# Regulation and controlled synchronization

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# Regulation and controlled synchronization[1]

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## Abstract

In this paper we investigate the problem of controlled synchronization as a regulator problem. In controlled synchronization one is given autonomous transmitter dynamics and controlled receiver dynamics. The question is to find a (output) feedback controller that achieves matching between transmitter and controlled receiver. Several variants of the problem where the standard solvability assumptions for the regulator problem are not met turn out to have a solution. Simulations on two standard synchronization examples are also included.

## 1 Introduction

The regulator problem is a central problem in control theory and deals with the asymptotic tracking of certain classes of *prescribed trajectories* and asymptotic rejection of *undesired disturbances*. Over the years, the problem has received a lot of attention. For linear systems the regulator problem was extensively studied in [4] and [5]. For nonlinear systems, the problem was first studied in [6] and afterwards in [7] on the one hand and [8] on the other hand. An account of the state of the art on the problem, including both the linear and the nonlinear setting, is given in the book [3].

Essential in the regulator problem is that in any of the solutions to the problem the required feedback compensator incorporates an internal model of the exosystem that generates the command signals and the exogenous disturbances, cf. [6]. Typically, the solution to the regulator problem presented in [8],[3] is a local one around an equilibrium point of the exosystem. Further, an important hypothesis in [8],[3] is that the equilibrium point is stable, while all points in the neighborhood of the equilibrium point are Poisson stable. Recall that a point $w_0$ is called Poisson stable for a flow $\phi_t(\cdot)$ if $\phi_t(w_0)$ is defined for all $t \in \mathbb{R}$ and for every neighborhood $U$ of $w_0$ and every $T > 0$ there exist $t_1 > T$, $t_2 < -T$ such that $\phi_{t_1}(w_0), \phi_{t_2}(w_0) \in U$. In particular, this hypothesis implies that the exosystem has a critically stable linearization.

It is generally accepted that various controller design problems can be cast into an appropriate variant of the regulator problem. The purpose of this note is to show that *synchronization* of two systems can, under suitable hypotheses, be formulated as a regulator problem. This would imply that in principle a number of (controlled) synchronization problems become solvable through the application of results from [3]. An easy example of the same idea was dealt with in [9]. Unfortunately, in most cases where one seeks a controller that achieves synchronization, the more or less standard hypothesis of Poisson stability is not fulfilled. Thus, further investigation of the solvability of the corresponding regulator problem is needed. We will show that this research can be done successfully, although no complete characterization of the solvability is at hand.

This note is organized as follows. In the next section we recapitulate essential background on synchronization and controlled synchronization. Afterwards, in Section 3, we recast this within the context of a regulator problem, and a general preliminary result is presented and illustrated by means of an example. As another example of controlled synchronization when the hypotheses in [8],[3] do not hold, we consider the controlled synchronization of two coupled Van der Pol oscillators in Section 4. In Section 5, some conclusions are drawn.

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## 2 Controlled synchronization

Although different formulations of what is called synchronization exist, we adopt here a definiton that captures the essential ideas of it, cf. [11],[2]. Suppose two systems are given:

$$\begin{cases} \dot{w} &=& s(w) \\ y &=& \phi(w) \end{cases} \qquad (1)$$

and

$$\begin{cases} \dot{x} &=& f(x,y) \\ \eta &=& \phi(x) \end{cases} \qquad (2)$$

where both $x$ and $w$ are in $I\!R^n$ (or, more generally, in the same Riemannian manifold). We will assume that the origin is one of the equilibrium points of (1), and that $\phi(0) = 0$. System (1) is the so-called *transmitter* (or *master*), and system (2) is the *receiver* (or *slave*). Synchronization of (1) and (2) occurs if, no matter how (1) and (2) are initialized, we have that asymptotically their states will match, i.e.,

$$\lim_{t \to +\infty} \|x(t) - w(t)\| = 0 \qquad (3)$$

Typically, the receiver (2) depends on (1) via the drive signal $y = \phi(w)$, which explains the transmitter/receiver terminology. It is clear that synchronization will only occur in particular cases. In a previous paper, [11], it was shown that an interpretation is to view (2) as an observer for (1) given the output signal $\phi(w)$. So in those applications where one is able to design (2) freely, this provides a potential solution for the synchronization problem. In most of the synchronization literature (see e.g. [12]) however, the systems (1) and (2) are systems that are given beforehand, so that no synchronization will occur in general. However, and this is the viewpoint that we take in this paper, we may consider a controlled version of the problem, in that we allow the receiver dynamics also to depend on a control variable $u$, that is, we replace (2) by

$$\dot{x} = f(x,y,u) \qquad (4)$$

The natural question now is to seek an appropriate dynamic feedback of the form

$$u = \alpha(z,\eta,y) \qquad (5)$$

where $z$ is driven by the dynamics

$$\dot{z} = k(z,\eta,y) \qquad (6)$$

such that the resulting closed loop dynamics (4,5,6) synchronizes with (1), i.e., (3) holds for (1,4,5,6), and some appropriate stability requirements are fulfilled. Note that this definition in a more general form is contained in [2], and typically brings the problem of controlled synchronization within the scope of the regulator problem.

## 3 Controlled synchronization as a regulator problem

The controlled synchronization problem as formulated above can be viewed as a regulator problem. The *exosystem* is given as (1) and generates the to-be-tracked signals. (We require full state regulation of $w$, but obviously variants are possible.) The plant is given as (4), and we seek a compensator of the form (6,5) such that the error $e(t) := x(t) - w(t)$ satisfies

$$\lim_{t \to +\infty} e(t) = 0 \qquad (7)$$

Additionally, we require the unforced closed loop dynamics

$$\begin{cases} \dot{x} &=& f(x,0,\alpha(z,\eta,0)) \\ \dot{z} &=& k(z,\eta,0) \end{cases} \qquad (8)$$

to be asymptotically stable. Note that usually this requirement is not *explicitly* included in the synchronization problem, but is very natural in our context. Also for synchronization purposes, this property seems quite natural (but not absolutely necessary!). In particular, in the secure communication context this requirement implies that the receiver becomes silent when the transmitter is silent.

It is clear that with the above conditions we have brought the problem of controlled synchronization in the framework of a regulator problem. Unfortunately, in most applications of chaos synchronization (like e.g. secure communication ([12])) the system (1) possesses a chaotic attractor in which several equilibrium points with un unstable linearization are embedded. This means that one of the standard hypotheses from [3] on the exosystem (Poisson stability of the equilibrium) is *not* met. Therefore, the results from [3] are not directly applicable, and this makes the controlled synchronization problem difficult.

In the following we want to show that despite the fact that the transmitter/receiver dynamics do not satisfy the standard Poisson stability requirements, it is possible to solve the corresponding regulator problem in several cases. Our work can thus be viewed as an extension of the standard solutions of the regulator problem that have been given in [5],[3],[7]. The first variant we treat consists of solving the regulator problem for Lur'e-like systems, i.e., linear systems with a nonlinear output dependent feedback loop.

**Theorem 3.1** *Consider the transmitter*

$$\begin{cases} \dot{w} &=& Aw + \Psi(y) \\ y &=& Cw \end{cases} \qquad (9)$$

*and receiver*

$$\begin{cases} \dot{x} &=& Ax + \Psi(y) + Bu \\ \eta &=& Cx \end{cases} \qquad (10)$$

with $w, x \in \mathbb{R}^n$ $u \in \mathbb{R}^m$, and $\Psi$ a mapping of appropriate dimensions, and $A$, $B$, $C$ matrices of appropriate dimensions. Under the assumptions that $(C, A)$ is detectable and $(A, B)$ is stabilizable, the controlled synchronization problem is solvable.

**Proof** Since $(C, A)$ is detectable, there exists a matrix $K$ such that all eigenvalues of $A + KC$ are in the open left half plane. Further, the fact that $(A, B)$ is stabilizable implies that there exists a matrix $F$ such that all eigenvalues of $A + BF$ are in the open left half plane. It is then readily checked that the following dynamic feedback solves the controlled synchronization problem:

$$\begin{cases} \dot{\hat{w}} &= A\hat{w} + K(\hat{y} - y) + \Psi(y) \\ \dot{\hat{x}} &= A\hat{x} + K(\hat{\eta} - \eta) + \Psi(y) + Bu \\ \hat{y} &= C\hat{w} \\ \hat{\eta} &= C\hat{x} \\ u &= F(\hat{x} - \hat{w}) \end{cases} \quad (11)$$

$\blacksquare$

Below, we give an example of a chaotic transmitter and receiver where the result can be applied.

**Example 3.2** In this example, we take as the transmitter the Chua circuit, which in dimensionless form is described by the equations

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\alpha(m_1 + 1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}}_{A} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \underbrace{\begin{pmatrix} -\alpha(m_0 - m_1)\mathrm{sat}(w_1) \\ 0 \\ 0 \end{pmatrix}}_{\Psi(w_1)}$$

$$(12)$$

where $\mathrm{sat}(\cdot)$ is the saturation function given by $\mathrm{sat}(w_1) = \frac{1}{2}(\mid w_1 + 1 \mid - \mid w_1 - 1 \mid)$. For the parameter values $\alpha = 15.6$, $m_0 = -\frac{8}{7}$, $m_1 = -\frac{5}{7}$, $\beta = 25$, this system is known to have a so called double scroll chaotic attractor, in which three unstable equilibrium points are embedded (see e.g. [1]). We assume that $y = w_1$ is the transmitted signal, so that (12) takes the form (9). Note that with this choice of $y$ the pair $(C, A)$ is observable. As the receiver, we take the system

$$\dot{x} = Ax + \Psi(w_1) + bu \quad (13)$$

where $u \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Note that we now have that for a generic choice of $b$ the pair $(A, b)$ is stabilizable. Thus, all conditions of Theorem 3.1 are satisfied for a generic choice of $b$. In Figures 1,2,3 a simulation is given for (12),(13),(11) with $b = (1\ 1\ 1)^T$, $F = (-1\ -15.6\ 0)^T$, $K = (4.36\ 0\ 0)$.



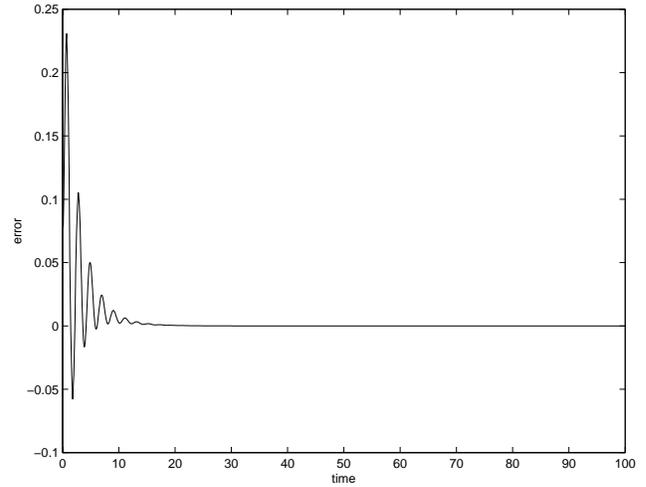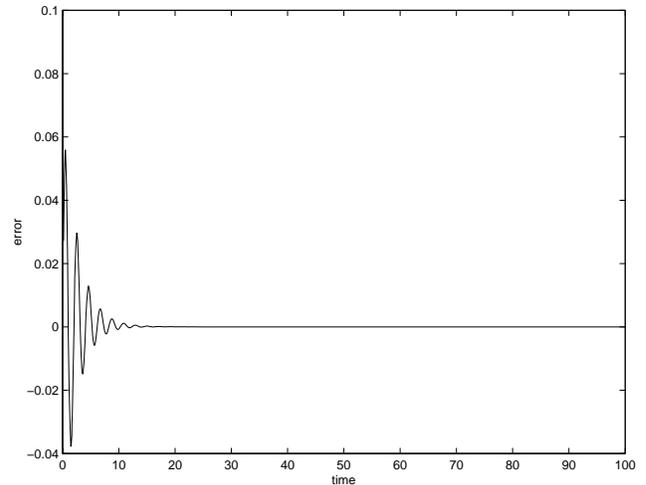**Figure 1:** $x_1 - w_1$ for Example 3.2



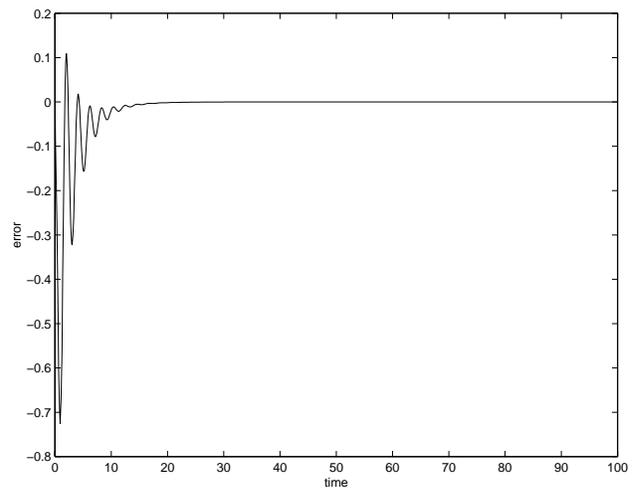**Figure 2:** $x_2 - w_2$ for Example 3.2



**Figure 3:** $x_3 - w_3$ for Example 3.2

# 4 Controlled synchronization of coupled Van der Pol systems

In this section we discuss the controlled synchronization problem for a (controlled) Van der Pol equation. Clearly, in this case the transmitter dynamics do not meet the Poisson stability hypothesis, nor does this system belong to the class of Lur'e-like systems of Section 3. Nevertheless, the regulator problem can in this case be solved by means of static linear error feedback, provided the gain is sufficiently large. The analysis needed involves some nontrivial classical results about time-varying linear differential equations.

As transmitter dynamics we take a Van der Pol system of the form

$$\begin{cases} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 - (w_1^2 - 1)w_2 \\ y &= w_1 \end{cases} \quad (14)$$

The only equilibrium of this system is the origin, which is an unstable focus. Thus, the system (14) does not satisfy the hypotheses in ([3]). Further, it is well known (see e.g. [1]) that this system has a limit cycle $C$ that is attracting for all initial points $w(0) \in \mathbb{R}^2 - \{0\}$. Consider a solution $(\tilde{w}_1(t), \tilde{w}_2(t))$ that starts on $C$, and let $T$ denote its period. Define $p(t) := \tilde{w}_1(t)^2 - 1$. By modifying an argument in [13], it may then be shown that

$$\bar{p} := \frac{1}{T} \int\limits_0^T p(\tau)d\tau > 0 \quad (15)$$

As receiver dynamics, we take the following controlled "copy" of (14):

$$\begin{cases} \dot{x}_1 &= x_2 + \alpha u \\ \dot{x}_2 &= -w_1 - (w_1^2 - 1)x_2 + \beta u \end{cases} \quad (16)$$

We are now interested in the question whether or not both systems will synchronize by applying a static (high-gain) error feedback $u = -c(x_1 - w_1)$. To study this question, we introduce the error signals $e_i := x_i - w_i$ $(i = 1, 2)$, which, after feedback, satisfy:

$$\begin{cases} \dot{e}_1 &= -\alpha c e_1 + e_2 \\ \dot{e}_2 &= -\beta c e_1 - (w_1^2 - 1)e_2 \end{cases} \quad (17)$$

Further, it follows from the fact that $C$ is attracting and the time-invariance of (14) that (17) is exponentially stable (and thus (14) and (16) synchronize) if and only if the following linear periodic differential equation is exponentially stable:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\alpha c & 1 \\ -\beta c & -p(t) \end{pmatrix}}_{A(t)} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \quad (18)$$

We now study the exponential stability of this differential equation for three different cases.

**Case 1:** $\beta = 0$

Note that one may assume without loss of generality that $\alpha = 1$. The fundamental matrix $\Phi(t, t_0)$ of (18) is then given by

$$\Phi(t, t_0) = \begin{pmatrix} \exp(-c(t - t_0)) & \psi(t, t_0) \\ 0 & \exp(-\int_{t_0}^t p(\tau)d\tau) \end{pmatrix}$$

where $\psi(t, t_0) := \int_{t_0}^t \exp(-c(t - \tau) - \int_{t_0}^\tau p(\sigma)d\sigma)d\tau$. From this form and (15), it is readily seen that (18) is (uniformly) exponentially stable if and only if $c > 0$. Thus, in this case synchronization of (14) and (16) may be achieved by an error feedback $u = -c(x_1 - w_1)$, where $c > 0$. Note, however, that in this case the unforced closed loop dynamics (8) will have the form (18) with $\alpha = 1$, $\beta = 0$, $p(t) = -1$. From this it is easily seen that the unforced closed loop dynamics will be unstable, no matter how $c$ is chosen.

**Case 2:** $\alpha = 0$

One may now assume without loss of generality that $\beta = 1$. The unforced closed loop dynamics now have the form (18) with $\alpha = 0$, $\beta = 1$, $p(t) = -1$. From this we see that also in this case the unforced closed loop dynamics will be unstable, no matter how $c$ is chosen. Defining $\tilde{e}_1 := e_1$, $\tilde{e}_2 := e_2/c$, $\epsilon := 1/c$, we obtain the following singularly perturbed differential equation:

$$\begin{cases} \epsilon\dot{\tilde{e}}_1 &= \tilde{e}_2 \\ \dot{\tilde{e}}_2 &= -\tilde{e}_1 - p(t)\tilde{e}_2 \end{cases} \quad (19)$$

Using methods from singular perturbation theory (see e.g. [14]) it may then be shown that (18) is (uniformly) exponentially stable when $\epsilon \downarrow 0$ (i.e., when $c \to +\infty$). Thus, in this case synchronization of (14) and (16) may be achieved by high-gain error feedback.

We next investigate whether a reasonable lower bound $c_*$ may be given so that (18) is exponentially stable for all $c > c_*$. To this end, we assume that $p(t)$ is differentiable, and define

$$q(t) := \frac{1}{4}p(t)^2 + \frac{1}{2}\dot{p}(t), \quad \bar{q} := \frac{1}{T} \int_0^T |q(\tau)|d\tau$$

Define $\xi(t) := \exp(\frac{1}{2} \int_0^t p(\tau)d\tau)e_1(t)$. It then follows that $\xi(t)$ satisfies the Hill equation

$$\ddot{\xi} + (c - q(t))\xi = 0$$

A result from [10] gives that the growth behavior of solutions of this Hill equation is given by

$$\xi(t) = O\left(\exp\left(\frac{1}{2\sqrt{c}} \int_0^t |q(\tau)|d\tau\right)\right)$$

From this result and the definition of $\xi(t)$, also a result on the growth behavior of solutions of (18) may be obtained. This result then gives that (18) is (uniformly)
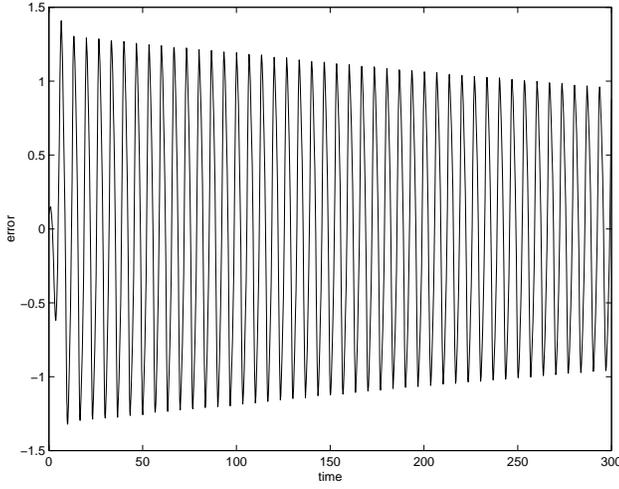
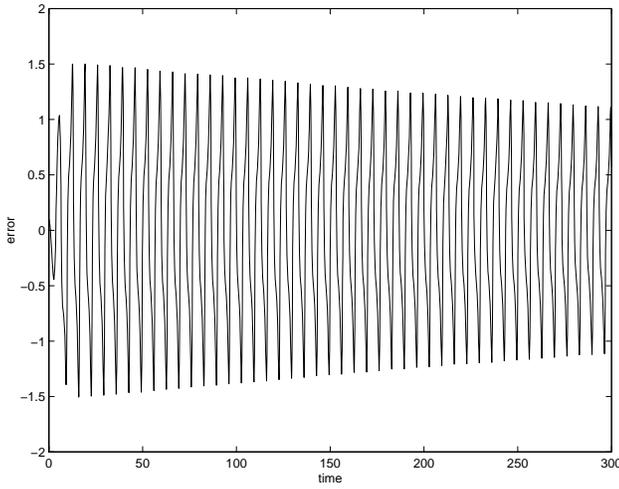**Figure 4:** $x_1 - w_1$ for Case 2 with $c = 1.4$



**Figure 5:** $x_2 - w_2$ for Case 2 with $c = 1.4$

exponentially stable for all $c > c_*$, where

$$c_* := \left( \frac{\bar{q}}{\bar{p}} \right)^2$$

Simulations give that for (14) we have $\bar{p} \approx 1.06$, $\bar{q} \approx 1.45$, and thus $c_* \approx 1.87$. Further, simulations indicate that in fact (14) and (16) synchronize for all $c > 1.39$, as is illustrated by means of a simulation in Figures 4,5. Thus, the bound $c_*$ given above is a conservative one.

**Case 3:** $\alpha \neq 0, \beta \neq 0$
Without loss of generality, we may assume that $\alpha = 1$. The unforced closed loop dynamics now have the form (18) with $\alpha = 1$, $p(t) = -1$. This gives that the unforced closed loop dynamics is asymptotically stable if and only if $\beta, c > 1$.

Defining $\tilde{e}_1$, $\tilde{e}_2$, $\epsilon$ as in Case 2, we obtain the following singularly perturbed differential equation:

$$\begin{cases} \epsilon \dot{\tilde{e}}_1 &= -\tilde{e}_1 + \tilde{e}_2 \\ \dot{\tilde{e}}_2 &= -\beta \tilde{e}_1 - p(t) \tilde{e}_2 \end{cases} \tag{20}$$

Application of Tikhonov's Theorem (see e.g. [14]) gives that (20) is (uniformly) exponentially stable for $\epsilon \downarrow 0$ (i.e., when $c \rightarrow +\infty$) if and only if the following differential equation is (uniformly) exponentially stable:

$$\dot{\xi} = -(\beta + p(t))\xi$$

It is readily checked that this differential equation is uniformly exponentially stable when

$$\beta > -\bar{p} \tag{21}$$

Thus, in this case synchronization of (14) and (16) may be achieved by high-gain error feedback when (21) holds.

Define $p_{\max} := \max\{p(t) \mid t \in [0, T]\}$, $p_{\min} := \min\{p(t) \mid t \in [0, T]\}$. Note that, since $p(t) = \tilde{w}_1(t)^2 - 1$, and there is a $t \in [0, T]$ such that $\tilde{w}_1(t) = 0$, we have that $p_{\min} = -1$. When $\beta > 1$, it may then be shown that (14) and (16) synchronize when $c > c_*$, where

$$c_* = p_{\max} + 2\beta + \sqrt{2\beta p_{\max} + 4\beta^2} \tag{22}$$

The proof of this claim proceeds as follows. Consider the Riccati differential equation

$$\dot{q} = q^2 + (c - p(t))q + \beta c \tag{23}$$

It may then be shown that when $c > c_*$ we have that

$$q_{\max} := \frac{1}{2}(-(c - p_{\max}) - \sqrt{(c - p_{\max})^2 - 4\beta c})$$

and

$$q_{\min} := \frac{1}{2}(-(c - p_{\min}) - \sqrt{(c - p_{\min})^2 - 4\beta c})$$

are real and satisfy $q_{\min} < q_{\max} < 0$. Further, it may be shown that

$$\dot{q} \mid_{q=q_{\min}} > 0, \quad \dot{q} \mid_{q=q_{\max}} < 0$$

which implies that the interval $[q_{\min}, q_{\max}]$ is an invariant set for (23). Let $q(t)$ be a solution of (23) that starts on $[q_{\min}, q_{\max}]$. Then we will have as a consequence of the above that the matrix

$$Q(t) := \begin{pmatrix} q(t) & 1 \\ 0 & \frac{1}{q(t)} \end{pmatrix}$$

is uniformly bounded and invertible for all $t > 0$. Thus, we will have that the time-varying coordinate change $y := Q(t)e$ is well-defined. It may be shown that $y$
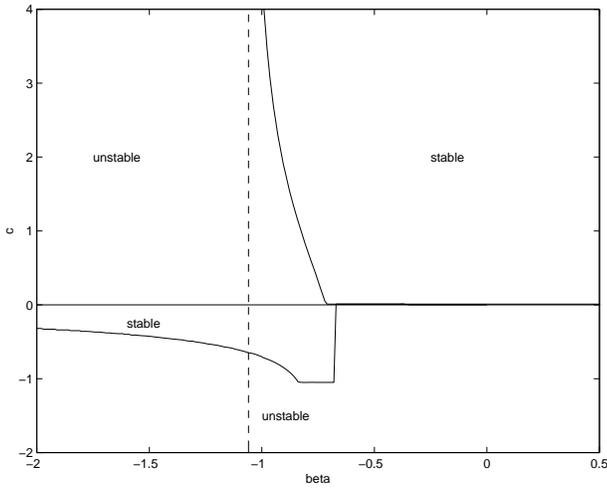
**Figure 6:** Stability regions in $(\beta, c)$-plane for Case 3

satisfies the linear time-varying differential equation in triangular form

$$\dot{y} = \begin{pmatrix} q(t) - p(t) & 0 \\ -\frac{\beta c}{q(t)^2} & -(q(t) + c) \end{pmatrix} y \qquad (24)$$

Further, it may be shown that when $c > c_*$ we will have that there exists a $\delta > 0$ such that $q(t) - p(t), -(q(t) + c) < -\delta$ for all $t > 0$, which gives that (24) is uniformly exponentially stable. Since $Q(t)$ is uniformly bounded, this immediately implies that also (18) is uniformly exponentially stable.

In Figure 6, the numerically determined regions in the $(\beta, c)$-plane where (14) and (16) synchronize are indicated. From this figure, we draw the conclusion that (14) and (16) synchronize for all values of $\beta$ and $c$ for which the unforced closed loop dynamics are uniformly exponentially stable. Further, when comparing Figure 6 to the lower bound $c_*$ given in (22), we see that this lower bound is very conservative.

## 5 Conclusions

We have shown that the controlled synchronization problem can be treated as a regulator problem. However, in most applications where synchronization plays a role some of the standard assumptions for the solvability of the regulator problem are not fulfilled, and we are thus asked to find separate solvability conditions. In a few case we have established that it is possible to achieve such controlled synchronization and, in particular, we have shown that a few standard examples from the synchronization literature admit a solution. Simulations support our findings, and, in fact, suggest that for the synchronization in the Van der Pol example in Section 4 the bounds obtained are relatively conservative. It is therefore interesting to continue this research and to see whether a more general set of solvability conditions can be derived.

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