{F}(\boldsymbol{u})\boldsymbol{w}, \boldsymbol{v} > .$$

If $\boldsymbol{u}$ is a nonsingular solution of (2.1), $D\mathcal{F}(\boldsymbol{u})$ is an isomorphism from V onto V^* (see [7],[1]), and we can assert that there must be some constant $\alpha_0 > 0$ such that

$$\inf_{\boldsymbol{w} \in V} \sup_{\boldsymbol{v} \in V} \frac{\mathcal{L}_1(\boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{w}\| \, \|\boldsymbol{v}\|}, \ \inf_{\boldsymbol{v} \in V} \sup_{\boldsymbol{w} \in V} \frac{\mathcal{L}_1(\boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{w}\| \, \|\boldsymbol{v}\|} \geq \alpha_0. \tag{3.2}$$

Thus, by the Lax-Milgram theorem, the variational problem

$$\begin{cases} \text{Find } \ \boldsymbol{w} \in V \ \text{ such that} \\ \mathcal{L}_1(\boldsymbol{w}, \boldsymbol{v}) =< \boldsymbol{g}, \boldsymbol{v} >, \quad \forall \boldsymbol{v} \in V \end{cases}$$

has an unique soluton for any $\boldsymbol{g} \in V^*$.

From (2.5) and (2.6), we can similarly define an operator $\mathcal{F}_m$ from H_m to V_m^*. Then (2.5) is equivalent to

$$\mathcal{F}_m(\boldsymbol{u}_m) = 0.$$

From (3.1), we can get a new bilinear form $\mathcal{L}_{m,1}(\cdot, \cdot)$:

$$\begin{aligned} \mathcal{L}_{m,1}(\boldsymbol{w}, \boldsymbol{v}) :=& a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}_m; \boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}; \boldsymbol{u}_m, \boldsymbol{v}) \\ =& \mathcal{L}_1(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}_m - \boldsymbol{u}; \boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{w}; \boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{v}), \quad \forall \boldsymbol{w}, \boldsymbol{v} \in V. \end{aligned} \tag{3.3}$$

Obviously, if we restrict $\boldsymbol{w}, \boldsymbol{v}$ to H_m, (3.3) can be seen as the definition of $< D\mathcal{F}_m(\boldsymbol{u}_m)\boldsymbol{w}, \boldsymbol{v} >$.

Under the assumption that $\boldsymbol{u}$ is a nonsingular point of $\mathcal{F}$, the following lemma and its corollary will tell us $\boldsymbol{u}_m$ is a nonsingular point of $\mathcal{F}_m$ if $\boldsymbol{u}_m$ is very close to $\boldsymbol{u}$(see [6]).

Lemma 3.1. *Assume $\tilde{V} \subset V$ is a finite dimensional subspace and $\tilde{\mathcal{F}}$ is a smooth mapping from $\tilde{V}$ to $\tilde{V}^*$. Let $\boldsymbol{u}$ be a nonsingular point of $\mathcal{F}$ and denote*

$$\sigma(\boldsymbol{u}) = \|D\mathcal{F}(\boldsymbol{u})^{-1}\|_{\mathcal{L}(V^*, V)}, \quad \mu(\tilde{\boldsymbol{u}}) = \|D\mathcal{F}(\boldsymbol{u}) - D\tilde{\mathcal{F}}(\tilde{\boldsymbol{u}})\|_{\mathcal{L}(\tilde{V}, \tilde{V}^*)}.$$

If $\tilde{\boldsymbol{u}}$ is closed to $\boldsymbol{u}$ such that

$$\sigma(\boldsymbol{u})\mu(\tilde{\boldsymbol{u}}) < 1,$$

$D\tilde{\mathcal{F}}(\tilde{\boldsymbol{u}})$ is an isomorphism from $\tilde{V}$ onto $\tilde{V}^*$. Hence, $\tilde{\boldsymbol{u}}$ is a nonsingular point of $\tilde{\mathcal{F}}$.

The following corollary of the above lemma will tell us how to guarantee that $\boldsymbol{u}_m$ is also a nonsingular point of $\mathcal{F}_m$ if $\boldsymbol{u}$ is a nonsingular point of $\mathcal{F}$.

Corollary 3.1. *Assume $\boldsymbol{u}$ is a nonsingular solution of the Navier-Stokes equations (2.1). If m is large enough such that*

$$\lambda_{m+1} \geq \frac{16c_b^2c_1^2}{\alpha_0^2}, \tag{3.4}$$

$\boldsymbol{u}_m$ is a nonsingular solution of its standard Galerkin approximate equations. And we have

$$\inf_{\boldsymbol{w}\in H_m}\sup_{\boldsymbol{v}\in H_m}\frac{\mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|},\ \inf_{\boldsymbol{v}\in H_m}\sup_{\boldsymbol{w}\in H_m}\frac{\mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|} \geq \frac{\alpha_0}{2}. \tag{3.5}$$

Furthermore,

$$\inf_{\boldsymbol{w}\in V}\sup_{\boldsymbol{v}\in V}\frac{\mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|},\ \inf_{\boldsymbol{v}\in V}\sup_{\boldsymbol{w}\in V}\frac{\mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|} \geq \frac{\alpha_0}{2}. \tag{3.6}$$

Proof. Thanks to (3.2), we have

$$\sigma(\boldsymbol{u}) = \|D\mathcal{F}(\boldsymbol{u})^{-1}\|_{\mathcal{L}(V^*,V)} \leq \alpha_0^{-1}.$$

On the other hand,

$$< (D\mathcal{F}(\boldsymbol{u}) - D\mathcal{F}_m(\boldsymbol{u}_m))\boldsymbol{w}, \boldsymbol{v} >= b(\boldsymbol{u}-\boldsymbol{u}_m;\boldsymbol{w},\boldsymbol{v}) + b(\boldsymbol{w};\boldsymbol{u}-\boldsymbol{u}_m,\boldsymbol{v}), \quad \forall \boldsymbol{w},\boldsymbol{v}\in H_m.$$

Therefore, by using (2.7)

$$\begin{aligned}\mu(\boldsymbol{u}_m) =& \|D\mathcal{F}(\boldsymbol{u}) - D\mathcal{F}_m(\boldsymbol{u}_m)\| = \sup_{\boldsymbol{w},\boldsymbol{v}\in V_m}\frac{< (D\mathcal{F}(\boldsymbol{u}) - D\mathcal{F}_m(\boldsymbol{u}_m))\boldsymbol{w}, \boldsymbol{v} >}{\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|}\\ \leq& 2c_b\|\boldsymbol{u}-\boldsymbol{u}_m\| \leq 2c_bc_1\lambda_{m+1}^{-\frac{1}{2}}.\end{aligned}$$

From lemma 3.1, we can conclude (3.4)∼(3.5). And (3.6) is obvious if we notice (3.3). □

Now, it is time for us to construct our first projection.

Projection 1. Q_m^1: $H \to H_m$

$$\begin{cases} \text{for any } \ \boldsymbol{w}\in H, \ \text{find } \ Q_m^1\boldsymbol{w}\in H_m \ \text{ such that}\\ \mathcal{L}_{m,1}(\boldsymbol{w}-Q_m^1\boldsymbol{w},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v}\in H_m.\end{cases} \tag{3.7}$$

Thanks to (3.5), the following problem

$$\mathcal{L}_{m,1}(Q_m^1\boldsymbol{w},\boldsymbol{v}) = \mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v}), \quad \forall \boldsymbol{v}\in H_m$$

has an unique solution. Now for any vector $\boldsymbol{w}\in H$, we have

$$\boldsymbol{w} = Q_m^1\boldsymbol{w} + \hat{\boldsymbol{w}}^1, \qquad \text{with } \ \hat{\boldsymbol{w}}^1 = \boldsymbol{w} - Q_m^1\boldsymbol{w}.$$

(3.7) implies that $\hat{\boldsymbol{w}}^1$ is orthogonal to H_m in $\mathcal{L}_{m,1}(\cdot,\cdot)$. We identify $Q_m^1\boldsymbol{w}$ and $\hat{\boldsymbol{w}}^1$ as lower and higher frequency components of $\boldsymbol{w}$ with respect to projection Q_m^1 respectively. Furthermore, we define an adjoint bilinear form $\mathcal{L}_{m,1}^*(\cdot,\cdot)$ by

$$\mathcal{L}_{m,1}^*(\boldsymbol{w},\boldsymbol{v}) := \mathcal{L}_1(\boldsymbol{v},\boldsymbol{w}) + b(\boldsymbol{u}_m-\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) + b(\boldsymbol{v};\boldsymbol{u}_m-\boldsymbol{u},\boldsymbol{w}), \quad \forall \boldsymbol{w},\boldsymbol{v}\in V.$$

Of course

$$\mathcal{L}_{m,1}(\boldsymbol{w},\boldsymbol{v}) = \mathcal{L}_{m,1}^*(\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{w},\boldsymbol{v}\in V.$$

Set $\hat{H}^1 = (I-Q_m^1)H$, $\hat{V}^1 = (I-Q_m^1)V$. Then

$$\mathcal{L}_{m,1}(\hat{\boldsymbol{w}}^1,\boldsymbol{v}) = \mathcal{L}_{m,1}^*(\boldsymbol{v},\hat{\boldsymbol{w}}^1) = 0, \quad \forall \boldsymbol{v}\in H_m, \hat{\boldsymbol{w}}^1\in\hat{H}^1. \tag{3.8}$$

Similarly, we define

$$\mathcal{L}_1^*(\boldsymbol{w},\boldsymbol{v}) = \mathcal{L}_1(\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{w},\boldsymbol{v}\in V.$$

The next lemma shows that the norm of higher frequency components in $\hat{V}^1$ is very small.

Lemma 3.2. *Under the assumptions of corollary 3.1 and the adjoint linearized Navier-Stokes equations: for* $\boldsymbol{g} \in H$*, find* $\boldsymbol{\phi} \in V$ *such that*

$$\mathcal{L}_1^*(\boldsymbol{\phi}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V \tag{3.9}$$

is H^2*-regular. Then the projection* Q_m^1 *defined by (3.7) satisfies*

$$\|\boldsymbol{w} - Q_m^1 \boldsymbol{w}\| \le c_2 \|\boldsymbol{w}\|, \quad \forall \boldsymbol{w} \in V, \tag{3.10}$$

$$|\boldsymbol{w} - Q_m^1 \boldsymbol{w}| \le c_3 \lambda_{m+1}^{-\frac{1}{2}} \|\boldsymbol{w} - Q_m^1 \boldsymbol{w}\|, \quad \forall \boldsymbol{w} \in V, \tag{3.11}$$

where c_2, c_3 *are positive constants independent of* $\boldsymbol{w}$ *and* m*.*

Proof. Using (3.5) and (3.8), we have

$$\begin{aligned}\|Q_m^1 \boldsymbol{w}\| \le & \frac{2}{\alpha_0} \sup_{\boldsymbol{v} \in H_m} \frac{\mathcal{L}_{m,1}(Q_m^1 \boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{v}\|} = \frac{2}{\alpha_0} \sup_{\boldsymbol{v} \in H_m} \frac{\mathcal{L}_{m,1}(\boldsymbol{w}, \boldsymbol{v})}{\|\boldsymbol{v}\|} \\ \le & \frac{2(\nu \|\boldsymbol{w}\| + 2c_b \|\boldsymbol{u}_m\| \, \|\boldsymbol{w}\|)}{\alpha_0} \triangleq (c_2 - 1)\|\boldsymbol{w}\|.\end{aligned}$$

This proves (3.10) by the triangular inequality and shows that Q_m^1 is a bounded projection.

As a result of the H^2- regularity assumption of (3.9), there must be some positive constant c' such that

$$\|\boldsymbol{\phi}\|_2 \le c' |\boldsymbol{g}|.$$

Now we set $\boldsymbol{g} = \boldsymbol{v} = \hat{\boldsymbol{w}}^1$ in (3.9). From (3.3), we have

$$\mathcal{L}_{m,1}^*(\boldsymbol{\phi}, \hat{\boldsymbol{w}}^1) = |\hat{\boldsymbol{w}}^1|^2 + b(\boldsymbol{u}_m - \boldsymbol{u}; \hat{\boldsymbol{w}}^1, \boldsymbol{\phi}) + b(\hat{\boldsymbol{w}}^1; \boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{\phi}).$$

Taking $\boldsymbol{\phi}_m = P_m \boldsymbol{\phi} \in H_m$ and using (3.8) and the classical property of L^2- orthogonal projection P_m, we have

$$\begin{aligned}|\mathcal{L}_{m,1}^*(\boldsymbol{\phi}, \hat{\boldsymbol{w}}^1)| =& |\mathcal{L}_{m,1}^*(\boldsymbol{\phi} - \boldsymbol{\phi}_m, \hat{\boldsymbol{w}}^1)| \le (\nu + 2c_b \|\boldsymbol{u}_m\|) \|\boldsymbol{\phi} - \boldsymbol{\phi}_m\| \, \|\hat{\boldsymbol{w}}^1\| \\ \le & \lambda_{m+1}^{-\frac{1}{2}} (\nu + 2c_b \|\boldsymbol{u}_m\|) \|\boldsymbol{\phi}\|_2 \|\hat{\boldsymbol{w}}^1\| \le c' \lambda_{m+1}^{-\frac{1}{2}} (\nu + 2c_b \|\boldsymbol{u}_m\|) |\hat{\boldsymbol{w}}^1| \, \|\hat{\boldsymbol{w}}^1\|.\end{aligned}$$

Here we used the well known property

$$|(I - P_m)\phi| \le \lambda_{m+1}^{-\frac{1}{2}} \|\phi\|, \quad \forall \phi \in V.$$

On the other hand,

$$|b(\boldsymbol{u}_m - \boldsymbol{u}; \hat{\boldsymbol{w}}^1, \boldsymbol{\phi})| = |b(\boldsymbol{u}_m - \boldsymbol{u}; \boldsymbol{\phi}, \hat{\boldsymbol{w}}^1)| \le c_b |\boldsymbol{u}_m - \boldsymbol{u}| \, \|\boldsymbol{\phi}\|_2 \|\hat{\boldsymbol{w}}^1\| \le c_b c_1 c' \lambda_{m+1}^{-1} |\hat{\boldsymbol{w}}^1| \, \|\hat{\boldsymbol{w}}^1\|,$$

$$|b(\hat{\boldsymbol{w}}^1; \boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{\phi})| = |b(\hat{\boldsymbol{w}}^1; \boldsymbol{\phi}, \boldsymbol{u}_m - \boldsymbol{u})| \le c_b \|\boldsymbol{\phi}\|_2 \|\hat{\boldsymbol{w}}^1\| \, |\boldsymbol{u}_m - \boldsymbol{u}| \le c_b c_1 c' \lambda_{m+1}^{-1} |\hat{\boldsymbol{w}}^1| \, \|\hat{\boldsymbol{w}}^1\|.$$

Finally, we can get (3.11) for $c_3 = c'(\nu + 2c_b \|\boldsymbol{u}_m\| + 2c_b c_1)$. □

Note that the construction of the above projection Q_m^1 is based on the assumption that $\boldsymbol{u}$ is a nonsingular solution of (2.1). Of course, the usable range of this projection is restricted by this condition. To overcome this disadvantage, we will construct other two kinds of projection from H onto H_m which can always make sense whether $\boldsymbol{u}$ is a singular solution of (2.1) or not. The only difference between these three projections is that the associated inertial algorithms may have different accuracy.

Projection 2. Q_m^2: $H \to H_m$

$$\left\{\begin{array}{l} \textit{for any } \ \boldsymbol{w} \in H, \ \textit{ find } \ Q_m^2 \boldsymbol{w} \in H_m \ \textit{ such that} \\ \mathcal{L}_{m,2}(\boldsymbol{w} - Q_m^2 \boldsymbol{w}, v) = 0, \quad \forall \boldsymbol{v} \in H_m, \end{array}\right. \tag{3.12}$$

where

$$\mathcal{L}_{m,2}(\boldsymbol{w}, \boldsymbol{v}) = a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}_m, \boldsymbol{w}, \boldsymbol{v}).$$

For further discussion, we define

$$\mathcal{L}_2(\boldsymbol{w}, \boldsymbol{v}) = a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \tag{3.13}$$

where $\boldsymbol{u}$ is the solution of (2.2). Of course, we see

$$\mathcal{L}_{m,2}(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_2(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}).$$

At this moment, the bilinear forms $\mathcal{L}_2(\cdot,\cdot)$ and $\mathcal{L}_{m,2}(\cdot,\cdot)$ are all positive. In fact

$$\mathcal{L}_2(\boldsymbol{w}, \boldsymbol{w}) = a(\boldsymbol{w}, \boldsymbol{w}) = \nu\|\boldsymbol{w}\|^2, \quad \mathcal{L}_{m,2}(\boldsymbol{w}, \boldsymbol{w}) = a(\boldsymbol{w}, \boldsymbol{w}) = \nu\|\boldsymbol{w}\|^2.$$

So, (3.12) can define a projection from H onto H_m whenever $\boldsymbol{u}$ is a singular or nonsingular solution.

Projection 3. Q_m^3: $H \to H_m$

$$\left\{\begin{array}{l} \textit{for any } \ \boldsymbol{w} \in H, \ \textit{ find } \ Q_m^3 \boldsymbol{w} \in H_m \ \textit{ such that} \\ \mathcal{L}_{m,3}(\boldsymbol{w} - Q_m^3 \boldsymbol{w}, v) = 0, \quad \forall \boldsymbol{v} \in H_m, \end{array}\right. \tag{3.14}$$

where

$$\mathcal{L}_{m,3}(\boldsymbol{w}, \boldsymbol{v}) = a(\boldsymbol{w}, \boldsymbol{v}).$$

Also, we define

$$\mathcal{L}_3(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_{m,3}(\boldsymbol{w}, \boldsymbol{v}).$$

They are obvious symmetric and positive, so (3.14) makes sense whenever $\boldsymbol{u}$ is a singular or nonsingular solution, too.

It is very easy to prove that the projections Q_m^2 and Q_m^3 have similar properties as Q_m^1. To illustrate this, we define the following adjoint bilinear forms corresponding to $\mathcal{L}_i(\cdot,\cdot)$ and $\mathcal{L}_{m,i}(\cdot,\cdot)$, $i = 2, 3$, as

$$\left\{\begin{array}{ll} \mathcal{L}_2^*(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_2(\boldsymbol{v}, \boldsymbol{w}), & \mathcal{L}_{m,2}^*(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_{m,2}(\boldsymbol{v}, \boldsymbol{w}) = \mathcal{L}_2(\boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{u}_m - \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \\ \mathcal{L}_3^*(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_3(\boldsymbol{v}, \boldsymbol{w}), & \mathcal{L}_{m,3}^*(\boldsymbol{w}, \boldsymbol{v}) = \mathcal{L}_{m,3}(\boldsymbol{v}, \boldsymbol{w}). \end{array}\right.$$

It is easy to know that $\mathcal{L}_3^*(\cdot,\cdot) \equiv \mathcal{L}_3(\cdot,\cdot)$. So does $\mathcal{L}_{m,3}^*(\cdot,\cdot)$.

For $\mathcal{L}_3^*(\cdot,\cdot)$, the variational problem

$$\mathcal{L}_3^*(\boldsymbol{\phi}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V$$

is H^2- regular under the assumption of $\partial\Omega$ being class C^2. And we can easily prove

$$\mathcal{L}_2^*(\boldsymbol{\phi}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V$$

is H^2- regular, too. Also, orthogonality properties like (3.8) hold, e.g.

$$\mathcal{L}_{m,i}(\hat{\boldsymbol{w}}^i, \boldsymbol{v}) = \mathcal{L}_{m,i}^*(\boldsymbol{v}, \hat{\boldsymbol{w}}^i) = 0, \quad \forall \boldsymbol{v} \in H_m, \hat{\boldsymbol{w}}^i \in \hat{H}^i.$$

Having the above knowledge of these two kinds of bilinear forms, we can easily prove that Q_m^i, $i = 2, 3$, defined by (3.12) and (3.14) have similar properties as Q_m^1, which we only state without proof in the following lemma.

Lemma 3.3. *The projections Q_m^2 and Q_m^3 defined by (3.12) and (3.14) satisfy*

$$\|\boldsymbol{w} - Q_m^i \boldsymbol{w}\| \le c_2 \|\boldsymbol{w}\|, \quad \forall \boldsymbol{w} \in V,$$
$$|\boldsymbol{w} - Q_m^i \boldsymbol{w}| \le c_3 \lambda_{m+1}^{-\frac{1}{2}} \|\boldsymbol{w} - Q_m^i \boldsymbol{w}\|, \quad \forall \boldsymbol{w} \in V,$$

where $i = 2, 3$ and c_2, c_3 are positive constants independent of $\boldsymbol{w}$ and m.

To avoid having too many symbols, we still use c_2 and c_3 in the results of this lemma although we used them in lemma 3.2 because they have very similar forms and would not change our discussion significantly.

Remark. Besides a general assumption on the smoothness of $\partial\Omega$, Q_m^1 can only be used when $\boldsymbol{u}$ is a nonsingular solution of (2.1). So its usable range is limited. That is the reason why we like to introduce Q_m^2 and Q_m^3, which always make sense whenever $\boldsymbol{u}$ is a singular or nonsingular solution of (2.1). In this sense, they should be regarded as some generalization of Q_m^1. Of course, this kind of generalization has its own cost. That is the inertial algorithms based on them which will be given later will lose some accuracy compared with the algorithm based on Q_m^1.

4. Lower Frequency Analysis and Finite Dimensional Mappings

As we said in the introduction, our projections constructed in the previous section should satisfy

$$\|Q_m^i \boldsymbol{u} - \boldsymbol{u}_m\| = o(\|\boldsymbol{\eta}\|), \quad i = 1, 2, 3. \tag{4.1}$$

Only when these conditions are satisfied, it is possible for us to construct some finite dimensional mappings $\boldsymbol{\Phi}^i$ corresponding to different projections which can generate higher order approximations of the higher frequency components of the true solution with respect to the different projections

$$\|\boldsymbol{\Phi}^i(\boldsymbol{u}_m) - \hat{\boldsymbol{u}}^i\| = o(\|\boldsymbol{\eta}\|), \quad i = 1, 2, 3, \tag{4.2}$$

such that

$$\|\boldsymbol{u} - (\boldsymbol{u}_m + \boldsymbol{\Phi}^i(\boldsymbol{u}_m))\| \le \|Q_m^i \boldsymbol{u} - \boldsymbol{u}_m\| + \|\hat{\boldsymbol{u}}^i - \boldsymbol{\Phi}^i(\boldsymbol{u}_m)\| = o(\|\boldsymbol{\eta}\|), \quad i = 1, 2, 3. \tag{4.3}$$

So, the first thing we should do is to make sure that (4.1) is valid for our projections. In the following, we will show that (4.1) is satisfied by our projections one by one. We recall the following decomposition

$$\boldsymbol{w} = Q_m^i \boldsymbol{w} + \hat{\boldsymbol{w}}^i, \quad Q_m^i \boldsymbol{w} \in H_m, \ \hat{\boldsymbol{w}}^i \in \hat{V}^i, \ \forall \boldsymbol{w} \in V, \ i = 1, 2, 3. \tag{4.4}$$

We will call $Q_m^i \boldsymbol{w}$ the lower frequency components of $\boldsymbol{w}$ of the ith projection, simply, the lower frequency components of $\boldsymbol{w}$. And we will call $\hat{\boldsymbol{w}}^i$ the higher frequency components of $\boldsymbol{w}$ in the sense of the ith projection, simply, the higher frequency components of $\boldsymbol{w}$.

Before verifying (4.1) for our projections, we give a novel property of the trilinear form $b(\cdot, \cdot, \cdot)$:

Lemma 4.1. *For any $\boldsymbol{\phi} \in D(A^{-\frac{1}{2}})$, $\boldsymbol{w} \in D(A^{\frac{1}{2}})$ and $\boldsymbol{v} \in H_m$, we have*

$$|b(\boldsymbol{\phi}, \boldsymbol{v}, \boldsymbol{w})| \le c_b L_m^d |A^{-\frac{1}{2}} \boldsymbol{\phi}| \, \|\boldsymbol{v}\|_2 \|\boldsymbol{w}\|,$$

where

$$L_m^d := \begin{cases} 3c_0(1 + \sqrt{\ln(1 + \lambda_m^{\frac{1}{2}})}), & d = 2, \\ 3c_0 \lambda_{m+1}^{\frac{1}{4}}, & d = 3. \end{cases}$$

Proof. Suppose

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Then

$$(\boldsymbol{\phi}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}=\left[\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\cdot\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}\right]\begin{pmatrix}v_1\\ v_2\end{pmatrix}\cdot\begin{pmatrix}w_1\\ w_2\end{pmatrix}$$
$$=\phi_1 w_1\partial_x v_1+\phi_2 w_1\partial_y v_1+\phi_1 w_2\partial_x v_2+\phi_2 w_2\partial_y v_2=(\boldsymbol{w}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{\phi},$$

where $\boldsymbol{w}\cdot\nabla\boldsymbol{v}$ means $w_1\nabla v_1+w_2\nabla v_2$. Now we define a new bilinear operator $\tilde{B}$

$$\tilde{B}(\boldsymbol{w},\boldsymbol{v})=P(\boldsymbol{w}\cdot\nabla\boldsymbol{v}),$$

and its associated trilinear form $\tilde{b}$

$$\tilde{b}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{\phi})=(\tilde{B}(\boldsymbol{w},\boldsymbol{v}),\boldsymbol{\phi})=b(\boldsymbol{\phi},\boldsymbol{v},\boldsymbol{w}).$$

In fact $\tilde{b}$ and b are the same thing with different format, so all the estimations for b are also valid for $\tilde{b}$.

Let δ be any derivative operator on $\mathcal{R}^2$, i.e.

$$\delta(\boldsymbol{w}\cdot\nabla\boldsymbol{v})=(\delta\boldsymbol{w}\cdot\nabla\boldsymbol{v})+(\boldsymbol{w}\cdot\nabla\delta\boldsymbol{v}).$$

Because of $\boldsymbol{w}\in D(A^{\frac{1}{2}})$ and $\boldsymbol{v}\in H_m$, we have

$$|A^{\frac{1}{2}}\tilde{B}(\boldsymbol{w},\boldsymbol{v})|\le\|\tilde{B}(\boldsymbol{w},\boldsymbol{v})\|\le|\delta\tilde{B}(\boldsymbol{w},\boldsymbol{v})|\le|(\delta\boldsymbol{w}\cdot\nabla\boldsymbol{v})|+|(\boldsymbol{w}\cdot\nabla\delta\boldsymbol{v})|$$
$$\le\frac{|\tilde{b}(\delta\boldsymbol{w},\boldsymbol{v},\boldsymbol{\psi})|+|\tilde{b}(\boldsymbol{w},\delta\boldsymbol{v},\boldsymbol{\psi})|}{|\boldsymbol{\psi}|}\le c_b|A^{\frac{1}{2}}\boldsymbol{w}|\,|A^{\frac{1}{2}}\boldsymbol{v}|_{L^\infty}+\frac{|\tilde{b}(\boldsymbol{w},\delta\boldsymbol{v},\boldsymbol{\psi})|}{|\boldsymbol{\psi}|}.$$

We decompose $\boldsymbol{w}$ as

$$\boldsymbol{w}=P_m\boldsymbol{w}+(I-P_m)\boldsymbol{w}:=\boldsymbol{w}_m+\bar{\boldsymbol{w}}.$$

Then

$$|\tilde{b}(\boldsymbol{w},\delta\boldsymbol{v},\boldsymbol{\psi})|\le|\tilde{b}(\boldsymbol{w}_m,\delta\boldsymbol{v},\boldsymbol{\psi})|+|\tilde{b}(\bar{\boldsymbol{w}},\delta\boldsymbol{v},\boldsymbol{\psi})|\le c_b|\boldsymbol{w}_m|_{L^\infty}|A\boldsymbol{v}|\,|\boldsymbol{\psi}|+|\tilde{b}(\bar{\boldsymbol{w}},\delta\boldsymbol{v},\boldsymbol{\psi})|.$$

Therefore

$$|A^{\frac{1}{2}}\tilde{B}(\boldsymbol{w},\boldsymbol{v})|\le c_b|A^{\frac{1}{2}}\boldsymbol{w}|\,|A^{\frac{1}{2}}\boldsymbol{v}|_{L^\infty}+c_b|\boldsymbol{w}_m|_{L^\infty}|A\boldsymbol{v}|+\frac{|\tilde{b}(\bar{\boldsymbol{w}},\delta\boldsymbol{v},\boldsymbol{\psi})|}{|\boldsymbol{\psi}|}.\tag{4.5}$$

Now let us estimate terms on the right hand side of (4.5) for $d=2$ and $d=3$ respectively. From (2.4) and the classical property of P_m, we know for any $\boldsymbol{v}\in H_m$,

$$|\boldsymbol{v}|_{L^\infty}\le\begin{cases}c_0(1+\sqrt{\ln(1+\lambda_m^{\frac{1}{2}})})\|\boldsymbol{v}\|, & d=2,\\ c_0\|\boldsymbol{v}\|^{\frac{1}{2}}\|\boldsymbol{v}\|_2^{\frac{1}{2}}\le c_0\lambda_m^{\frac{1}{4}}\|\boldsymbol{v}\|, & d=3.\end{cases}$$

If we denote

$$L_m^d:=\begin{cases}3c_0(1+\sqrt{\ln(1+\lambda_m^{\frac{1}{2}})}), & d=2,\\ 3c_0\lambda_m^{\frac{1}{4}}, & d=3,\end{cases}$$

we have

$$|A^{\frac{1}{2}}\boldsymbol{v}|_{L^\infty}\le\frac{L_m^d}{3}\|\boldsymbol{v}\|_2,\quad|\boldsymbol{w}_m|_{L^\infty}\le\frac{L_m^d}{3}\|\boldsymbol{w}\|.$$

Then (4.5) becomes

$$|A^{\frac{1}{2}}\tilde{B}(\boldsymbol{w},\boldsymbol{v})|\le\frac{2c_bL_m^d}{3}\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|_2+\frac{|\tilde{b}(\bar{\boldsymbol{w}},\delta\boldsymbol{v},\boldsymbol{\psi})|}{|\boldsymbol{\psi}|}.\tag{4.6}$$

As for the last term on the right hand side of (4.6), we have

$$\frac{|\tilde{b}(\bar{\boldsymbol{w}},\delta\boldsymbol{v},\boldsymbol{\psi})|}{|\boldsymbol{\psi}|}\leq\begin{cases} c_b\|\bar{\boldsymbol{w}}\|_{\frac{1}{2}}|A\boldsymbol{v}|_{\frac{1}{2}}\leq c_b\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|_2, & d=2,\\ c_b\|\bar{\boldsymbol{w}}\|\,|A\boldsymbol{v}|_{\frac{1}{2}}\leq c_b\lambda_m^{\frac{1}{4}}\|\boldsymbol{w}\|\,\|\boldsymbol{v}\|_2, & d=3.\end{cases}$$

Combining this estimation with (4.6) and noting

$$|b(\boldsymbol{\phi},\boldsymbol{v},\boldsymbol{w})|=|\tilde{b}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{\phi})|\leq|A^{-\frac{1}{2}}\boldsymbol{\phi}|\,|A^{\frac{1}{2}}\tilde{B}(\boldsymbol{w},\boldsymbol{v})|,$$

we can get the result. □

Now, let us consider the projection Q_m^1. We can rewrite the Navier-Stokes equations (2.2) as

$$\mathcal{L}_{m,1}(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{u},\boldsymbol{u}_m,\boldsymbol{v})-b(\boldsymbol{u}_m,\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in V.$$

Using (4.4) and (2.3), we have

$$\mathcal{L}_{m,1}(Q_m^1\boldsymbol{u},\boldsymbol{v})+\mathcal{L}_{m,1}(\hat{\boldsymbol{u}}^1,\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})-b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall v\in V. \tag{4.7}$$

Meanwhile, the standard spectral Galerkin approximate equations (2.6) can also be rewritten as

$$\mathcal{L}_{m,1}(\boldsymbol{u}_m,\boldsymbol{v})-b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in H_m. \tag{4.8}$$

Restricting (4.7) to H_m then subtracting (4.8) from (4.7) and using (3.8), we have

$$\mathcal{L}_(\boldsymbol{e}^1,\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in H_m, \tag{4.9}$$

where $\boldsymbol{e}^1=Q_m^1\boldsymbol{u}-\boldsymbol{u}_m$. For the sake of convenience, we sometimes use $\boldsymbol{e}^i$ to denote $Q_m^i\boldsymbol{u}-\boldsymbol{u}_m$ for $i=1,2,3$ in the rest. Now by corollary 3.1, we know

$$\|\boldsymbol{e}^1\|\leq\frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in H_m}\frac{\mathcal{L}_{m,1}(\boldsymbol{e}^1,\boldsymbol{v})}{\|\boldsymbol{v}\|}\leq\frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in H_m}\frac{|b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})|}{\|\boldsymbol{v}\|}.$$

For $d=2$, we have

$$|b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})|=|b(\boldsymbol{\eta},\boldsymbol{v},\boldsymbol{\eta})|\leq c_b|\boldsymbol{\eta}|\,\|\boldsymbol{\eta}\|\,\|\boldsymbol{v}\|,$$

and for $d=3$,

$$|b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})|\leq c_b\|\boldsymbol{\eta}\|_{\frac{1}{2}}\|\boldsymbol{\eta}\|\,\|\boldsymbol{v}\|\leq c_0c_b|\boldsymbol{\eta}|^{\frac{1}{2}}\|\boldsymbol{\eta}\|^{\frac{3}{2}}\|\boldsymbol{v}\|.$$

Then we have

$$\|\boldsymbol{e}^1\|\leq\frac{2c_4}{\alpha_0}|\boldsymbol{\eta}|^{1-\varepsilon}\,\|\boldsymbol{\eta}\|^{1+\varepsilon}, \tag{4.10}$$

where

$$\varepsilon:=\begin{cases}0, & \text{for } d=2,\\ \frac{1}{2}, & \text{for } d=3,\end{cases}\qquad c_4=\max\{c_b,c_0c_b\}.$$

For the second projection Q_m^2, the Navier-Stokes equations (2.2) and its standard spectral Galerkin approximate equations (2.6) can be rewritten as

$$\mathcal{L}_{m,2}(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in V, \tag{4.11}$$

$$\mathcal{L}_{m,2}(\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in H_m. \tag{4.12}$$

Then by using (4.4) and (3.12), we have

$$\mathcal{L}_{m,2}(\boldsymbol{e}^2,\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v})=0,\quad \forall \boldsymbol{v}\in H_m.$$

Taking $\boldsymbol{v}=\boldsymbol{e}^2$ yields

$$\nu\|\boldsymbol{e}^2\|^2\le|b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{e}^2)|+|b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{e}^2)|\le c_4|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}\|\boldsymbol{e}^2\|+|b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{e}^2)|.$$

For the second term on the right hand side of the last inequality, we have from lemma 4.1

$$|b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{e}^2)|\le c_bL_m^d\|\boldsymbol{u}_m\|_2|A^{-\frac{1}{2}}\boldsymbol{\eta}|\,\|\boldsymbol{e}^2\|.$$

Then,

$$\|\boldsymbol{e}^2\|\le\frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}+\frac{c_bL_m^d\|\boldsymbol{u}_m\|_2}{\nu}|A^{-\frac{1}{2}}\boldsymbol{\eta}|. \tag{4.13}$$

The last projection Q_m^3 is indeed the standard L^2- orthogonal projection P_m. Now, (2.2) and (2.6) can be rewritten as

$$\mathcal{L}_{m,3}(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{u}_m,\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v})+b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in V, \tag{4.14}$$

$$\mathcal{L}_{m,3}(\boldsymbol{u}_m,\boldsymbol{v})+b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in H_m. \tag{4.15}$$

Thus

$$\mathcal{L}_{m,3}(\boldsymbol{e}^3,\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{u}_m,\boldsymbol{\eta},\boldsymbol{v})+b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in H_m.$$

Taking $\boldsymbol{v}=\boldsymbol{e}^3$ in the above equality yields

$$\nu\|\boldsymbol{e}^3\|^2\le|b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{e}^3)|+|b(\boldsymbol{u}_m,\boldsymbol{\eta},\boldsymbol{e}^3)|+|b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{e}^3)|\le c_4|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}\|\boldsymbol{e}^3\|+2c_b\|\boldsymbol{u}_m\|_{L^\infty}|\boldsymbol{\eta}|\,\|\boldsymbol{e}^3\|.$$

Therefore,

$$\|\boldsymbol{e}^3\|\le\frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}+\frac{2c_b\|\boldsymbol{u}_m\|_{L^\infty}}{\nu}|\boldsymbol{\eta}|. \tag{4.16}$$

Now, we can summarize (4.10), (4.13) and (4.16) into following

Theorem 4.1. *Suppose $\boldsymbol{f}\in H$, $\partial\Omega$ be of class C^2. Then*

$$\|Q_m^2\boldsymbol{u}-\boldsymbol{u}_m\|\le\frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}+\frac{c_bL_m^d\|\boldsymbol{u}_m\|_2}{\nu}|A^{-\frac{1}{2}}\boldsymbol{\eta}|,$$

$$\|Q_m^3\boldsymbol{u}-\boldsymbol{u}_m\|\le\frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}+\frac{2c_b\|\boldsymbol{u}_m\|_{L^\infty}}{\nu}|\boldsymbol{\eta}|.$$

Furthermore, if the assumptions in lemma 3.2 holds, we have

$$\|Q_m^1\boldsymbol{u}-\boldsymbol{u}_m\|\le\frac{2c_4}{\alpha_0}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}.$$

From (2.7), we easily see that (4.1) is satisfied for all our projections. In addition, it is worth paying attention to Q_m^3. As we said, $Q_m^3\equiv P_m$. Then this result shows that

$$\|P_m\boldsymbol{u}-\boldsymbol{u}_m\|=o(\|\boldsymbol{u}-\boldsymbol{u}_m\|).$$

That is

$$\|P_m\boldsymbol{u}-\boldsymbol{u}_m\|=o(\|(I-P_m)\boldsymbol{u}\|).$$

This result tells us that the error of the standard Galerkin method is dominated by the truncation error and also indicates the standard spectral Galerkin method has some superconvergence property.

The previous theorem 4.1 shows that $\|\boldsymbol{e}^i\|$ has a higher order than $\|\boldsymbol{\eta}\|$. So, it is the higher frequency components of the particular projection Q_m^i that restricts the approximate order, that is $\hat{\boldsymbol{u}}^i$. Also as we said before, if we could find some finite dimensional mapping $\boldsymbol{\Phi}^i$ from H_m to $\hat{V}^i$ such that (4.2) are satisfied, we can get a more accuracy approximation of $\boldsymbol{u}$ based on $\boldsymbol{u}_m$, that is (4.3).

Now we give the definitions of our finite dimensional mappings and some basic properties of them. For any $\boldsymbol{w} \in H_m$, find $\boldsymbol{\Phi}^i(\boldsymbol{w}) \in \hat{V}^i$, $i = 1, 2, 3$, such that

$$\mathcal{L}_{m,1}(\boldsymbol{\Phi}^1(\boldsymbol{w}), \boldsymbol{v}) = b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) - \mathcal{L}_{m,1}(\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^1, \tag{4.17}$$

$$\mathcal{L}_{m,2}(\boldsymbol{\Phi}^2(\boldsymbol{w}), \boldsymbol{v}) = -\mathcal{L}_{m,2}(\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^2, \tag{4.18}$$

$$\mathcal{L}_{m,3}(\boldsymbol{\Phi}^3(\boldsymbol{w}), \boldsymbol{v}) = -b(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}) - \mathcal{L}_{m,3}(\boldsymbol{w}, \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^3. \tag{4.19}$$

Theorem 4.2. *i) Under the assumptions of lemma 3.2 and the following restriction on m*

$$\lambda_{m+1} \geq \frac{4c_b^2 c_3^2 \|\boldsymbol{u}_m\|^2}{\nu^2}, \tag{4.20}$$

(4.17) can define a single valued mapping

$$\boldsymbol{\Phi}^1 : H_m \to \hat{V}^1.$$

ii) (4.18) and (4.19) can define the following single valued mappings respectively.

$$\boldsymbol{\Phi}^2 : H_m \to \hat{V}^2, \quad \boldsymbol{\Phi}^3 : H_m \to \hat{V}^3.$$

iii) If we denote

$$\mathcal{M}^i = Graph(\boldsymbol{\Phi}^i), \quad i = 1, 2, 3,$$

$\mathcal{M}^i$ is a Lipschitz manifold with

$$\|\boldsymbol{\Phi}^i(\boldsymbol{w}_1) - \boldsymbol{\Phi}^i(\boldsymbol{w}_2)\| \leq l^i \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|, \quad \forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in H_m \cap B_\rho,$$

where $B_\rho = \{\boldsymbol{v} \in V, \|\boldsymbol{v}\| \leq \rho\}$ and l^i is a Lipschitz constant depending on ρ, $\|\boldsymbol{u}_m\|$, ν and α_0.

Proof. i) From lemma 3.2 and (4.20), for any $\boldsymbol{\phi} \in \hat{V}^1$,

$$\mathcal{L}_{m,1}(\boldsymbol{\phi}, \boldsymbol{\phi}) = \nu\|\boldsymbol{\phi}\|^2 + b(\boldsymbol{\phi}, \boldsymbol{u}_m, \boldsymbol{\phi}) \geq \nu\|\boldsymbol{\phi}\|^2 - c_b c_3 \lambda_{m+1}^{-\frac{1}{2}} \|\boldsymbol{u}_m\| \, \|\boldsymbol{\phi}\|^2 \geq \frac{\nu}{2}\|\boldsymbol{\phi}\|^2.$$

Then by Lax-Milgram theorem, we can immediately get the result.

ii) It is obvious that for any $\boldsymbol{\phi} \in \hat{V}^i$, $i = 2, 3$

$$\mathcal{L}_{m,2}(\boldsymbol{\phi}, \boldsymbol{\phi}) = \nu\|\boldsymbol{\phi}\|^2, \quad \mathcal{L}_{m,3}(\boldsymbol{\phi}, \boldsymbol{\phi}) = \nu\|\boldsymbol{\phi}\|^2.$$

Again, by Lax-Milgram theorem, we can get the result.

iii) Let us introduce some symbols.

$$\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in H_m \cap B_\rho, \quad \text{denote } \; \boldsymbol{\phi}_1^i = \boldsymbol{\Phi}^i(\boldsymbol{w}_1), \; \boldsymbol{\phi}_2^i = \boldsymbol{\Phi}^i(\boldsymbol{w}_2), \; i = 1, 2, 3.$$

First, we consider $\mathbf{\Phi}^1$. It is easily to see that

$$\begin{aligned}\mathcal{L}_{m,1}(\boldsymbol{\phi}_1^1-\boldsymbol{\phi}_2^1,\boldsymbol{v}) =& b(\boldsymbol{w}_1,\boldsymbol{w}_1,\boldsymbol{v})-b(\boldsymbol{w}_2,\boldsymbol{w}_2,\boldsymbol{v})-\mathcal{L}_{m,1}(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v})\\ =& b(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{w}_1,\boldsymbol{v})+b(\boldsymbol{w}_2,\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v})-\mathcal{L}_{m,1}(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v}),\quad \forall \boldsymbol{v}\in\hat{V}^1.\end{aligned}$$

From corollary 3.1, (3.8) and the above expression, we obtain

$$\begin{aligned}\|\boldsymbol{\phi}_1^1-\boldsymbol{\phi}_2^1\| \leq& \frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in V}\frac{\mathcal{L}_{m,1}(\boldsymbol{\phi}_1^1-\boldsymbol{\phi}_2^1,\boldsymbol{v})}{\|\boldsymbol{v}\|}=\frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in\hat{V}^1}\frac{\mathcal{L}_{m,1}(\boldsymbol{\phi}_1^1-\boldsymbol{\phi}_2^1,\boldsymbol{v})}{\|\boldsymbol{v}\|}\\ \leq& \frac{2}{\alpha_0}(2c_b\rho+\nu+2c_b\|\boldsymbol{u}_m\|)\|\boldsymbol{w}_1-\boldsymbol{w}_2\| \triangleq l^1\|\boldsymbol{w}_1-\boldsymbol{w}_2\|.\end{aligned}$$

This proves the result for $\mathbf{\Phi}^1$.

For $\mathbf{\Phi}^2$, we have

$$\mathcal{L}_{m,2}(\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2,\boldsymbol{v})=-\mathcal{L}_{m,2}(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v}).$$

Taking $\boldsymbol{v}=\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2$,

$$\nu\|\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2\|^2\leq\nu\|\boldsymbol{w}_1-\boldsymbol{w}_2\|\,\|\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2\|+c_b\|\boldsymbol{u}_m\|\,\|\boldsymbol{w}_1-\boldsymbol{w}_2\|\,\|\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2\|.$$

Thus, we have

$$\|\boldsymbol{\phi}_1^2-\boldsymbol{\phi}_2^2\|\leq(1+\frac{c_b\|\boldsymbol{u}_m\|}{\nu})\|\boldsymbol{w}_1-\boldsymbol{w}_2\|\triangleq l^2\|\boldsymbol{w}_1-\boldsymbol{w}_2\|.$$

At last, for $\mathbf{\Phi}^3$,

$$\mathcal{L}_{m,3}(\boldsymbol{\phi}_1^3-\boldsymbol{\phi}_2^3,\boldsymbol{v})=-b(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{w}_1,\boldsymbol{v})-b(\boldsymbol{w}_2,\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v})-\mathcal{L}_{m,3}(\boldsymbol{w}_1-\boldsymbol{w}_2,\boldsymbol{v}).$$

We can easily get

$$\|\boldsymbol{\phi}_1^3-\boldsymbol{\phi}_2^3\|\leq(1+\frac{2c_b\rho}{\nu})\|\boldsymbol{w}_1-\boldsymbol{w}_2\|\triangleq l^3\|\boldsymbol{w}_1-\boldsymbol{w}_2\|.$$

□

From the proof of this theorem, we see that $\mathbf{\Phi}^1$ and $\mathbf{\Phi}^3$ are local Lipschitz continuous and $\mathbf{\Phi}^2$ is global Lipschitz continuous.

5. Inertial Algorithms

After we get the finite dimensional mappings $\mathbf{\Phi}^i$, $i=1,2,3$, we can construct the following three inertial algorithms easily with respect to each projection to get a more accuracy approximation of $\boldsymbol{u}$ based on $\boldsymbol{u}_m$.

Inertial Algorithms 1.

(Step 1)

Solve (2.6) to get the standard spectral Galerkin approximation $\boldsymbol{u}_m\in H_m$.

(Step 2)

$$\begin{cases}\textit{find}\ \ \tilde{\boldsymbol{u}}^1=\mathbf{\Phi}^1(\boldsymbol{u}_m)\in\hat{V}^1\ \ \textit{such that}\\ \mathcal{L}_{m,1}(\tilde{\boldsymbol{u}}^1,\boldsymbol{v})=b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{v})-\mathcal{L}_{m,1}(\boldsymbol{u}_m,\boldsymbol{v})+(\boldsymbol{f},\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\hat{V}^1.\end{cases}$$

(Step 3)

Get the new approximation: $\boldsymbol{u}^1=\boldsymbol{u}_m+\tilde{\boldsymbol{u}}^1$.

Theorem 5.1. *Suppose the assumptions in lemma 3.2 are fulfilled. Then the inertial algorithm 1 admits*

$$\|\boldsymbol{u}-\boldsymbol{u}^1\| \leq c_5|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon},$$

where c_5 is a positive constant.

Proof. Let $\boldsymbol{v} \in \hat{V}^1$ in (4.7) and subtract the equations of (Step 2) from it, we have

$$\mathcal{L}_{m,1}(\hat{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^1, \boldsymbol{v}) = -b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v}) - \mathcal{L}_{m,1}(\boldsymbol{e}^1,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^1.$$

By (3.6), (3.8) and noticing theorem 4.1, we find

$$\begin{aligned}
\|\hat{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^1\| \leq & \frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in V}\frac{\mathcal{L}_{m,1}(\hat{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^1, \boldsymbol{v})}{\|\boldsymbol{v}\|} = \frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in \hat{V}^1}\frac{\mathcal{L}_{m,1}(\hat{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^1, \boldsymbol{v})}{\|\boldsymbol{v}\|} \\
= & \frac{2}{\alpha_0}\sup_{\boldsymbol{v}\in \hat{V}^1}\frac{-b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v}) - \mathcal{L}_{m,1}(\boldsymbol{e}^1,\boldsymbol{v})}{\|\boldsymbol{v}\|} \leq \frac{2c_4}{\alpha_0}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon} + \frac{2(\nu + 2c_b\|\boldsymbol{u}_m\|)}{\alpha_0}\|\boldsymbol{e}^1\| \\
\leq & \frac{2c_4}{\alpha_0^2}(\alpha_0 + 2\nu + 4c_b\|\boldsymbol{u}_m\|)|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}.
\end{aligned}$$

If we denote $c_5 := \dfrac{2c_4}{\alpha_0^2}(2\alpha_0 + 2\nu + 4c_b\|\boldsymbol{u}_m\|)$, we can get the result. □

Inertial Algorithms 2.

(Step 1)

Solve (2.6) to get the standard spectral Galerkin approximation $\boldsymbol{u}_m \in H_m$.

(Step 2)

$$\begin{cases} \textit{find} \quad \tilde{\boldsymbol{u}}^2 = \boldsymbol{\Phi}^2(\boldsymbol{u}_m) \in \hat{V}^2 \quad \textit{such that} \\ \mathcal{L}_{m,2}(\tilde{\boldsymbol{u}}^2,\boldsymbol{v}) = -\mathcal{L}_{m,2}(\boldsymbol{u}_m,\boldsymbol{v}) + (\boldsymbol{f},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^2. \end{cases}$$

(Step 3)

Get the new approximation: $\boldsymbol{u}^2 = \boldsymbol{u}_m + \tilde{\boldsymbol{u}}^2$.

Theorem 5.2. *The solution of inertial algorithm 2 admits*

$$\|\boldsymbol{u}-\boldsymbol{u}^2\| \leq c_6 L_m^d |A^{-\frac{1}{2}}\boldsymbol{\eta}| + c_7|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon},$$

where c_6, c_7 are positive constants.

Proof. We can rewrite the Navier-Stokes equations (4.11) as

$$\mathcal{L}_{m,2}(Q_m^2\boldsymbol{u},\boldsymbol{v}) + \mathcal{L}_{m,2}(\hat{\boldsymbol{u}}^2,\boldsymbol{v}) + b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v}) + b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V. \tag{5.1}$$

Restricting (5.1) to $\hat{V}^2$ and subtracting the equations of (Step 2) in inertial algorithm 2, we have

$$\mathcal{L}_{m,2}(\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2,\boldsymbol{v}) = -b(\boldsymbol{\eta},\boldsymbol{\eta},\boldsymbol{v}) - b(\boldsymbol{\eta},\boldsymbol{u}_m,\boldsymbol{v}) - \mathcal{L}_{m,2}(\boldsymbol{e}^2,\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^2. \tag{5.2}$$

Taking $\boldsymbol{v} = \hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2 \in \hat{V}^2$, (5.2) yields

$$\begin{aligned}
\nu\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\|^2 \leq & c_4|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon}\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\| + c_b L_m^d\|\boldsymbol{u}_m\|_2|A^{-\frac{1}{2}}\boldsymbol{\eta}|\,\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\| \\
& + \nu\|\boldsymbol{e}^2\|\,\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\| + c_b\|\boldsymbol{u}_m\|\,\|\boldsymbol{e}^2\|\,\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\|.
\end{aligned}$$

By theorem 4.1, we have

$$\begin{aligned}
\|\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}}^2\| \leq & \frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon} + \frac{c_b L_m^d\|\boldsymbol{u}_m\|_2}{\nu}|A^{-\frac{1}{2}}\boldsymbol{\eta}| + (1 + \frac{c_b\|\boldsymbol{u}_m\|}{\nu})\|\boldsymbol{e}^2\| \\
\leq & \frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon} + \frac{c_b L_m^d\|\boldsymbol{u}_m\|_2}{\nu}|A^{-\frac{1}{2}}\boldsymbol{\eta}| + (1 + \frac{c_b\|\boldsymbol{u}_m\|}{\nu})\frac{c_4}{\nu}|\boldsymbol{\eta}|^{1-\varepsilon}\|\boldsymbol{\eta}\|^{1+\varepsilon} \\
& + (1 + \frac{c_b\|\boldsymbol{u}_m\|}{\nu})\frac{c_b L_m^d\|\boldsymbol{u}_m\|_2}{\nu}|A^{-\frac{1}{2}}\boldsymbol{\eta}|.
\end{aligned}$$

Now we denote $c_6 := \dfrac{3\nu c_4 + c_b c_4\|\boldsymbol{u}_m\|}{\nu^2}$ and $c_7 := \dfrac{3\nu c_b\|\boldsymbol{u}_m\|_2 + c_b^2\|\boldsymbol{u}_m\|\,\|\boldsymbol{u}_m\|_2}{\nu^2}$, then we can get the result by triangle inequality. □

Inertial Algorithm 3.

(Step 1)

Solve (2.6) to get the standard spectral Galerkin approximation $\boldsymbol{u}_m \in H_m$.

(Step 2)

$$\begin{cases} \textit{find} \quad \tilde{\boldsymbol{u}}^3 = \boldsymbol{\Phi}(\boldsymbol{u}_m) \in \hat{V}^3 \quad \textit{such that} \\ \mathcal{L}_{m,3}(\tilde{\boldsymbol{u}}^3, \boldsymbol{v}) = -b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v}) - \mathcal{L}_{m,3}(\boldsymbol{u}_m, \boldsymbol{v}) + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^3. \end{cases}$$

(Step 3)

Get the new approximation: $\boldsymbol{u}^3 = \boldsymbol{u}_m + \tilde{\boldsymbol{u}}^3$.

Theorem 5.3. *The solution of inertial algorithm 3 admits*

$$||\boldsymbol{u} - \boldsymbol{u}^3|| \leq c_8 |\boldsymbol{\eta}|^{1-\varepsilon} ||\boldsymbol{\eta}||^{1+\varepsilon} + c_9 |\boldsymbol{\eta}|,$$

where c_8, c_9 *are positive constants.*

Proof. The proof of this theorem is very similar to that of theorem 5.2. First, we can get a new form of the Navier-Stokes equations just like (5.1) and then restrict it in $\hat{V}^3$ and subtract the equations of (Step 2) of this algorithm from it. Then we get

$$\mathcal{L}_{m,3}(\hat{\boldsymbol{u}}^3 - \tilde{\boldsymbol{u}}^3, \boldsymbol{v}) = -b(\boldsymbol{\eta}, \boldsymbol{\eta}, \boldsymbol{v}) - b(\boldsymbol{\eta}, \boldsymbol{u}_m, \boldsymbol{v}) - b(\boldsymbol{u}_m, \boldsymbol{\eta}, \boldsymbol{v}) - \mathcal{L}_{m,3}(\boldsymbol{e}^2, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \hat{V}^3. \tag{5.3}$$

Taking $\boldsymbol{v} = \hat{\boldsymbol{u}}^3 - \tilde{\boldsymbol{u}}^3$, (5.3) admits

$$||\hat{\boldsymbol{u}}^3 - \tilde{\boldsymbol{u}}^3|| \leq \frac{c_4}{\nu} |\boldsymbol{\eta}|^{1-\varepsilon} ||\boldsymbol{\eta}||^{1+\varepsilon} + \frac{2c_b ||\boldsymbol{u}_m||_{L^\infty}}{\nu} |\boldsymbol{\eta}| + ||\boldsymbol{e}^3|| \leq \frac{2c_4}{\nu} |\boldsymbol{\eta}|^{1-\varepsilon} ||\boldsymbol{\eta}||^{1+\varepsilon} + \frac{4c_b ||\boldsymbol{u}_m||_{L^\infty}}{\nu} |\boldsymbol{\eta}|.$$

Let us introduce $c_8 := \dfrac{3c_4}{\nu}$ and $c_9 := \dfrac{6c_b ||\boldsymbol{u}_m||_{L^\infty}}{\nu}$, then we can get the result. □

Remark. From the above three convergence theorems, we obtain

$$||\boldsymbol{u} - \boldsymbol{u}^i|| = o(||\boldsymbol{\eta}||), \quad i = 1, 2, 3.$$

That is the inertial algorithms can really be more accurate for $\boldsymbol{u}$. Comparing the above three algorithms, inertial algorithm 1 has the highest order of accuracy but its applicable range is restricted by the nonsingularity condition and that m also must be great than some lower bound. The inertial algorithm 3 has the simplest form and also is the easiest one to be implemented numerically. In fact, we can directly get $\tilde{\boldsymbol{u}}^3$. Another advantage compared with algorithm 1 is that it is valid whether $\boldsymbol{u}$ is a nonsingular solution of the Navier-Stokes equations or not. But its disadvantage is also obvious. Its accuracy is of course the worst one. All kinds of properties of inertial algorithm 2 are just between the inertial algorithm 1 and the inertial algorithm 3. Meanwhile, its applicability is just as for algorithm 3 and its convergence rate is very close to algorithm 1. In fact, for a three dimensional case, it has the same approximation order as that of inertial algorithm 1.

There is another problem we need to cope with the numerical implementation of these three algorithms. As we said, algorithm 3 is the simplest one because of

$$\hat{V}^3 \equiv (I - P_m)V.$$

For algorithm 1 and 2, the situation is quite different. If we consider the standard L^2- orthogonal projection P_m, the elements of $\hat{V}^1$ and $\hat{V}^2$ may contain the usual higher frequency components as well as the usual lower frequency components. Indeed, their orthogonal basis functions with respect to projections Q_m^1 and Q_m^2 are not at hand and deriving the basis functions is not easier than solving any Navier-Stokes equations. So they will be rather theoretical algorithms till we can find a good way to get $\hat{V}^1$ and $\hat{V}^2$. In the following, we will modify inertial algorithm 1 and 2 above such that their numerical implementation becomes possible without loss of accuracy they have.

Inertial Algorithm 1'.

(Step 1)

Solve (2.6) to get the standard spectral Galerkin approximation $\boldsymbol{u}_m \in H_m$.

(Step 2)

$$\begin{cases} \textit{find} \ \ \tilde{\boldsymbol{u}}^4 \in V \ \ \textit{such that} \\ \mathcal{L}_{m,1}(\tilde{\boldsymbol{u}}^4, \boldsymbol{v}) = ((I - P_m)\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V. \end{cases}$$

(Step 3)

Get the new approximation: $\boldsymbol{u}^4 = \boldsymbol{u}_m + \tilde{\boldsymbol{u}}^4$.

Inertial Algorithm 2'.

(Step 1)

Solve (2.6) to get the standard spectral Galerkin approximation $\boldsymbol{u}_m \in H_m$.

(Step 2)

$$\begin{cases} \textit{find} \ \ \tilde{\boldsymbol{u}}^5 \in V \ \ \textit{such that} \\ \mathcal{L}_{m,2}(\tilde{\boldsymbol{u}}^5, \boldsymbol{v}) = ((I - P_m)\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V. \end{cases}$$

(Step 3)

Get the new approximation: $\boldsymbol{u}^5 = \boldsymbol{u}_m + \tilde{\boldsymbol{u}}^5$.

In fact, we only enlarged the test function space from $\hat{V}^1$ and $\hat{V}^2$ to V in each (Step 2). At this time, the right hand side terms of the equations in (Step 2) can be simplified by using the standard Galerkin approximation equations (2.6). Of course, we cannot restrict $\tilde{\boldsymbol{u}}^4$ and $\tilde{\boldsymbol{u}}^5$ in the original higher frequency spaces now. But a simple analysis will show that this kind of modification will not influence the convergence rates of the original inertial algorithm 1 and 2. On the other hand, because $\tilde{\boldsymbol{u}}^4$ and $\tilde{\boldsymbol{u}}^5$ are now sought in the whole space V, whose dimension is a little larger than that of $\tilde{V}^1$ and $\tilde{V}^2$. In addition, the right hand side of the equations of (Step 2) in inertial algorithm 3 can also be rewritten as

$$\mathcal{L}_{m,3}(\tilde{\boldsymbol{u}}^3, \boldsymbol{v}) = ((I - P_m)\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in (I - P_m)V$$

because of $Q_m^3 \equiv P_m$ and $\hat{V}^3 \equiv (I - P_m)V$.

Theorem 5.4. *i) Under the assumptions of theorem 5.1, the inertial algorithm 1' yields*

$$||\boldsymbol{u} - \boldsymbol{u}^4|| \le c_5 |\boldsymbol{\eta}|^{1-\varepsilon} ||\boldsymbol{\eta}||^{1+\varepsilon}.$$

ii) The inertial algorithm 2' admits

$$||\boldsymbol{u} - \boldsymbol{u}^5|| \le c_6 L_m^d |A^{-\frac{1}{2}}\boldsymbol{\eta}| + c_7 |\boldsymbol{\eta}|^{1-\varepsilon} ||\boldsymbol{\eta}||^{1+\varepsilon}.$$

The proof of this theorem is very similar to that of theorem 5.1 and 5.2, thus we omit it.

6. Numerical Examples

To illustrate the better convergence rate of the proposed inertial algorithms, we will give some numerical results for the two-dimensional Kolmogorov flow on $\Omega = [-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}] \times [-\pi, \pi]$,

$$\begin{cases} \text{find } \boldsymbol{u} \in V \ \ \text{such that} \\ a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V, \end{cases} \tag{6.1}$$

where

$$V := \{\boldsymbol{v} = \sum_{k \in Z^2, k \neq 0} \boldsymbol{v}_k e^{i(k_1 \alpha x + k_2 y)}, \boldsymbol{v}_k = \overline{\boldsymbol{v}_{-k}}, \sum_{k \in Z^2, k \neq 0} |\boldsymbol{v}_k|^2 (k_1^2 \alpha^2 + k_2^2) < +\infty, \text{div}\boldsymbol{v} = 0\},$$

$\boldsymbol{f} := \frac{1}{Re}(\sin y, 0)^T$, $Re > 0$ is the Reynolds number defined in [11]. From the literature of Kolmogorov flows (e.g. [11], [12]), the above equation has a trivial solution $\boldsymbol{u} = (\sin y, 0)^T$ for any Reynolds number, and when we take $\alpha = 0.7$, bifurcation occurs at $Re^* = 3.01119\cdots$. For $\alpha = 1$, there will be no bifurcation points for any $Re > 0$.

In the following, we will give some numerical results of inertial algorithm 1', 2' and 3.

Remark. Note that step 2 in these algorithms has to be solved in the whole space V. To solve them numerically, we should project them onto another larger finite dimensional space H_M with $M \gg m$ and so we can get new approximations

$$\boldsymbol{u}_M^i = \boldsymbol{u}_m + \tilde{\boldsymbol{u}}_M^i, \quad i = 3, 4, 5. \tag{6.2}$$

Generally, if we choose M sufficiently large, then the error of these finite approximations given by (6.2) should be equivalent to the error order in theorem 5.4 and theorem 5.3 plus a truncation error. That is

$$\|\boldsymbol{u} - \boldsymbol{u}_M^i\| = o(\|\boldsymbol{u} - \boldsymbol{u}^i\|) + o(\lambda_{M+1}^{-\frac{1}{2}}), \quad i = 3, 4, 5. \tag{6.3}$$

For this concrete problem, the true solution has only two modes, so we only choose $M = 2m$ and $m = 9$. The associated algebraic equations are solved by some LINPACK subroutines.

In Table 1, we give some results of these three algorithms for $\alpha = 0.7$ and $Re \to Re^*$. It is obvious that the inertial algorithm 1'(inertial algorithm 1) will lose its higher convergence rate near the singular point because α_0 may tend to zero when Re tends to Re^*. And the other two algorithms will still be valid. And our numerical results in table 1 just indicates this kind of phenomena. And it seems inertial algorithm 2' and 3 has a better performance than the inertial algorithm 1' when Re tends to the bifurcation point.

We denote by IA1', IA2', IA3 and SGM the names of inertial algorithm 1', 2', 3 and standard Galerkin method respectively.

Table 1

Re	IA1' ($\|\boldsymbol{u} - \boldsymbol{u}_M^4\|$)	IA2' ($\|\boldsymbol{u} - \boldsymbol{u}_M^5\|$)	IA3 ($\|\boldsymbol{u} - \boldsymbol{u}_M^3\|$)	SGM ($\|\boldsymbol{u} - \boldsymbol{u}_m\|$)
1.00	$2.06E-06$	$2.15E-06$	$1.02E-06$	$1.88E-04$
2.00	$1.17E-05$	$7.48E-06$	$4.03E-06$	$3.69E-04$
2.50	$2.86E-05$	$1.09E-05$	$6.25E-06$	$4.59E-04$
3.00	$3.54E-04$	$1.49E-05$	$8.96E-06$	$5.48E-04$
3.01	$4.62E-04$	$1.52E-05$	$9.15E-06$	$5.53E-04$
Re^*	$----$	$1.54E-05$	$9.26E-06$	$5.64E-04$

Here '−−' means that the condition number of the associated matrix is very close to zero and the Gauss elimination cannot process it.

For $\alpha = 1$, we know that there will be no bifurcation in the system. So we can observe the performance of these three algorithms when Re becomes more and more large. The following table 2 gives the numerical results related to this procedure.

Table 2

Re	IA1' ($\|\boldsymbol{u} - \boldsymbol{u}_M^4\|$)	IA2' ($\|\boldsymbol{u} - \boldsymbol{u}_M^5\|$)	IA3 ($\|\boldsymbol{u} - \boldsymbol{u}_M^3\|$)	SGM ($\|\boldsymbol{u} - \boldsymbol{u}_m\|$)
1.00	$0.00E-00$	$0.00E-00$	$0.00E-00$	$2.07E-04$
10.0	$6.35E-07$	$2.61E-07$	$2.63E-07$	$2.24E-03$
100.	$2.78E-05$	$2.98E-05$	$2.45E-04$	$2.25E-02$
200.	$1.06E-04$	$1.21E-04$	$1.92E-03$	$4.50E-02$
300.	$2.67E-04$	$2.88E-04$	$6.30E-03$	$6.75E-02$
1000	$1.53E-02$	$2.89E-02$	$1.90E-01$	$2.30E-01$

From Table 2, we see that all algorithms lose some accuracy when Re becomes large as expected. And the numerical tests also tell us that when there is no bifurcation point along the path of Re, the performance of inertial algorithm 1' and 2' seems to be more accurate than that of inertial algorithm 3 when Re tends to infinity. So when we want to perform a numerical simulation at high Reynolds number, we prefer inertial algorithm 1' and 2', especially the inertial algorithm 2' which can process whenever there will be a bifurcation point or not along the path of Re and has almost the same accuracy as inertial algorithm 1'.

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