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CIRCULAR MARKOV CHAINS

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Abstract. In this note we explicitly determine the steady state probabilities for an arbitrary circular Markov chain by successively reducing the number of states of the Markov chain.

Key words. Markov chains, steady state probabilities

AMS subject classifications. 60J10

1. Introduction. In this note we explicitly determine the steady state probabilities π_i for circular Markov chains. In these chains, the states are located on a circle and transitions can only be made to neighbouring states. In the sequel we shall denote the transition probability matrix of the circular Markov chain with N states by

$$P_N = \begin{pmatrix} 0 & p_1 & & & q_1 \\ q_2 & 0 & p_2 & & \\ & \ddots & \ddots & \ddots & \\ & & q_{N-1} & 0 & p_{N-1} \\ p_N & & & q_N & 0 \end{pmatrix}.$$

The way we calculate the steady state probabilities is by successively reducing the number of states of the Markov chain. In the literature, this procedure is often used to develop numerical algorithms for calculating steady state probabilities for Markov chains or Markov processes (see e.g. [1, 7]). However, in the case of circular Markov chains it enables us to find an explicit expression for these probabilities.

Two special cases of circular Markov chains are well-known. On one hand, in the symmetrical case ($p_i = p$ and $q_i = q$ for all i) we are dealing with a so-called circular random walk (see [2]), for which we clearly have $\pi_i = 1/N$, $i = 1, \dots, N$. On the other hand, if $p_N = q_1 = 0$ we are dealing with a Markov chain *on a line*, for which it is easy to show, by balancing the flow between two neighbouring states, that

$$\pi_i = C \prod_{j=1}^{i-1} \frac{p_j}{q_{j+1}}$$

with C a normalization constant. However, this local balance property is not valid for an arbitrary circular Markov chain.

Circular Markov chains play a role in the context of nearest neighbour random walks on inhomogeneous periodic lattices, i.e. lattices which consist of a periodically repeated unit cell, where each unit cell contains a number of non-equivalent sites (see [3, 4, 5]). An example of circular Markov processes occurs in the evaluation of $M/M/c$ queueing systems in which the servers, after an idle period, only restart work when enough customers arrived to the system.

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2. Steady state probabilities. Our main result is stated in the next theorem.

THEOREM 2.1. *For the steady state probabilities π_i of the circular Markov chain with state space $\{1, \dots, N\}$ and transition probability matrix P_N , we have*

$$(1) \quad \pi_i = C \sum_{k=1}^N \prod_{j=1}^{k-1} q_{i+j} \prod_{j=k}^{N-1} p_{i+j},$$

where C is a normalization constant and q_i (resp. p_i) should be read as q_{i-N} (resp. p_{i-N}) if $i > N$.

Proof. The proof is by induction. The statement is clearly true for $N = 2$. Assume that (1) holds for a circular Markov chain with $N - 1$ states. Then we will prove (1) for a chain with N states by relating the steady state probabilities of this chain to those of another chain with one state less, to which we can apply the induction hypothesis. The latter chain is obtained as follows. First we condition the original chain to the set $\{1, \dots, N - 1\}$. The resulting chain has cycles in states 1 and $N - 1$. After removal of these cycles, we obtain the desired circular Markov chain with $N - 1$ states. So, let P'_N be the transition probability matrix of the circular Markov chain with N states, conditioned on the set $\{1, 2, \dots, N - 1\}$, i.e.

$$P'_N = \begin{pmatrix} q_1 p_N & p_1 & & & q_1 q_N \\ & q_2 & 0 & p_2 & \\ & & \ddots & \ddots & \ddots \\ & & & q_{N-2} & 0 & p_{N-2} \\ p_{N-1} p_N & & & & q_{N-1} & p_{N-1} q_N \end{pmatrix}.$$

Furthermore, let P''_N be the transition probability matrix obtained from the matrix P'_N by removing the cycles in state 1 and state $N - 1$, i.e.

$$P''_N = \begin{pmatrix} 0 & \frac{p_1}{p_1 + q_1 q_N} & & & \frac{q_1 q_N}{p_1 + q_1 q_N} \\ q_2 & 0 & p_2 & & \\ & \ddots & \ddots & \ddots & \\ & & q_{N-2} & 0 & p_{N-2} \\ \frac{p_{N-1} p_N}{q_{N-1} + p_{N-1} p_N} & & & \frac{q_{N-1}}{q_{N-1} + p_{N-1} p_N} & 0 \end{pmatrix}.$$

The matrix P''_N is the transition probability matrix of the desired circular Markov chain with only $N - 1$ states. Now it is easily checked that, if $(\alpha_1, \dots, \alpha_{N-1})$ is a solution of $\alpha P''_N = \alpha$, then

$$\left(\frac{\alpha_1}{p_1 + q_1 q_N}, \alpha_2, \dots, \alpha_{N-2}, \frac{\alpha_{N-1}}{q_{N-1} + p_{N-1} p_N} \right)$$

is a solution of $\alpha P'_N = \alpha$, and

$$\left(\frac{\alpha_1}{p_1 + q_1 q_N}, \alpha_2, \dots, \alpha_{N-2}, \frac{\alpha_{N-1}}{q_{N-1} + p_{N-1} p_N}, \frac{q_1 \alpha_1}{p_1 + q_1 q_N} + \frac{p_{N-1} \alpha_{N-1}}{q_{N-1} + p_{N-1} p_N} \right)$$

is a solution of $\alpha P_N = \alpha$. The latter solution relates (upto normalization) the steady state probabilities of the chain with N states to those of the related chain with $N - 1$ states. By combining this relation with the induction hypothesis, it is straightforward to prove the theorem. \square

