An Improved Model of Nonlinear Fiber Propagation in the Presence of Kerr Nonlinearity and Stimulated Raman Scattering

*Citation for published version (APA):*
A General Analytical Model of Nonlinear Fiber Propagation in the Presence of Kerr Nonlinearity and Stimulated Raman Scattering

Hami Rabbani, Gabriele Liga, Member, IEEE, Vinícius Oliari, Lotfollah Beygi, Erik Agrell, Fellow, IEEE, Magnus Karlsson, Senior Member, IEEE, Fellow, OSA, Alex Alvarado, Senior Member, IEEE

Abstract—Ultra-wideband fiber optical transmission suffers from nonlinear interference (NLI) noise caused by both Kerr nonlinearity and stimulated Raman scattering (SRS). Mathematical models that address the interplay between Kerr nonlinearity and SRS exist. Most of these models are based on the Gaussian noise (GN) model which is based on the assumption that the transmitted signal, after transmission on a dispersive fiber, is Gaussian. Similarly to the case where SRS is not present or neglected, these models overestimate the NLI power. This problem can be partially solved by adding modulation format-dependent correction terms. In this paper, we introduce a general model that accounts for both Kerr nonlinearity and SRS, accounting for all terms of nonlinear interactions, including self-channel interference, cross-channel interference, and multi-channel interference. The model lifts the Gaussianity assumption and can handle different modulation formats over different wavelength channels, different symbol rates, multi-span systems with different fibers, and hybrid amplification schemes. Numerical results indicate that when both SRS and arbitrary modulation formats are considered, previous models may inaccurately predict the NLI power. This difference could be up to 4.3 dB for a 10.011 THz system with 1001 channels at 10 Gbaud. Split-step Fourier simulations support our analytical results.

Index Terms—Coherent transmission, C+L band transmission, Gaussian noise model, Kerr nonlinearity, nonlinear stimulated Raman scattering effect, Optical fiber communications.

I. INTRODUCTION

The tremendous growth in the demand for high data rates is gradually leading to a capacity crunch of optical networks operating transmission in the C-band [1]. To cope with this capacity shortage, transmission in the C+L band and beyond is currently seen by the optical communication community as one of the most promising solutions (see e.g., [2]–[4]). The most dominant factor currently restricting the capacity of optical fiber transmission systems is the Kerr nonlinearity [5], which leads to signal distortion and decreased transmission quality. Although wideband optical transmission provides a clear path to a linear scaling of the system throughput, stronger nonlinear interference is also incurred as the number of channels is increased. Moreover, due to the large optical bandwidth, these systems are significantly affected by the stimulated Raman scattering (SRS) effect, which causes the power profile of the transmitted signal to change as a function of the channel location in the optical spectrum [6].

Finding efficient ways to estimate the transmission performance of optical transmission systems in the presence of Kerr and SRS effects is then of key importance for modern optical links. Brute-force numerical approaches such as the split-step Fourier method to solve the nonlinear Schrödinger equation (NLSE) are not a viable option due to the high computational complexity caused by the wide transmission bandwidth considered. On the other hand, many approximated analytical models for nonlinear fiber propagation are currently available in the literature [7]–[11]. All of these models aim to accurately predict the nonlinear interference (NLI) power caused by the Kerr effect, in order to quantify the system transmission performance. This remarkable modelling effort enables NLI power prediction in a wide variety of system scenarios such as multiple optical channels, flexible channel symbol rates and frequency spacing, different modulation formats, different amplification schemes, etc. Among the models, the Gaussian noise (GN) model [7], [13], [14], and the enhanced Gaussian noise (EGN) model [12], [15] have risen to popularity due to the wide range of applicability of the EGN model with the Kerr-SRS interplay.

The importance of the effect of SRS on the NLI power has only recently been recognised and modelled in [16]–[20]. The SRS models proposed so far mainly extend the scope of the GN model to wideband transmission scenarios. Also, correction terms to include modulation format dependency of the NLI in the Kerr-SRS context have been derived in [21]. However, no previous work has attempted to fully incorporate the wide range of applicability of the EGN model with the Kerr-SRS phenomenon.

In this paper, we propose a general analytical model which accurately captures the effect on NLI of the main features of interest for modern wideband optical communication systems. These include: flexible modulation formats across different wavelength-division multiplexing (WDM) channels, varying symbol rate, heterogeneous fiber spans and power profiles, and finally SRS. In what follows, we briefly review some of the
main models available in the literature. We then explain our contributions.

A. Main NLI Models in the Absence of SRS

Since the early 2010s a large amount of analytical models based on perturbation methods have been proposed to estimate the effect of the fiber Kerr nonlinearity on the transmission performance. The GN model was derived based on the assumption that the field at the input of the fiber can be modelled as a Gaussian process [13], [22]. Similar derivations to GN model were also presented in [9], [23]. One drawback of all the aforementioned GN-based models is that they often significantly overestimate the NLI power due to the assumption that the transmitted signal, after transmission, statistically behaves as stationary Gaussian noise.

The first modulation-format dependent model was introduced in [10], [24], using a time-domain perturbational approach. This model only considers cross-phase modulation (XPM) as a dominant nonlinear effect. The advantages of such a model in accurately capturing the effect of the modulation format on the NLI were highlighted in [10], [25].

Following a similar approach as in [10], the authors of [12] derived a new perturbation model (in the frequency domain) dropping the assumption of Gaussianity of the transmitted signal. This model was labelled enhanced Gaussian noise (EGN) model. The EGN model resulted in a number of additional correction terms compared to the GN model formulation, which fully captured the modulation format dependency of the NLI. Moreover, the frequency-domain approach in [12] allows the model to fully account for all the different contributions of the NLI in a WDM spectrum, including: the self-channel interference (SCI), and unlike [10], all cross-channel interference (XCI) and multi-channel interference (MCI) terms. In [26] the time domain, GN, and EGN models were compared in subcarrier-multiplexed systems via simulation results, and it was found that both the GN and time-domain model in [10], [24] failed to accurately predict the NLI falling over the channel of interest (COI), whilst the EGN model was able to capture both the modulation format and the symbol rate dependency of the NLI. In Table I we show a summary of the GN-like channel models with applicability to a bandwidth regime where SRS can be safely neglected (e.g. C-band transmission).

B. GN and EGN Models with SRS

All of the works discussed in the previous section are based on the assumption that all frequency components attenuate in the same manner. This assumption is no longer satisfied for ultra-wideband transmission systems due to the SRS effect. In this scenario, the power evolution of signal substantially depends on the SRS loss/gain that each frequency component experiences during propagation along a link. In order to include the SRS effect, the conventional NLSE equation that governs pulse propagation in the presence of Kerr nonlinearity needs to be modified to include the Raman term [37, eq. (3)].

Channel models in the presence of SRS, which stems from the mathematical description in [37, eq. (3)], are also available in the literature [16], [17], [19], [38], [39]. Such models generalized the approach followed in the GN model derivation to include the effect of SRS. A closed-form expression was presented in [40] to compute the NLI power for first- and second-order backward-pumped Raman amplified links. The achievable information rate (AIR) degradation in coherent ultra-wideband systems was studied in [38], using a modified GN model in order to simultaneously take into account both SRS and Kerr nonlinearity such that the approximated NLI coefficient [14] for each channel was obtained by defining an effective attenuation coefficient. An effective attenuation coefficient for each channel matches the actual effective length of the corresponding channel in the presence of SRS. In [16], the signal power profile was obtained based on the linearity assumption of the attenuation profile in frequency. A discrete GN model expression was extended to include SRS in [41]. Another derivation of GN model in the presence of SRS was presented in [17], which is capable of taking into account an arbitrary frequency-dependent signal power profile. The model derived in [17] is valid for Gaussian-modulated signals such as probabilistically-shaped high-order modulation signals. Very recently, [21], [35] proposed an approximate GN model for SCI and XPM. The authors of [21] added a modulation format correction term to XPM, while SCI was computed under a Gaussian assumption. A summary of the channel models is given in Table I.

C. Contribution of the Proposed Model

All previous analytical works on SRS have either entirely neglected the modulation format dependence of the NLI when SRS is present [16]–[18], or have only partially accounted for it, using simple, yet approximated, closed-form expressions [21]. In this work, we present a new analytical model which fully captures the interplay between Kerr effect and SRS, enabling fully accurate predictions within the framework of first-order perturbative models. The proposed model takes into account any frequency-dependent signal power profile when arbitrary modulation formats are transmitted. Unlike the previous models addressing ultra-wideband transmission, all nonlinear interference terms (SCI, XCI, and MCI) are factored in. The model can be interpreted as a generalization of [12] to the SRS scenario. However, unlike [12], the proposed model is analytically formulated on a channel-by-channel basis, allowing for a more straight-forward computation of the channel-dependent NLI. The model works for multiple different spans and coherently evaluates the NLI in the presence of SRS. A frequency-dependent fiber attenuation coefficient can also be accounted for in the model, as well as hybrid amplification schemes. As we will discuss in Sec. II-C an accurate signal power profile can be obtained by solving a set of coupled ordinary differential equations, however, for our model, a closed-form expression for lumped amplification systems with constant fiber attenuation coefficient was used. Such a closed-form formula is obtained via linear regression of Raman gain profile for bandwidths up to 14 THz, as already showed in [17].

The rest of this paper is organized as follows: in Sec. II we describe the system model and the SRS phenomenon.
main result of this work is presented in Sec. III. Numerical results are presented in Section IV, where our results are benchmarked against previous models accounting for SRS. Finally, Sec. V concludes this paper.

D. Notation Convention

We have three delta functions in this paper: \( \delta(f) \) is used for the continuous domain, i.e., \( \int_{-\infty}^{\infty} df \delta(f) = 1 \), and \( \delta_{i,j} \) is used for the discrete domain (Kronecker delta), i.e.,

\[
\delta_{i,j} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{otherwise}
\end{cases}
\]

In this paper we also use \( \delta_{i,j} = 1 - \delta_{i,j} \). The Fourier transform of \( s(t) \) is defined as

\[
S(f) = \int_{-\infty}^{\infty} dt s(t) \exp(-j2\pi ft).
\]

The imaginary unit is denoted by \( j \).

Throughout this paper we use \( (\cdot)_x \) and \( (\cdot)_y \) to represent variables associated to polarizations x and y, resp. We also use the notation \( (\cdot)_{x/y} \) to show that a certain expression is valid for both x and y polarizations. Expectations are denoted by \( \mathbb{E}\{\cdot\} \), and two-dimensional complex (time- and frequency-domain) functions are denoted using boldface symbols.

II. PRELIMINARIES

A. System Model

We consider multi-channel optical transmission of independent and identically distributed (i.i.d.) random complex symbol sequences \( (b_{x,\kappa,1}, b_{x,\kappa,2}, \ldots) \) and \( (b_{y,\kappa,1}, b_{y,\kappa,2}, \ldots) \), selected from arbitrary dual-polarization (DP) constellations, where \( \kappa = -M, -M + 1, \ldots, M - 1, M \) is the channel index. We further assume that the transmitted symbols on polarization x and y are independent of each other. We also assume that different channels across the spectrum can use different modulation formats, and that all formats have zero mean.

The low-pass equivalent of the DP transmitted signal is denoted by \( a_\kappa(t) = (a_{x,\kappa}(t), a_{y,\kappa}(t)) \), which is assumed to be periodic with an arbitrarily large signal period \( T_0 \), i.e.,

\[
a_\kappa(t) = e^{j2\pi \kappa R t} \sum_{n=-\infty}^{\infty} p_\kappa(t - nT_0),
\]

where \( \kappa R \) is the center frequency of channel \( \kappa \) with symbol rate of \( R = 1/T_s \) and \( T_0 = WT_s \). The signal \( p_\kappa(t) = (p_{x,\kappa}(t), p_{y,\kappa}(t)) \) consists of W symbols, where

\[
p_{x/y,\kappa}(t) = \sum_{w=1}^{W} b_{x/y,\kappa,w} s(t - wT_s),
\]

in which \( s(t) \) is assumed to have a sinc pulse shape. As discussed in [7] Sec. II-B), the assumption of a periodic signal results in no loss of generality, as an aperiodic signal can be seen as the limit of a periodic signal for its period tending to infinity.

The Fourier transform of the signal \( a_\kappa(t) \) in (3), denoted by \( A_\kappa(f) = (A_{x,\kappa}(f), A_{y,\kappa}(f)) \), can be expressed as [22]

<table>
<thead>
<tr>
<th>Year</th>
<th>Ref.</th>
<th>MD?</th>
<th>FD/TD</th>
<th>SCI, XCI, MCI</th>
<th>Signal</th>
<th>Accounting for SRS?</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1993</td>
<td>[22]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Finding bit error rate</td>
</tr>
<tr>
<td>2012</td>
<td>[24]</td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>GN model</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SCI</td>
<td>NG</td>
<td>No</td>
<td>Interpreting channel as a white Gaussian channel for high enough symbol rate</td>
</tr>
<tr>
<td>2013</td>
<td>[25]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Inclusion of higher order dispersion and a more formal derivation</td>
</tr>
<tr>
<td></td>
<td>[26]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Alternative GN model</td>
</tr>
<tr>
<td></td>
<td>[27]</td>
<td>Yes</td>
<td>TD</td>
<td>SCI</td>
<td>NG</td>
<td>No</td>
<td>Alternative GN model</td>
</tr>
<tr>
<td>2014</td>
<td>[28]</td>
<td>Yes</td>
<td>FD</td>
<td>SCI, XPM</td>
<td>G</td>
<td>No</td>
<td>Adding a correction term to XPM by comparing [23], [24]</td>
</tr>
<tr>
<td></td>
<td>[12]</td>
<td>Yes</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Valid for flex-grid WDM systems</td>
</tr>
<tr>
<td></td>
<td>[29]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>NG</td>
<td>No</td>
<td>EGN model. Adding correction terms to SCI, XCI, and MCI</td>
</tr>
<tr>
<td>2015</td>
<td>[30]</td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XPM</td>
<td>NG</td>
<td>No</td>
<td>A simple approximate closed-form for XPM</td>
</tr>
<tr>
<td></td>
<td>[31]</td>
<td>Yes</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Comparing time domain, GN and EGN models in sub-carrier multiplexed systems</td>
</tr>
<tr>
<td>2016</td>
<td>[32]</td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>GN model for multimode fiber</td>
</tr>
<tr>
<td></td>
<td>[33]</td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XCI, MCI</td>
<td>NG</td>
<td>No</td>
<td>Modulation dependent model for multimode fiber</td>
</tr>
<tr>
<td>2017</td>
<td>[34]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>Yes</td>
<td>Frequencies attenuate as the COI center frequency</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>Yes</td>
<td>Frequencies attenuate as the COI center frequency</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SCI</td>
<td>NG</td>
<td>No</td>
<td>A comprehensive model for multimode fiber</td>
</tr>
<tr>
<td>2018</td>
<td>[35]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>Yes</td>
<td>Frequencies attenuate as the COI center frequency</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Yes</td>
<td>TD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>Yes</td>
<td>Frequencies attenuate in an individual manner</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>SCI</td>
<td>NG</td>
<td>No</td>
<td>Frequencies attenuate in an individual manner</td>
</tr>
<tr>
<td>2019</td>
<td>[36]</td>
<td>No</td>
<td>FD</td>
<td>SCI, XPM</td>
<td>G</td>
<td>Yes</td>
<td>Simple closed-forms for SPM and XPM</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Yes</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>G</td>
<td>No</td>
<td>Ignoring other XCI terms and MCI's as well</td>
</tr>
<tr>
<td>This Work</td>
<td>Yes</td>
<td>FD</td>
<td>SCI, XCI, MCI</td>
<td>NG</td>
<td>Yes</td>
<td>An extension of [12] in the presence of SRS, Frequencies attenuate in an individual manner</td>
<td></td>
</tr>
</tbody>
</table>
Figure 1. Fiber-optic system studied in this work: an ideal WDM multiplexing is used to transmit \(2M+1\) optical channels, followed by an optical link comprising \(N\) different spans with span lengths \(L_n\) and EDFAs with gains \(g_n\) at the end of each span. At the receiver, ideal demultiplexing and down-conversion are performed. The output for channel \(\kappa\) includes a linear component \(E(0)_\kappa(f) + n_{\text{ASE,}\kappa}(f)\) and a nonlinear component \(E(1)_\kappa(f)\), where \(n_{\text{ASE,}\kappa}(f)\) is the ASE noise, originated from the EDFAs along the fiber. The main goal of the model is to derive the power spectral density (PSD) of \(E(1)_\kappa(f)\) in order to compute the NLI power.

B. Nonlinear Propagation

The propagation of DP signals in an optical fiber is governed by the Manakov equation [42, Ch. 2], [19, eq. (5)], [17, eq. (13)] which in the frequency domain can be written as

\[
\frac{\partial}{\partial z} E(z, f) = \Gamma(z, f) E(z, f) + Q(z, f),
\]

where \(\Gamma(z, f) = \frac{g(z, f)}{2} + \gamma 2\pi^2 \beta_2 f^2 + \frac{4}{3} \pi^3 \beta_3 f^3\),

and

\[
Q(z, f) = \gamma 8 \left[ E_x(z, f) * E_x^*(z, -f) + E_y(z, f) * E_y^*(z, -f) \right] * E(z, f).
\]

is the “Kerr term”.

In [5], \(E = (E_x, E_y, E_0)\) is the spectrum of the electrical field of the propagating DP signal. We model the effect of SRS through the generic frequency- and distance-dependent gain coefficient \(g(z, f)\) (see [19, Appendix] and [17, Appendix A]). The term \(Q\) in (11) is the DP Kerr-term vector \(Q = (Q_x, Q_y, Q_0)\), where \(*\) stands for convolution and \((\cdot)^*\) denotes the complex conjugate.

An analytical approximation to (5) can be written as

\[
E(z, f) \approx E(0)_\kappa(z, f) + E(1)_\kappa(z, f),
\]

In (12), \(E(0)_\kappa = (E_x(0)_\kappa, E_y(0)_\kappa)\) is almost the linear solution in the absence of Kerr nonlinearity, which is given by

\[
E(0)_\kappa(z, f) = e^{\tilde{\Gamma}(z, f)} E(0, f),
\]

where

\[
\tilde{\Gamma}(z, f) = \int_0^z dz' \Gamma(z', f),
\]

and \(E(0, f)\) is the spectrum of the electrical field of the DP signal at the input of the fiber-optic link, which can be

The results in this paper can also be used for near rectangular signal spectral shape such as a root raised cosine with small roll off factor.
expressed as

\[ \mathbf{E}(0, f) = \sum_{\kappa = -M}^{M} \mathbf{A}_\kappa(f) \]

(15)

\[ = \sum_{\kappa = -M}^{M} \sqrt{f_0} \sum_{n = -\infty}^{\infty} \xi_{\kappa,n} \delta(f - nf_0 - \kappa R), \]

(16)

where (16) follows from (5).

To compute the nonlinear solution (perturbative term) \( \mathbf{E}^{(1)} \) in (12), we use the well-known perturbation approach (similar to [9], [12], [13], [29]) which gives

\[ Q(z, f) = \gamma \left[ \frac{8}{9} \left( E^{(0)}_\kappa(z, f) * E^{(0)}_\kappa(z, f) + E^{(0)}_\gamma(z, f) * E^{(0)}_\gamma(z, f) \right) \right] * E^{(0)}_\kappa(z, f). \]

(17)

We then insert (12) into (9), and use (13) and (17) to obtain

\[ \mathbf{E}^{(1)}(z, f) = \mathbf{E}^{(0)}(z, f) \int_0^z dz' Q(z', f) e^{-\Gamma(z', f)}. \]

(18)

It is noticeable that \( \mathbf{E}^{(0)} \) is not strictly the linear solution of the Manakov equation (9), as it accounts for the nonlinear Raman part. Although the model in this paper is not totally consistent with the perturbation approach since the nonlinear Raman part is also perturbative, we follow the approaches proposed in [16], [17], [19] which are based on the perturbation approach. The investigation of this problem goes beyond the scope of this paper and is left to future research.

C. Stimulated Raman Scattering

In optical WDM systems, low wavelength channels act as low power pump channels and provide gain for high wavelength channels, an effect known as SRS. Raman optical amplifiers are built based on this phenomenon. The frequency dependent attenuation coefficient and the coupling between short and long wavelengths which stems from the SRS process result in each frequency component having different power evolutions. To evaluate the power profile of channel \( \kappa \) in a WDM system, the set of coupled ordinary differential equations [43] eq. (1), [16] eq. (1), [17] eq. (6),

\[ \frac{\partial P_\kappa}{\partial z} = - \sum_{i = -M}^{M} g_r(\Delta f) P_\kappa P_i + \sum_{i = k+1}^{M} g_r(\Delta f) P_\kappa P_i - 2\alpha(\kappa R) P_\kappa, \]

(19)

must be solved for \( \kappa = -M, \ldots, M \), where \( \Delta f = |(i - \kappa)R| \), \( g_r(\Delta f) \) is the Raman gain spectrum (see [44], Fig. 2) and [38], Fig. 1), and \( \alpha(\kappa R) \) is the field attenuation coefficient of channel \( \kappa \). The first term in the right hand side of (19) accounts for depletion of channel \( \kappa \) by channels whose central frequencies are smaller than \( \kappa R \), while the second term accounts for depletion of channels with central frequencies above \( \kappa R \). The factor \( i/\kappa \) in the first term of (19) accounts for the energy difference between channels \( i \) and \( \kappa \). Here, following [43] we assume this ratio is equal to one, i.e., \( i/\kappa \approx 1 \). Since the deviation between attenuation coefficients in systems including C+L band is lower than 0.01 dB/km [17], the effect of the frequency-varying attenuation coefficient across the entire spectrum is negligible and the dominant effect that yields the frequency-dependent signal profile is SRS. We therefore assume \( \alpha(\kappa R) = \alpha \) to be constant.

Eq. (19) can be obtained from [9] following the procedures given in [45–47]. The \( 2M + 1 \)-coupled equations given in [19] can be written as one single differential equation [43] eq. (2) whose solution is \( \mathbf{E}^{(0)}(z, f) \) where (16) follows from (5).

\[ \rho(z, f) = \frac{B_{tot} P_{tot} C_r L_{eff}(z) \cdot e^{-2\alpha z - P_{tot} C_r L_{eff}(z)f}}{2 \sinh \left( \frac{P_{tot} B_{tot} C_r L_{eff}(z)}{2} \right)}, \]

(20)

where \( P_{tot} \) is the total launch power within the entire WDM spectrum, \( B_{tot} = (2M + 1) \cdot R \) is the whole WDM spectrum, \( C_r \) is the slope of the Raman gain spectrum and

\[ L_{eff}(z) = (1 - e^{-2\alpha z})/2\alpha \]

(21)

is the effective length of each fiber span. Eq. (20) describes the normalized signal power profile of each frequency component. Considering (15) and (10), we can express (20) as [19] Appendix A

\[ e^{\int_0^z dz' g(z', f)} = \rho(z, f). \]

(22)

**Example 1 (Raman Gain/Loss):**

By excluding the fiber attenuation from (20), the SRS gain [43] eq. (10),

\[ SRS_G(z, f) = \frac{B_{tot} P_{tot} C_r L_{eff}(z) \cdot e^{-P_{tot} C_r L_{eff}(z)f}}{2 \sinh \left( \frac{P_{tot} B_{tot} C_r L_{eff}(z)}{2} \right)}, \]

(23)

is obtained. Fig. 2 shows the SRS gain versus channel number for various launch powers. Here, we assume that the C+L band spectrum (approximately 10 THz) can accommodate 251 Nyquist rectangular spectral shape channels with bandwidth of 40 GHz. As shown in this figure, the low frequency channels are amplified at the expense of high frequency channels due to the SRS. The influence of SRS is larger for high launch powers.

The power transfer across the optical bandwidth in dB can be expressed as [43], eq. (8).

\[ \Delta \rho(z) = 4.3 P_{tot} B_{tot} C_r L_{eff}(z). \]

(24)

III. KEY RESULT: NONLINEAR NOISE POWER

The NLI power on the COI caused by \( \mathbf{E}^{(1)}(z, f) \) is given by

\[ \sigma_{NLI,\kappa}^2 = \int_{R - R/2}^{R + R/2} df G_{NLI,\kappa}(f), \]

(25)

where \( G_{NLI,\kappa}(f) \) is the PSD of the dual-polarization (DP) nonlinear electrical field of channel \( \kappa \) at the input of the receiver. This PSD is

\[ G_{NLI,\kappa}(f) = G_{NLI,x,\kappa}(f) + G_{NLI,y,\kappa}(f) = 2G_{NLI,x,\kappa}(f), \]

(26)

\[ \rho(z, f) = |E^{(0)}(z, f)|^2 / P_{tot}. \]

(27)
where we used the fact that the NLI PSD is equal on both polarizations.

The following theorem is the main result of the paper, which gives an analytical expression for the NLI power in (25).

**Theorem 1 (Nonidentical Spans):** The NLI power on channel \( \kappa \) in (25) is given by

\[
\sigma_{\text{NLI, } \kappa}^2 = \sum_{\kappa_1, \kappa_2, \kappa_3 \in T_\kappa} \mathcal{D}_{\kappa_1, \kappa_2, \kappa_3} + \Phi_{\kappa_1} \cdot \delta_{\kappa_1, \kappa_3 + \kappa_2, \kappa - \kappa + l} E_{\kappa_1} + \Phi_{\kappa_2} \cdot \delta_{\kappa_2, \kappa_3 + \kappa - \kappa + l} F_{\kappa_2} + \Phi_{\kappa_1} \cdot \delta_{\kappa_1, \kappa_2} G_{\kappa_1} + \delta_{\kappa_1, \kappa_2} \delta_{\kappa_3, \kappa_2 + \kappa_3, \kappa - \kappa + l} \Psi_{\kappa_1, H_{\kappa}},
\]

(27)
Table IV

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(f_1, f_2, f) )</td>
<td>( \varphi(f_1, f_2, f', z') = \frac{b_0 f_{\text{tot}} C_r L_{\text{eff}}(z')e^{-\frac{2\pi z'}{\alpha(z')}} - b_{\text{tot}} C_r L_{\text{eff}}(z')f}{2 \sinh \left( \frac{b_{\text{tot}} C_r L_{\text{eff}}(z')}{2} \right)} )</td>
</tr>
<tr>
<td>( \mu(f_1, f_2, f) )</td>
<td>( \int_0^L dz' \rho(z', f_1 + f_2 - f) e^{i \varphi(f_1, f_2, f', z')} )</td>
</tr>
<tr>
<td>( \rho(z', f) )</td>
<td>( \frac{b_0 f_{\text{tot}} C_r L_{\text{eff}}(z')e^{-\frac{2\pi z'}{\alpha(z')}} - b_{\text{tot}} C_r L_{\text{eff}}(z')f}{2 \sinh \left( \frac{b_{\text{tot}} C_r L_{\text{eff}}(z')}{2} \right)} )</td>
</tr>
</tbody>
</table>

as in [22] eq. (6). The expressions \( \mathbb{E}\{b_i^4\}/\mathbb{E}\{b_i^2\} \) and \( \mathbb{E}\{b_i^6\}/\mathbb{E}\{b_i^2\} \) are referred to in [28] as the second and third order modulation factors, respectively. The authors of [21] called the expression in (29) the excess kurtosis of the modulation format.

The next corollary shows how Theorem 1 particularizes to the case of multiple identical spans where span loss fully compensated for by the EDFA at the end of span.

Corollary 1 (Identical Spans): For systems with multiple identical spans, where amplifiers perfectly compensate for the span loss, the NLI power is given by (27) and Table IV where the terms \( T, \mu, \varphi, \) and \( \rho \) are given by the expressions in Table IV.

Proof: See Appendix B.

IV. Numerical Results

In this section, our model is numerically validated. A comparison with a previously published model on SRS is also presented. The numerical study was conducted for a transmission in a single \( L = 100 \) km long fiber span with identical transmitted powers per channel. Two different scenarios were considered: i) \( B_{\text{tot}} = 1 \) THz with an artificially increased Raman gain slope of 1.12 \([1/W/km/THz]\), and ii) \( B_{\text{tot}} = 10 \) THz with a more typical Raman gain slope of 0.028 \([1/W/km/THz]\). In both cases, the product \( C_r B_{\text{tot}} \) was fixed to 0.089 \([1/km]\), resulting in a power profile gap of \( \Delta \rho(L) \approx 8.2 \) dB between the outermost channels for both scenarios, consistently with the approach employed in [17] Sec. III]. The system parameters for the two cases are shown in Table VI.

Our model’s analytical expressions for

\[ \eta_\kappa \triangleq \frac{\sigma_\text{NLI,}\kappa}{P^3}, \]

assuming \( P_\kappa = P \) for all \( \kappa \), were first evaluated using Monte-Carlo (MC) numerical integration. As the span loss was fully compensated at the end of the link by a noiseless amplifier, we numerically evaluated the expression in Corollary I instead of Theorem 1. Split-step Fourier method (SSFM) simulations using the Manakov equation (9)–(11) were then performed to provide an arbitrarily accurate reference for the true \( \eta_\kappa \) values. The function \( g(z, f) \) in (10) is equal to \( d \ln (\rho(z, f))/dz \) according to (22), where \( \rho(z, f) \) is obtained from (20). The group velocity dispersion and third-order dispersion given in (10) are expressed as \( \beta_2 = -DX^2/(2\pi c) \) and \( \beta_3 = (2D + \lambda S^3)/2(2\pi c)^2 \), respectively, where \( \lambda \) is the operating wavelength, \( c \) is the light velocity, and the dispersion coefficient \( D \) and dispersion slope \( S \) are given in Table VI. The SSFM results were compared to our model in Corollary I and the closed-form expression in (36) eq. (3). Three modulation formats were investigated: PM-QPSK, PM-16QAM and PM-64QAM. Moreover, the \( \eta_\kappa \) performance of a polarization-multiplexed, two-dimensional Gaussian-distributed constellation (PM-2D-Gauss) was also studied as a reference case. The results for PM-2D-Gauss were obtained using both Corollary I and the closed-form approximation proposed in (33) eq. (3). We note that, when a Gaussian constellation is selected, our model’s expression in Corollary I coincides, as expected, to the interchannel SRS-GN model presented in [17].

Table V

<table>
<thead>
<tr>
<th>Format</th>
<th>( \Phi )</th>
<th>( \Psi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PM-QPSK</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>PM-16QAM</td>
<td>0.68</td>
<td>2.08</td>
</tr>
<tr>
<td>PM-64QAM</td>
<td>0.619</td>
<td>1.7972</td>
</tr>
</tbody>
</table>

Gaussian

0

0

Table VI

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loss (( \alpha )) [dB/km]</td>
<td>0.2</td>
</tr>
<tr>
<td>Dispersion (( D )) [ps/nm/km]</td>
<td>17</td>
</tr>
<tr>
<td>Dispersion slope (( S )) [ps/nm/km/THz]</td>
<td>0</td>
</tr>
<tr>
<td>Nonlinear coefficient (( \gamma )) [1/W/km]</td>
<td>1.2</td>
</tr>
<tr>
<td>Raman gain slope (( C_r )) [1/W/km/THz]</td>
<td>1.12</td>
</tr>
<tr>
<td>Span length (( L )) [km]</td>
<td>100</td>
</tr>
<tr>
<td>Total launch power (( P_{\text{tot}} )) [dBm]</td>
<td>19</td>
</tr>
<tr>
<td>Symbol rate (( R )) [Gbaud]</td>
<td>1000</td>
</tr>
<tr>
<td>Roll-off factor [%]</td>
<td>0.01</td>
</tr>
<tr>
<td>Channel spacing (( H )) [GHz]</td>
<td>10.1</td>
</tr>
<tr>
<td>Number of channels (( 2M + 1 ))</td>
<td>101</td>
</tr>
<tr>
<td>Optical bandwidth (( B_{\text{tot}} )) [THz]</td>
<td>10.011</td>
</tr>
</tbody>
</table>

\[ \text{Corollary I} \]

\[ \text{Table IV} \]

\[ \text{Table V} \]

\[ \text{Table VI} \]

Figure 3. The parameter $\eta_\kappa$ as a function of channel number $\kappa$ for 1 THz transmission and after a single $L = 100$ km fiber span. Results are shown for transmission with (a), and without (b) SRS, and with a spectrally flat input power profile for PM-QPSK, PM-16QAM, and PM-2D-Gauss constellations. SSFM simulations are represented by circles whilst MC integrations of Corollary 1 are represented by solid lines. Dashed lines represent the closed-form expression in (33).
is symmetric with respect to x and y polarizations, both the polarization channels were used for the SNR estimation.

\[ C. \ 1 \ \text{THz Results} \]

In the 1 THz scenario, 101 WDM channels with symbol rate \( R = 10 \) Gbaud are transmitted, each spaced by 10.001 GHz, as shown in Table IV. Each channel was shaped by an ideal root-raised-cosine with 0.01% roll-off factor. The launch power was set to -1 dBm per channel, yielding a total launch power of 19 dBm.

Fig. 3 shows \( \eta_\kappa \) in dB (W^-2) = 10 log_{10}(\eta_\kappa \cdot 1W^{-2}) as a function of channel number for systems without (a) and with (b) SRS for PM-QPSK (red), PM-16QAM (blue), and PM-2D-Gauss (black) modulation formats. The results presented in solid line were obtained using our model in Corollary \[ \text{IV} \] which are compared to the closed-form expression (dashed lines) in [36, eq. (3)]. SSFM simulation results (circles) are presented for PM-QPSK and PM-16QAM.

As expected, we observe the largest values of \( \eta_\kappa \) for the PM-2D-Gauss constellation (black). The agreement between Corollary \[ \text{IV} \] and the closed-form approximation in [36, eq. (3)] is within 0.6 dB both in the absence and in the presence of SRS. In particular, the closed-form systematically underestimates the NLI power. This is explained by the fact that this approximated expression only accounts for the SCI and XPM, and neglects other NLI contributions. As depicted in Fig. 3 changing the modulation format significantly impacts \( \eta_\kappa \). For example, \( \eta_\kappa \) for PM-2D-Gauss (black dashed line) is approximately 4.1 and 3.9 dB higher than PM-QPSK (red circles) for the center channel frequency \( f = 0 \) GHz in the systems with and without SRS, resp. The gap between PM-QPSK and PM-16QAM (blue circles) is approximately 1.6 dB for no SRS. This comes from the fact that PM-QPSK has the lowest excess kurtosis (given in Table IV) among the exploited modulation formats.

The modulation format dependence of \( \eta_\kappa \) in the presence of SRS is well predicted by the model presented in this paper. The SSFM results are practically coinciding with the curves obtained using Corollary \[ \text{IV} \] for both systems. Average gaps between our model and SSFM simulations are approximately 0.18 dB for PM-QPSK in the presence of SRS. The same match is not observed for the results using the model in [36, eq. (3)]. For PM-16QAM in the presence of SRS, the model in [36, eq. (3)] (blue dashed lines) predicts \( \eta_\kappa \) 2.2 dB lower than the SSFM simulation results at \( f = -400 \) GHz (blue circles). For PM-QPSK, this gap increases to 4.3 dB. This remarkable discrepancy stems from the fact that [36, eq. (3)] only considers SCI (for Gaussian signal) and XPM (for non-Gaussian signal) nonlinear terms and discards the XCI and MCI terms whose contributions at low symbol rates are substantial.

\[ D. \ 10 \ \text{THz Results} \]

The results for a 10 THz optical transmission bandwidth are presented in Fig. 4. The same modulation formats as in Sec. IV.C are shown also for this scenario. Figs. 4(a)-(c) show \( \eta_\kappa \) as a function of the channel frequency, where the total optical bandwidth is partitioned in (a) 1001×10 Gbaud, (b) 251×40 Gbaud, and (c) 101×100 Gbaud channels. The rest of the system parameters are listed in Table IV. In Figs. 4(d)-(f), the gaps of the expressions presented in Figs. 4(a)-(c) to their corresponding SSFM estimates are shown for each investigated symbol rate.

For the 10 Gbaud case (Fig. 4(a)), our model is in very good agreement with the SSFM results across the entire transmitted optical bandwidth for both PM-QPSK (red) and PM-16QAM (blue) modulation formats. The closed-form expression in [36, eq. (3)] results in a significant underestimation of the \( \eta_\kappa \), which is more pronounced for PM-QPSK. The level of accuracy of the compared analytical expressions is shown in more detail in Fig. 4(d), where gaps with SSFM estimates are illustrated as a function of the channel frequency for both our model and [36, eq. (3)]. The average gap across the entire optical bandwidth is also shown (horizontal lines). The model in Corollary \[ \text{IV} \] (solid lines with no markers) is on average approximately 0.2 dB and 0.3 dB above the SSFM estimates for PM-QPSK and PM-16QAM formats, resp. The closed-form formula in [36, eq. (3)] can be seen to underestimate on average the SSFM results by 1.2 dB and 3 dB for PM-QPSK and PM-16QAM, resp. To explain the source of this inaccuracy, we observe that for the PM-2D-Gauss format the closed-form expression (black dashed line in Fig. 4) follows closely the prediction given by the integral form (black solid line) for frequencies around the center of the optical spectrum. However, an increasing gap is observed as we move away from the central channel frequency \( f = 0 \) (up to 1 dB at \( f = 4 \) THz). We conclude that for the 10 Gbaud transmission scenario the inaccuracy of [36, eq. (3)] is due to the missing MCI terms in the modulation format correction term and, to a minor extent, to its Gaussian component [35]. As discussed in Sec. IV.C for the 1 THz transmission case, the MCI terms bring a significant contribution in relatively low symbol rate scenarios such as 10 Gbaud channels (see [26, Sec. II]). As confirmed by results in Figs. 4(a) and (d), this contribution is still very noticeable for 10 THz transmission.

The results on the 251×40 Gbaud channel transmission case are shown in Figs. 4(b) and (d). In general, it can be noticed from Fig. 4(b) that all compared models are in good agreement with the SSFM results. This is due to the increased dominance of the SCI and XPM terms (see, e.g., [26, Sec. II]) over the MCI ones, which is confirmed by the fact that the GN closed-form expression [35] agrees very well with its integral form (black lines with squares) across the whole optical spectrum. Fig. 4(b) shows an average gap of the model in Corollary \[ \text{IV} \] from SSFM \( \eta_\kappa \) estimates of 0.6 dB and 0.5 dB for PM-QPSK and PM-16QAM, resp. It can also be observed that the closed form in [36, eq. (3)] is fairly accurate (approx 0.5 dB away from SSFM estimates) for PM-16QAM, but still showing an average 1 dB gap from SSFM estimates for PM-QPSK, where the modulation-format correction term is more dominant.

Finally, the 101×100 Gbaud transmission results are shown in Figs. 4(c) and (f). In Fig. 4(c), the model in Corollary \[ \text{IV} \] can be observed to be still in very good agreement with SSFM results for both PM-QPSK and PM-16QAM. Moreover, for the
In the first row, $\eta_k$ as a function of channel number $\kappa$ for 10 THz transmission and after a single $L = 100$ km fiber span. PM-QPSK (red), PM-16QAM (blue), and PM-2D-Gauss constellation (black) performance are shown for (a) 10 Gbaud, (b) 40 Gbaud, and (c) 100 Gbaud. Circles refer to SSFM simulation results, whilst solid lines and dashed lines refer to MC integrations and closed-form expressions, resp. In the second row, $\eta_k$ gap from SSFM estimates for Corollary 1 (solid lines) and the closed-form expression in [36, eq. (3)] (dashed lines) for (d) 10 Gbaud, (e) 40 Gbaud, and (f) 100 Gbaud. Horizontal lines indicate the average gap across the whole optical spectrum.

PM-2D-Gauss case [36, eq. (3)] approximates very well the expression in Corollary 1. However, for PM-QPSK and PM-16QAM formats [36, eq. (3)] significantly overestimates the $\eta_k$ across the entire optical bandwidth. This can be attributed to the increasingly dominant SCI terms as the symbol rate is increased for a fixed total optical bandwidth. In Fig. 4 (f), it can be seen that this results in an average gap of the closed-form expression compared to SSFM results of approximately 2 dB and 1 dB for PM-QPSK and PM-16QAM, resp. Only an average 0.4 dB gap is instead observed for the model in Corollary 1 in both PM-QPSK and PM-16QAM cases.

E. Different Modulation Formats

To further confirm the validity of our model, in Fig. 5, $\eta_k$ is shown as a function of the channel frequency for a scenario where different modulation formats are transmitted over different WDM channels. Both 10 Gbaud and 40 Gbaud transmission scenarios are analyzed for a total 10.011 THz and 10.041 THz bandwidth, resp. PM-64QAM channels are transmitted over the first third of the optical bandwidth, PM-16QAM channels are transmitted over the second third, and
PM-QPSK channels are transmitted over the last third. This results in a channel distribution of for the 3 modulation formats PM-64QAM, PM-16QAM and PM-QPSK of (334,333,334) and (84,83,84) for the 10 Gbaud case, and 40 Gbaud case, resp. For the 10 Gbaud channel transmission (purple lines), it can be seen that our model (solid line) is matching quite well the SSFM results (circles), with deviations within 0.4 dB across the entire optical bandwidth. The closed-form expression in \[\text{(35)}\] increasingly underestimates \(\eta_6\) as we move towards the right side of the spectrum, where lower-order formats are transmitted. For the right-most part of the spectrum (PM-QPSK format transmitted) the gap between SSFM results and \[\text{(35)}\] can in excess of 3 dB. For 40 Gbaud channels (green lines), estimates, \textit{Corollary 1} and \[\text{(35)}\] are in good agreement, with maximum deviations across the entire optical spectrum of about 0.7 dB.

V. CONCLUSIONS

We conclude that the presented model is able to predict the NLI power with a good level of accuracy (within 0.6 dB) across all symbol rates and modulation formats. This is particularly true whenever relatively low or high symbol rates (e.g., 10 Gbaud and 100 Gbaud, resp.) are used in combination with low-order modulation formats. In these scenarios, available closed-form approximations can result in a marked underestimation or overestimation of the NLI power potentially exceeding 3 dB.

We remark, however, that closed-form expressions, whenever achieving the required accuracy, are orders of magnitude faster to evaluate than integral forms such as the one in Corollary 1. More work is, thus, needed to precisely characterize the accuracy/complexity trade-offs arising when using a closed-form as opposed to an integral form in different transmission regimes. Future works also include convex power optimization and physical layer impairment-aware optical networking using the proposed model across the C+L band. In addition, the derived model provides a powerful tool for efficient design of capacity-maximizing modulation formats over C+L band systems.

VI. ACKNOWLEDGMENTS

The authors would like to thank Daniel Semrau (University College London) for useful discussions on earlier versions of this manuscript. The work of H. Rabbani, G. Liga, and A. Alvarado has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 757791). The work of A. Alvarado and V. Oliari is supported by the Netherlands Organisation for Scientific Research (NWO) via the VIDI Grant ICONIC (project number 5685).

APPENDIX A

PROOF OF THEOREM 1

This appendix contains two Lemmas. In \textit{Lemma 1} we derive the nonlinear electrical field at the end of a link with different spans. \textit{Lemma 2} uses this result to derive the NLI PSD in \[\text{(26)}\], which is then used to compute the nonlinear power in \[\text{(27)}\] via \[\text{(25)}\]. The proofs are relegated to the end of this appendix.

In the following Lemma we derive, following the regular perturbation (RP) approach \[\text{(50)}\], the total nonlinear electrical field at the end of a link with different spans for one of the 2 transmitted orthogonal polarizations (here referred to as x-polarization). The same result can also be used for the y-polarization field under the substitution \(x \rightarrow y, y \rightarrow x\).

\textit{Lemma 1:} The total nonlinear electrical field at the end of a link with \(N\) different spans can be written as

\[
E_{\eta_6}^{(s)}(L, f) = \frac{8}{9} \sum_{\kappa_1, \kappa_2, f \in \mathcal{T}_s} \delta(f - f_0 - (\kappa - l)R) \sum_{\kappa_1, \kappa_2, f \in \mathcal{T}_s} \sum_{m, n, p} \xi_{\kappa_1, \kappa_2, f}^{\eta_6} \left( \xi_{\kappa_1, \kappa_2, f}^{\eta_6} \right) \sum_{m, n, p} \left( m \cdot n \cdot p = i \right) \sum_{m, n, p} \left( m \cdot n \cdot p = i \right)
\]

where \(f_0 = 1/T_0\) and \(T_0\) is the period of the transmitted signal. In \[\text{(35)}\], \(L = \sum_{s=1}^{N} L_s\) is the total length of the link, \(\mathcal{T}_s\) is given in \[\text{(28)}\], and

\[
\mathcal{T}_s' = \{ \left( m, n, p \right) \in \mathbb{Z}^3 : m + n + p = i, \text{ } R/2 \leq f_0 - 1R \leq \frac{R}{2} \}.
\]

The coefficient \(\gamma_{\kappa_1, \kappa_2, f}(m, n, p)\) in \[\text{(35)}\] is given by

\[
\gamma_{\kappa_1, \kappa_2, f}(m, n, p) = \sum_{s=1}^{N} \gamma_{\kappa_1, \kappa_2, f}^{(s)} (m, n, p) \cdot \gamma_{\kappa_1, \kappa_2, f}^{(s)} (m, n, p),
\]

in which

\[
\gamma_{\kappa_1, \kappa_2, f}^{(s)} (m, n, p) = \prod_{s'=1}^{s-1} g_{s'} \sqrt{\rho_{\kappa_1, \kappa_2, f} (L_{s'}, m f_0 + \kappa_1 R)} \cdot \rho_{\kappa_1, \kappa_2, f} (L_{s'}, p f_0 + \kappa_2 R)
\]

\[
\cdot \exp \left( i 4 \pi^2 \beta_{2, s'} / L_{s'} \left( (m - n) f_0 + (\kappa_1 + \kappa_2 - l)R \right) \right)
\]

\[
\cdot \exp \left( i 4 \pi^2 \beta_{3, s'} / L_{s'} \left( (p - n) f_0 + (\kappa_1 + \kappa_2 - l)R \right) \right)
\]

\[
\cdot \exp \left( i 4 \pi^2 \beta_{4, s'} / L_{s'} \left( (m + p) f_0 + (\kappa_1 + \kappa_2)R \right) \right)
\]

\[
\cdot \prod_{s'=1}^{N} g_{s'}^{1/2} \sqrt{\rho_{\kappa_1, \kappa_2, f} (L_{s'}, i f_0 + (\kappa - l)R)}
\]

\[
\cdot \prod_{s'=1}^{N} \exp \left( i 2 \pi^2 \beta_{2, s'} (i f_0 + (\kappa - l)R)^2 \right)
\]

\[
+ \frac{4}{3} \pi^2 \beta_{4, s'} (i f_0 + (\kappa - l)R)^3 L_{s'},
\]

\[\text{(38)}\]
This process gives the total PSD and (25). To this end, we multiply the PSD in (40) by

\[
E_{x,\kappa}^{(1)}(L, f) = \sum_{s=1}^{N} E_{x,\kappa}^{(1)}(L_s, f),
\]

where \(E_{x,\kappa}^{(1)}(L_s, f)\) is the nonlinear electrical field of channel \(\kappa\) generated in span \(s\), which linearly propagates until the end of the link of length \(L_s\), as schematically shown in Fig. 6. To derive each of the terms in (41), the linear electrical field is first needed.

By solving (9) in the absence of the forcing NL term \(Q(z, f)\) we obtain that the linear electrical field at the input of span \(s\) is

\[
E_{x,\kappa}^{(0)}(L'_s, f) = \prod_{s'=1}^{s-1} \beta_{s'} \cdot \int_{\mathbb{R}} \rho_s(z', f) \, dz',
\]

in which

\[
\beta_s(z, f) = \frac{g_s(z, f)}{2} + \frac{2\pi^2 \beta_{s,2} f^2}{3} + \frac{4}{3} \beta_{s,3} f^3,
\]

where \(g_s(z, f)\) is the generic frequency- and distance-dependent gain coefficient of span \(s\), and \(E_{x,\kappa}(0, f)\) is given by (16). Equations (42)–(44) represent the electrical field passing through \(s-1\) spans influenced only by chromatic dispersion, span losses, SRS gain/loss, and amplifier gains. The linear field in span \(s\) can then be written as

\[
E_{x,\kappa}^{(0)}(z, f) = e^{i\xi(z, f)} E_{x,\kappa}^{(0)}(L'_s, f),
\]

where \(E_{x,\kappa}^{(0)}(L'_s, f)\) is given by (42). In order to find the contribution at the \(s\)-th span to the nonlinear optical field in (41) we define, similar to (11),

\[
Q_s(z, f) = \frac{8}{9} \left[ E_{x,\kappa}^{(0)}(z, f) * E_{x,\kappa}^{(0)*}(z, -f) + E_{x,\kappa}^{(0)}(z, f) * E_{x,\kappa}^{(0)*}(z, -f) \right] * E_{x,\kappa}^{(0)}(z, f),
\]

whose first component (for the x polarization) can be written
Table VII
Integral Expressions for the Terms Used in Lemma. The term $\Upsilon(\cdot)$ is given in Tables III and IV

<table>
<thead>
<tr>
<th>Term</th>
<th>Integral Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>$\frac{8}{9} R^3 \int f R^2 f_1 \int f R^2 f_2 \int R^2 f_3 \Upsilon(f_1 + \kappa R, f_2 + \kappa R, f_3)(f_1^2</td>
</tr>
<tr>
<td>$E$</td>
<td>$R^2 \int f R^2 f_1 \int f R^2 f_2 \int R^2 f_3 \Upsilon(f_1 + \kappa R, f_2 + \kappa R, f_3)(f_1^2</td>
</tr>
<tr>
<td>$F$</td>
<td>$R^2 \int f R^2 f_1 \int f R^2 f_2 \int R^2 f_3 \Upsilon(f_1 + \kappa R, f_2 + \kappa R, f_3)(f_1^2</td>
</tr>
<tr>
<td>$G$</td>
<td>$R^2 \int f R^2 f_1 \int f R^2 f_2 \int R^2 f_3 \Upsilon(f_1 + \kappa R, f_2 + \kappa R, f_3)(f_1^2</td>
</tr>
<tr>
<td>$H$</td>
<td>$\frac{8}{9} R^2 \int f R^2 f_1 \int f R^2 f_2 \int R^2 f_3 \Upsilon(f_1 + \kappa R, f_2 + \kappa R, f_3)(f_1^2</td>
</tr>
</tbody>
</table>

using (16) as

$$Q_{s,x}(z, f) = \gamma_s \frac{8}{9} \left[ f_0 \left( \sum_{s' = 1}^{s-1} g_{s'}^{1/2} \cdot e^{\tilde{\Gamma}_{z,s'}(L_{s';f})} \sum_{\kappa' = -M}^{\infty} \sum_{m = -\infty}^{\infty} \xi_{s',\kappa',m} \cdot \delta(-m f_0 - \kappa R) \right) + f_0 \left( \sum_{s' = 1}^{s-1} g_{s'}^{1/2} \cdot e^{\tilde{\Gamma}_{z,s'}(L_{s';f})} \sum_{\kappa' = -M}^{\infty} \sum_{m = -\infty}^{\infty} \xi_{s',\kappa',m} \cdot \delta(-m f_0 - \kappa R) \right) \right]$$

where

$$\tilde{S}_{i,R} = \left\{ (m, n, p) \in \mathbb{Z}^3 : m - n + p = i, \kappa R - \frac{R}{2} \leq i f_0 + (\kappa - \kappa' + 2\kappa R) \leq \kappa R + \frac{R}{2} \right\}$$

which gives rise to

$$Q_{s,x}(z, f) = \gamma_s \frac{8}{9} \left[ f_0 \left( \sum_{s' = 1}^{s-1} g_{s'}^{1/2} \cdot e^{\tilde{\Gamma}_{z,s'}(L_{s';f})} \sum_{\kappa' = -M}^{\infty} \sum_{m = -\infty}^{\infty} \xi_{s',\kappa',m} \cdot \delta(-m f_0 - \kappa R) \right) + f_0 \left( \sum_{s' = 1}^{s-1} g_{s'}^{1/2} \cdot e^{\tilde{\Gamma}_{z,s'}(L_{s';f})} \sum_{\kappa' = -M}^{\infty} \sum_{m = -\infty}^{\infty} \xi_{s',\kappa',m} \cdot \delta(-m f_0 - \kappa R) \right) \right]$$

(47)

where we use the notation $Q_{s,x}(z, f)$ to show the Kerr term in channel $\kappa$. The rectangular spectral shape $S(n f_0)$ with center frequency $f = 0$ in (15) implies that $n f_0$ should satisfy $-\frac{R}{2} \leq n f_0 \leq \frac{R}{2}$. A similar interpretation can be used on $m f_0$ and $p f_0$, and thus,

$$-\frac{R}{2} \leq m f_0 \leq \frac{R}{2}, \quad -\frac{R}{2} \leq n f_0 \leq \frac{R}{2}, \quad -\frac{R}{2} \leq p f_0 \leq \frac{R}{2}.$$
which gives
\[ -\frac{3}{2} R \leq (m - n + p) f_0 \leq \frac{3}{2} R. \] (52)

Combining the inequalities in the definition of the set \( \tilde{S}_{i,\kappa} \) in (50) with (52), we obtain
\[ \kappa - 2 < \kappa_i - \kappa' + \kappa_2 < \kappa + 2. \] (53)

The expression in (53) in turn implies that for given values of \( \kappa, \kappa_1 \), and \( \kappa_2, \kappa' \) it can only take the three values
\[ \kappa' = \kappa_1 + \kappa_2 - \kappa + l, \quad l = -1, 0, 1. \] (54)

Using (54), we express (51) as
\[ Q_{s,x,n}(z, f) = \gamma_s \frac{8 f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_m} \sum_{i=\infty}^{\infty} \delta(f - i f_0 - (\kappa - l) R) \sum_{m,n,p \in \tilde{S}_{i,\kappa}} \left( \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l} \right) \cdot \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l, n} \right) \] (55)

where the set \( \tilde{S}_{i,\kappa} \) is defined in (50), which is obtained from (50) by replacing \( \kappa' \) with \( \kappa_1 + \kappa_2 - \kappa + l \) (see (54)). In what follows, we will use (55) to calculate the nonlinear electrical field on this channel, i.e., \( E^{(1)}_{s,x,n}(z, f) \). The nonlinear electrical field in the \( s \)-th span is obtained by combining (55) and (5), which gives
\[ E^{(1)}_{s,x,n}(z, f) = \gamma_s \frac{8 f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_m} \sum_{i=\infty}^{\infty} \left( \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l} \right) \cdot \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l, n} \right) \] (56)

in which the delta function property \( A(f) \delta(f - f_0) = A(f_0) \delta(f - f_0) \) is used. Considering the amplifier gain at the end of span \( s \), we have
\[ E^{(1)}_{s,x,n}(L_s' + L_s, f) = g_s^{1/2} \cdot E^{(1)}_{s,x,n}(L_s, f), \] (57)

which using (56) is equal to
\[ E^{(1)}_{s,x,n}(L_s' + L_s, f) = \gamma_s \frac{8 f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_m} \sum_{i=\infty}^{\infty} \left( \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l} \right) \cdot \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l, n} \right) \] (58)

From the RP approach, \( E^{(1)}_{s,x,n}(L_s' + L_s, f) \) in (58) propagates linearly over \( N - s \) spans (see Fig. 6), i.e.,
\[ E^{(1)}_{s,x,n}(L_s, f) = \prod_{s'=s+1}^{N} \gamma_s^{1/2} \cdot E^{(1)}_{s',x,n}(L_s' + L_s, f), \] (59)

which using (58) gives
\[ E^{(1)}_{s,x,n}(L_s, f) = \gamma_s \frac{8 f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_m} \sum_{i=\infty}^{\infty} \left( \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l} \right) \cdot \xi_{s,\kappa_1, \kappa_2, k+\kappa+k+\kappa+l, n} \right) \] (60)

\(^5To obtain (53), we use the fact that for any \( a, b, c, x \in \mathbb{R} \) the two inequalities \( b - a/2 \leq x \leq b + a/2 \) and \(-3a/2 \leq x \leq 3a/2 + c \), imply that \( b > -2a + c \) and \( b < 2a + c \).
Using (41) and (60) gives

\[
E^{(1)}_{s,\kappa}(L, f) = \frac{8f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_{\kappa}} \sum_{i = -\infty}^{\infty} \delta(f - if_0 - (k - l)R) \sum_{m, n, p \in \mathcal{S}_{i, i}}
\]

\[
\left( \xi_{s, \kappa_1, 1, m, \xi_{s, \kappa_1, 1, \kappa_2}} + \xi_{s_2, \kappa_1, 1, m, \xi_{s_2, \kappa_1, 1, \kappa_2}} + \xi_{s_3, \kappa_1, 1, m, \xi_{s_3, \kappa_1, 1, \kappa_2}} \right).
\]

\[
\sum_{s = 1}^{N} g_{s_0}^{3/2} e_{s_0}^{j} \left( L_{s_0}, \gamma_{s_0}^{j} \right) \int_{0}^{\infty} d\gamma_{s_0}^{j} \pi 
\]

development in (22) and the second from the fact that \( \beta_{s, s'} \) and \( \beta_{3, s'} \) are \( z \)-independent.

Using (66), the first product term in (64) can be written as

\[
\prod_{s = 1}^{N} g_{s}^{3/2} e_{s}^{j} \left( L_{s}, \gamma_{s}^{j} \right) = \prod_{s = 1}^{N} g_{s}^{3/2} \sqrt{\rho_{s}^{j} \left( L_{s}, \gamma_{s}^{j} \right)}
\]

to express the first exponential term in the right hand side of (67) as

\[
\exp \left( i2\pi^{2}\beta_{2, s'} L_{s'} \left[ (m - n + p)f_0 + (k - l)R \right] \right)
\]

\[
+ 4\pi^{2}\beta_{2, s'} L_{s'} \left[ (m - n + p)f_0 + (k - l)R \right]
\]

\[
\left. \cdot \left( (p - n)f_0 + (k - \kappa_1 - l)R \right) \right].
\]

We also use the equality

\[
x^{3} - y^{3} + z^{3} = (x - y + z)^{3} + 3(x - y)(z - y)(x + z),
\]

to express the second exponential term in the right hand side of (67) as

\[
\exp \left( i4\pi^{3}\beta_{3, s'} L_{s'} \left[ (m - n + p)f_0 + (k - l)R \right]^{3} \right)
\]

\[
+ 4\pi^{3}\beta_{3, s'} L_{s'} \left[ (m - n + p)f_0 + (k - l)R \right]^{3}
\]

\[
\left. \cdot \left( (p - n)f_0 + (k - \kappa_1 - l)R \right) \right].
\]
Using (72) and (73), (64) is equal to
\[
\prod_{s'=1}^{s-1} g_{s'}^{3/2} \sqrt{\rho_{s'}(L_{s'}, m f_0 + \kappa_1 R)} \\
\cdot \sqrt{\rho_{s'}(L_{s'}, n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} \left[ (m + p) f_0 + (\kappa + \kappa_2) R \right] \right) \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} \left[ (m + p) f_0 + (\kappa + \kappa_2) R \right] \right) \\
\cdot \sqrt{\rho_{s'}(L_{s'}, p f_0 + \kappa_2 R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} \left[ (m + p) f_0 + (\kappa + \kappa_2) R \right] \right). \tag{72}
\]

According to (66), the last product term in (64) can also be written as
\[
\prod_{s'=1}^{N} g_{s'}^{1/2} e_{L_{s'}}(i f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R) \\
\cdot \sqrt{\rho_{s'}(L_{s'}, i f_0 + (\kappa - l) R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} (i f_0 + (\kappa - l) R)^2 \right) \\
\cdot \sqrt{\rho_{s'}(L_{s'}, i f_0 + (\kappa - l) R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} (i f_0 + (\kappa - l) R)^2 \right). \tag{73}
\]

Using (72) and (73), (64) is equal to
\[
\prod_{s'=1}^{s-1} g_{s'}^{3/2} \sqrt{\rho_{s'}(L_{s'}, m f_0 + \kappa_1 R)} \\
\cdot \sqrt{\rho_{s'}(L_{s'}, n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} ((m - n) f_0 + (\kappa_1 - \kappa_2) R) \right) \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} ((m - n) f_0 + (\kappa_1 - \kappa_2) R) \right) \\
\cdot \sqrt{\rho_{s'}(L_{s'}, p f_0 + \kappa_2 R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} ((m + p) f_0 + (\kappa + \kappa_2) R) \right) \\
\cdot \sqrt{\rho_{s'}(L_{s'}, p f_0 + \kappa_2 R)} \\
\cdot \exp\left(i 2\pi^2 \beta_{2,s'} L_{s'} ((m + p) f_0 + (\kappa + \kappa_2) R) \right). \tag{74}
\]

We now use (66) to express the integral in (61) as
\[
\int_0^{L_s} d z' \exp\left(i \sum_{s'} (m f_0 + \kappa_1 R) e_{L_{s'}}(z', n f_0 + (\kappa_1 + \kappa_2 + \kappa l) R) \right) \\
\cdot \exp\left(i \sum_{s'} (p f_0 + \kappa_2 R) e_{L_{s'}}(z', i f_0 + (\kappa - l) R) \right) \\
\cdot \sqrt{\rho_{s'}(z', m f_0 + \kappa_1 R)} \\
\cdot \exp\left(i \sum_{s'} (p f_0 + \kappa_2 R) e_{L_{s'}}(z', i f_0 + (\kappa - l) R) \right) \\
\cdot \sqrt{\rho_{s'}(z', i f_0 + (\kappa - l) R)} \\
\cdot \exp\left(2\pi^2 \beta_{2,s'} (m f_0 + \kappa_1 R)^2 \right) \\
\cdot \exp\left(2\pi^2 \beta_{2,s'} (p f_0 + \kappa_2 R)^2 \right) \\
\cdot \exp\left(2\pi^2 \beta_{3,s'} (m f_0 + \kappa_1 R)^3 \right) \\
\cdot \exp\left(2\pi^2 \beta_{3,s'} (p f_0 + \kappa_2 R)^3 \right) \tag{75}
\]

The last step in the proof is therefore to show that the arguments of the four exponentials in (75) corresponds to the arguments of the two exponentials in (39). We do this by first grouping the quadratic and cubic terms in the exponentials in (75) as
\[
\exp\left(2\pi^2 \beta_{2,s'} z' \right) \\
\cdot \left[ (m f_0 + \kappa_1 R)^2 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^2 \right] \\
\cdot \exp\left(2\pi^2 \beta_{2,s'} z' \right) \\
\cdot \left[ (m f_0 + \kappa_1 R)^3 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^3 \right] \tag{76}
\]

We now use the equality in (68) to express the first exponential term in (76) as
\[
\exp\left(2\pi^2 \beta_{2,s'} z' \right) \\
\cdot \left[ (m f_0 + \kappa_1 R)^2 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^2 \right] \\
\cdot \left[ (m f_0 + \kappa_1 R)^3 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^3 \right] \tag{77}
\]

We also use (70) to express the second exponential term in (76) as
\[
\exp\left(2\pi^3 \beta_{3,s'} z' \right) \\
\cdot \left[ (m f_0 + \kappa_1 R)^2 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^2 \right] \\
\cdot \left[ (m f_0 + \kappa_1 R)^3 - (n f_0 + (\kappa_1 + \kappa_2 - \kappa + l) R)^3 \right] \tag{78}
\]

By replacing the exponential terms in (75) by the multiplication of (74) and (73), we can rewrite the right hand side of (75) as (39). This completes the proof.

\textbf{Proof of Lemma 2:} To evaluate the PSD of the nonlinear electrical field given in Lemma 1, we ignore the triplets (m, n, p) in which m = n or p = n, as these terms create a constant phase shift and can be interpreted as bias or non-fluctuating terms [Sec. VIII, [9] Sec. IV-B], [10] Sec. III], and thus, irrelevant for the noise variance we would like to
compute. The set $S_{i,l}$ in (56) can therefore be written as
$$S_{i,l} = \{ (m, n, p) \in \mathbb{Z}^3 : m - n + p = i, m \neq n, p \neq n, -\frac{R}{2} \leq (m - n + p)f_0 - lR \leq +\frac{R}{2} \}.$$  
(79)
The total nonlinear electrical field given in (35) has therefore the form
$$E_{S_{i,l}}^{(\lambda)} (L, f) = \sum_{l=-1}^{\infty} \sum_{i=-\infty}^{\infty} I_{i,l} \delta (f - if_0 - (\lambda - l)R)$$  
(80)
where
$$I_{i,l} = \frac{8f_0^{3/2}}{9} \sum_{\kappa_1, \kappa_2 \in T_{\kappa_1,l}} \sum_{m, n, p \in \mathbb{Z}} \left( \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\right) E_{\kappa_1,\kappa_2,m,n,p},$$  
(81)
and $T_{\kappa_1,l} = \{(\kappa_1, \kappa_2) \in \{-M, \ldots, M\}^2 : -M \leq \kappa_1 + \kappa_2 - \kappa + l \leq M \}$. Considering (51) Eqs. (60), (61), (62)], we find that the power spectral density of (80) can be expressed as
$$G_{\text{NLLN},\kappa}(f) = \sum_{l=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} E\{ |I_{i,l}|^2 \} \delta (f - if_0 - (\lambda - l)R).$$  
(82)
Using the fact that the symbols in the y polarization are independent from those in the x polarization (see Sec. II-A), we obtain from (81)
$$E\{ |I_{i,l}|^2 \} = \frac{64}{81} f_0^3 \sum_{\kappa_1, \kappa_2 \in T_{\kappa_1,l}} \sum_{m, n, p \in \mathbb{Z}} \left( \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\right) E_{\kappa_1,\kappa_2,m,n,p}$$  
(83)
where
$$\begin{align*}
C_{\text{sp}} &= E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} E_{\kappa_1,\kappa_2,m,n,p}, \\
C_{\text{xp}} &= E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} E_{\kappa_1,\kappa_2,m,n,p}.
\end{align*}$$  
(84)
and
$$C_{\text{xp}} = E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} E_{\kappa_1,\kappa_2,m,n,p}.$$  
(85)
We will now show that for any given $\kappa \in \{-M, \ldots, M\}$, $i \in \mathbb{Z}$, and $l \in \{-1, 0, 1\}$, $E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} = 0$ for all $i \neq 0$ and $l \neq 0$. Similarly, for any given $\kappa \in \{-M, \ldots, M\}$, $i \in \mathbb{Z}$, and $l \in \{-1, 0, 1\}$, we will also prove that $E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} = 0$ for all $i \neq 0$ and $l \neq 0$. These two cases will prove that only $C_{\text{sp}}$ and $C_{\text{xp}}$ contribute to (83).

We start by using (60), which gives
$$E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_2+n+2\lambda+l,m}\} = f_0 S^*(m' f_0) S(n' f_0)$$  
(86)
$$\cdot \sum_{w_1=1}^{W} \sum_{w_2=1}^{W} \sum_{\kappa_1, \kappa_2 \in T_{\kappa_1,l}} \sum_{m, n, p \in \mathbb{Z}} \sum_{m, n, p} E\{ b_{\kappa_1,\kappa_1,w_1} b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\} e^{i2\pi(m'-n')w_2},$$  
(87)
To show that (86) is indeed equal to zero, two cases should be taken into consideration: $\kappa_1' \neq \kappa_1 + \kappa_1' + \kappa_1 - \kappa + l$ and $\kappa_1' = \kappa_1 + \kappa_1' + \kappa_1 - \kappa + l$.

In the $\kappa_1' \neq \kappa_1 + \kappa_1' + \kappa_1 - \kappa + l$ case, the expectation term $E\{ b_{\kappa_1,\kappa_1,w_1} b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\}$ in (86) can be written as
$$E\{ b_{\kappa_1,\kappa_1,w_1} b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\} = 0$$  
(88)
because symbols in different WDM channels are independent. The expectation in (87) is zero because the constellations have zero mean.

In the $\kappa_1' = \kappa_1 + \kappa_1' + \kappa_1 - \kappa + l$ case, (86) is
$$f_0 S^*(m' f_0) S(n' f_0) \sum_{w_1=1}^{W} \sum_{w_2=1}^{W} \sum_{m, n, p} E\{ b_{\kappa_1,\kappa_1,w_1} b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\} e^{i2\pi(m'-n')w_2}$$  
(89)
with $w_1 \neq w_2$ we have
$$E\{ b_{\kappa_1,\kappa_1,w_1} b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\} = \mathcal{E}\{ b_{\kappa_1,\kappa_1,w_1}\} \mathcal{E}\{ b_{\kappa_2,\kappa_2+n+2\lambda+l,w_2}\} = 0$$  
(90)
which follows from the zero-mean and independence assumption on the symbols across WDM channels. Using (89), (88) becomes
$$f_0 S^*(m' f_0) S(n' f_0) \sum_{w_1=1}^{W} \sum_{m, n, p} \sum_{n'=1}^{W} e^{i2\pi(m'-n')w_1}$$  
(91)
where we have used the stationary $\mathcal{E}\{ b_{\kappa_1,\kappa_1,w_1}\}^2 = \mathcal{E}\{ b_{\kappa_1,\kappa_1,w_1}\}^2$ (see [52 Appendix E]). Furthermore, the sum in (91) is
$$\sum_{w_1=1}^{W} \sum_{m, n, p} e^{i2\pi(m'-n')w_1} = \begin{cases} 0 & m' \neq m' \neq pW \ \text{or} \ \text{W} \\
W & m' \neq m' = pW \end{cases}$$  
(91)
where $p \in \mathbb{Z} \setminus \{0\}$, $m' = n'$ is not included in $S_{i,l}$, and $W = R/f_0$ (see Sec. II-A). When $m' - n' = pW$ (i.e., the second case in (91)), the coefficients $S^*(m' f_0) S(n' f_0)$ in (90) are
$$S^*(m' f_0) S((m' - pW) f_0) = S^*(m' f_0) S(m' f_0 - pR) = 0$$  
(92)
where the second equality is due to the fact that $S(f) = 0$ for $|f| > R$.

The procedure above shows that indeed $E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_1+n+2\lambda+l,m}\} = 0$. The same procedure can be followed to prove that $E\{ \xi_{\kappa_1,\kappa_1,m}\xi_{\kappa_1,\kappa_1+n+2\lambda+l,m}\} = 0$. 

PREPRINT, JUNE 12, 2020 17
which makes (82) equal to
\[
\text{GNI}_{x}(f) = \frac{64}{81} f^3 \sum_{k_1, k_2 \in \mathcal{T}_h} \sum_{m, m'} \sum_{n, n'} \sum_{\xi, \xi'} \left( \delta(\xi - f) \right) R
\]
removing the bias terms from them assuming \( p = 0 \), we can write (93) as
\[
C_{sp} = \delta_{\kappa_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{k_1 \kappa_1 + k_2 - \kappa + l} \cdot A_0
\]
and hence, we can write \(84\) as
\[
C_{sp} = \delta_{\kappa_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{k_1 \kappa_1 + k_2 - \kappa + l} \cdot A_0
\]
where \( X_1 \) and \( X_2 \) are given in Table VIII and where the \( U \) functions in Table VIII are
\[
U_{mnp} \cdot \cdot = S(m f_0)S'(m f_0)S''(m f_0)S'(n f_0)S''(n f_0)
\]
and
\[
U_{m'np} \cdot \cdot = S'(m f_0)S'(n f_0)S''(m f_0)
\]
By substituting (94) and (95) into (82), the total nonlinear PSD can be written as
\[
\text{GNI}_{x}(f) = \frac{64}{81} \sum_{k_1, k_2 \in \mathcal{T}_h} \left( \delta_{\kappa_1 \kappa_2} \cdot A_0 \right)
\]
and
\[
\delta_{\kappa_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{n_1 \kappa_2} \delta_{k_1 \kappa_1 + k_2 - \kappa - l} \cdot A_0
\]
where \( D, E, F, G, \) and \( H \) are given in Table IX. Finally, by using Table VIII into Table IX we get Table X.
the terms given in Table IX and also canceling the delta function in Table X via the integral over \( f \), we can rewrite (102) in the continuous domain as (40). This completes the proof.

**APPENDIX B**

**PROOF OF COROLLARY I**

If the loss of each frequency is exactly compensated for at the end of the corresponding span, we have

\[
\prod_{s=t}^{s-1} \frac{3/2}{\sqrt{g^s(L'_s, f_1) \sqrt{\rho^s(L'_s, f_1 - f + f_2) \sqrt{\rho^s(L'_s, f_2)}}} = 1, \quad (103)
\]

and

\[
\prod_{s=t}^{s-N} \frac{1/2}{\sqrt{g^s(L'_s, f)} = 1, \quad (104)}
\]

and, hence, \( \Upsilon(\cdot) \) in Table III can be written as

\[
\Upsilon(f_1, f_2, f) = \sum_{s=1}^{N} \sum_{s'\leq s}^{s-1} \beta_{s, s'} L_{s'} + \pi(f_1 + f_2) \beta_{s, s'} L_{s'}.
\]

For multiple identical spans of homogeneous fiber \((\alpha_1 = \ldots = \alpha_N = \alpha, L_1 = \ldots = L_N = L, \gamma_1 = \ldots = \gamma_N = \gamma, \beta_{1, s} = \ldots = \beta_{2, N} = \beta_2, \beta_{3, s} = \ldots = \beta_{3, N} = \beta_3)\), (105) is equal to

\[
\Upsilon(f_1, f_2, f) = \gamma \mu(f_1, f_2, f) \cdot \left[ 1 + e^{4\pi^2(f_1-f)(f_2-f)}(\beta_2+\pi(f_1+f_2)\beta_3)L + e^{4\pi^2(f_1-f)(f_2-f)}(\beta_2+\pi(f_1+f_2)\beta_3)L + \ldots \right]
\]

where \( \mu(\cdot) \) is given in Table VI and using the fact that \( 1 + e^x + e^{2x} + \ldots = e^{(N-1)x} \), (106) can be written as

\[
\Upsilon(f_1, f_2, f) = \gamma \mu(f_1, f_2, f) \cdot \left[ 1 - e^{4\pi^2(f_1-f)(f_2-f)}(L[N\beta_2+\pi(f_1+f_2)] + \ldots \right]
\]

which is equivalent to \( \Upsilon(\cdot) \) given in Table VII.
Table X: The expansion of the expressions given in Table IX using the terms given in Table VIII

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>$f_0 \sum_{n=-\infty}^{\infty} \delta(f - f_0 - (\kappa - l)R) \sum_{m,n,p} \phi_{n,m,p} \phi_{n-\kappa,m,p} (m,n,p,\kappa) \phi_{n,m,p} (m,n,p) \cdot R(S(mf_0)) \cdot E { \phi_{n-\kappa,m,p} (m,n,p,\kappa) } \cdot R(S(mf_0)) \cdot E { \phi_{n,m,p} (m,n,p) }</td>
</tr>
<tr>
<td>E</td>
<td>$f_0 \sum_{n=-\infty}^{\infty} \delta(f - f_0 - (\kappa - l)R) \sum_{m,n,p} \phi_{n,m,p} \phi_{n-\kappa,m,p} (m,n,p,\kappa) \phi_{n,m,p} (m,n,p) \cdot R(S(mf_0)) \cdot E { \phi_{n-\kappa,m,p} (m,n,p,\kappa) } \cdot R(S(mf_0)) \cdot E { \phi_{n,m,p} (m,n,p) }</td>
</tr>
</tbody>
</table>


