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The geostrophic regime of rapidly rotating turbulent convection

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ABSTRACT
Rotating Rayleigh–Bénard convection is a simple model system used to study the interplay of buoyant forcing and rotation. Many recent studies have focused on the geostrophic regime of turbulent rotating convection where the principal balance of forces is between the Coriolis force and the pressure gradient. This regime is believed to be representative of conditions in geophysical and astrophysical flows. We hope to be able to extrapolate findings from laboratory experiments and numerical simulations towards these large-scale natural flows. In this paper I sketch the phase diagram of the geostrophic regime of rotating convection, put experimental and numerical studies in their place in these diagrams and discuss the partitioning into subranges characterized by different flow structures and heat transfer scaling. I also discuss some complications faced by experimentalists, such as constraints on the dimensions of the convection cell, wall modes near the sidewall and centrifugal buoyancy.

KEYWORDS
Turbulent rotating convection; flow structures; convective heat transfer; inverse energy cascade; laboratory experiments; direct numerical simulations.

1. Introduction

Turbulent, buoyancy-driven flows in geophysics and astrophysics are almost invariably and decisively shaped by rotation of the celestial body on which they reside. On our Earth, perhaps the most prominent and visible example is the atmosphere [1]. In fact, the notion that rotation shapes atmospheric flows can be traced back more than 280 years to Hadley’s work [2] on the trade winds. Oceanic deep convection in polar regions [3] drives the thermohaline circulation that is of eminent importance for our climate. Hidden to the eye is the convective flow in Earth’s liquid-metal outer core [4,5] that sustains the magnetic field [6,7] that protects our planet from harmful cosmic radiation. Moving away from our own planet, further examples are found in the giant gas planets with their characteristic zonal flows [8,9]. Our Sun also possesses a convective outer layer where its rotation shapes the flow [10].

Due to their vastness, remoteness and complexity it is exceedingly difficult to study the interaction of buoyancy-driven turbulence and rotation directly in these natural flows. A greatly simplified model to study is found in rotating Rayleigh–Bénard con-
vection (RRBC): a layer of fluid sandwiched between two parallel horizontal plates where the bottom plate is at a higher temperature than the top plate, rotating about a vertical axis parallel to gravity. This system is mathematically well-defined and eminently suited for both experimental study and direct numerical simulation (DNS). It is a flow that develops turbulence without any mechanical intrusions or prescribed forcing protocols. Despite its simplicity, a wide and diverse range of flow phenomenologies and scalings of various statistical flow quantities have been observed as a function of the control parameters.

Three principal parameters are required to describe RRBC; here I will use the Rayleigh number $Ra$, the Ekman number $E$ and the Prandtl number $Pr$, defined as

$$Ra = \frac{g \alpha \Delta T H^3}{\nu \kappa}, \quad E = \frac{\nu}{2 \Omega H^2}, \quad Pr = \frac{\nu}{\kappa},$$

with $g$ the gravitational acceleration, $\Delta T$ the temperature difference between the plates and $H$ their vertical separation, $\Omega$ is the rotation rate, and $\alpha$, $\nu$ and $\kappa$ respectively represent the thermal expansion coefficient, kinematic viscosity and thermal diffusivity of the fluid. The Rayleigh number $Ra$ quantifies the strength of thermal forcing; increasing $Ra$ while keeping the other parameters constant leads to a more intense convective flow with a higher transfer rate of heat. The Ekman number $E$ quantifies the relative importance of viscosity to rotation; it is notably inversely proportional to the rotation rate $\Omega$ and takes on small values $E \ll 1$ in all cases considered in this paper. Finally, the Prandtl number $Pr$ is a fluid property. Another parameter that is used to quantify the strength of rotation is the Taylor number $Ta = E^{-2}$. A parameter that is frequently used to directly compare the relative importance of buoyancy and rotation is the convective Rossby number $Ro = E \sqrt{Ra/Pr}$, comparing inertial to Coriolis forces where the inertial effects are estimated by the convective free-fall velocity $U = \sqrt{g \alpha \Delta TH}$. Note that $U$ is an overestimate of the mean convective flow velocity that will occur in the fluid layer and $Ro$ thus tends to overestimate inertial effects. Additionally, a geometrical parameter is needed. For confined domains (used in experiments as well as DNS) an upright cylinder or cuboid cell is most popular. Then the aspect ratio $\Gamma = \frac{D}{H}$ can be defined, where $D$ is either the diameter of the cylinder or the horizontal side length of the cuboid. The horizontally periodic domain is also a popular geometry for numerical simulations; in that case a similar aspect ratio $\Gamma$ can be defined, where now $D$ represents the horizontal periodicity length of the computational domain.

The primary output parameters are the efficacy of the convective heat flux and the magnitude of the velocity fluctuations (or intensity of turbulence). These are commonly expressed in dimensionless form as the Nusselt number $Nu$ and the Reynolds number $Re$:

$$Nu = \frac{qH}{k \Delta T}, \quad Re = \frac{u' H}{\nu},$$

where $q$ is the total heat flux, $k$ is the thermal conductivity of the fluid and $u'$ is a fluctuation velocity scale. The Nusselt number $Nu$ expresses the total heat flux $q$ (convection and conduction) normalized by the conductive contribution $q_{\text{cond}} = k \Delta T / H$. In absence of convection $Nu = 1$ while vigorously turbulent convection lets $Nu$ increase by several orders of magnitude. For the intensity of the velocity fluctuations it is customary to employ the root-mean-square velocity, which may be subdivided
Rayleigh numbers are \(14, 16, 17\). The critical Rayleigh number for onset of steady convection is also directly affected by the small Ekman numbers \(E\) that develop at onset, and the Prandtl number \(Pr\) whenever \(Pr > 0.68\). For finite Prandtl numbers, \(Pr\) may take on larger values for highly viscous liquids.

The addition of rotation to Rayleigh–Bénard convection has an overall stabilizing effect. Linear stability analysis by Chandrasekhar [14] has revealed that, as rotation is applied, the critical Rayleigh number \(Ra_C\) for onset of convection (i.e. \(Nu > 1\) when \(Ra > Ra_C\)) increases from Lord Rayleigh’s seminal results [15] for nonrotating convection. Depending on the value of the Prandtl number, convection sets in either as steady (for \(Pr \geq 0.68\)) or as oscillatory convection (for \(Pr < 0.68\)). In the limit of small \(E\), relevant for geophysical and astrophysical flows, the corresponding critical Rayleigh numbers are [14,16,17]

\[
Ra_C^o = 6 \frac{(\frac{1}{4} \pi^2)^{2/3}}{(1 + Pr)^{1/3}} (E/Pr)^{-4/3} \approx 17.4 (E/Pr)^{-4/3}, \quad (Pr < 0.68)
\]  

(2a)

for oscillatory convection (where in the last approximation \(Pr \ll 1\) is implied) and

\[
Ra_C^s = (\frac{27}{4} \pi^4)^{1/3} E^{-4/3} = 8.70 E^{-4/3}, \quad (Pr \geq 0.68)
\]  

(2b)

for steady convection. This asymptotic result is valid for both no-slip and stress-free boundary conditions on the plates [18]. However, for finite \(E\) and no-slip conditions a correction is required that slightly lowers the prefactor [18]:

\[
Ra_C = \left[ \left( \frac{27}{4} \pi^4 \right)^{1/3} - 4 \sqrt{2} \left( \frac{1}{4} \pi^2 \right)^{1/3} E^{1/6} \right] E^{-4/3} \approx 8.70 - 9.63 E^{1/6} E^{-4/3}.
\]  

(3)

Whenever \(Ra_C\) is mentioned in the remainder of this document, I refer to \(Ra_C^s\), the critical Rayleigh number for onset of steady convection. Depending on the boundary conditions this may either consider equation (2b) for stress-free or equation (3) for no-slip.

In addition to the delayed instability to higher \(Ra\), the horizontal scale \(\ell_C\) of convection that develops at onset is also directly affected [14,16,17]; again the small-\(E\) limit is implied:

\[
\ell_C^o/H = (1 + Pr)^{1/3} (2\pi^4)^{1/6} (E/Pr)^{1/3} \approx 2.41 (E/Pr)^{1/3}, \quad (Pr < 0.68)
\]  

(4a)

\[
\ell_C^s/H = (2\pi^4)^{1/6} E^{1/3} = 2.41 E^{1/3}. \quad (Pr \geq 0.68)
\]  

(4b)

(Again, the last approximation for \(\ell_C^s\) assumes \(Pr \ll 1\).) Convection sets in as narrow
convective cells in rotating convection at small $E$.

Another property of flows at very small Ekman and Rossby numbers is that the so-called geostrophic force balance is imposed, a balance between Coriolis force and pressure gradient (e.g. [19,20]). In case all other forces remain negligibly small, this balance directly leads to the Taylor–Proudman constraint [21,22], which states that no axial variation of the velocity vector $\mathbf{u}$ is allowed ($\partial \mathbf{u} / \partial z = 0$). In that case the flow field is expected to be practically two-dimensional (2D) and vertically aligned along the axis of rotation. However, small violations of the geostrophic balance induced by effects of inertia or buoyancy can still lead to features with vertical gradients in the flow, and even turbulence. Nonetheless, the flow field in these flows at small $E$ clearly shows vertical straightening and organization along the rotation axis.

Many geophysical and astrophysical convective flows are expected to belong to this so-called geostrophic regime of rotating convection: geostrophy is the principal force balance, but turbulence is still eminently possible (e.g. [23]), even though it is strongly constrained by rotation. It is thus our hope that insights and scaling arguments found in the geostrophic regime of turbulent RRBC can be extrapolated to better understand and model these vast natural flows.

The geostrophic regime of RRBC is studied both experimentally and numerically. In experiments convection cells, typically upright cylinders, are rotated about their axis. Simulations can consider the full governing equations (the Navier–Stokes and heat equations for incompressible fluids) in a direct numerical simulation (DNS) on a similar cylindrical domain for comparison, or employ a horizontally periodic domain for ease of numerical solution and to mimic a laterally unbounded layer. A useful simplification can be found in making an asymptotic expansion in the limit of rapid rotation, i.e. $E \to 0$ and $Ro \to 0$, and then retaining only the dominant terms [16,23–26]; the so-called non-hydrostatic quasi-geostrophic model (NH-QGM). This asymptotic model is a closed set of governing equations that can be simulated on horizontally periodic domains with guaranteed rotational constraint.

In this paper I want to give an overview of the works aimed at the exploration of the geostrophic regime of turbulent RRBC. It is rather recent that studies have been able to decisively enter into this regime that poses challenges to experimentalists and numericists alike. In section 2 I will explain what exactly constitutes the geostrophic regime of RRBC and discuss the various flow phenomenologies that can occur. Then, section 3 gives an overview of the studies aimed at exploring this regime. I will treat the results for various flow and heat-flux regimes observed in these works and consider the transition predictions. In section 4 I will focus on the experimental realization of geostrophic RRBC and the challenges and complications faced there. Section 5 concludes and sketches perspectives for future work.

### 2. A sketch of the geostrophic regime

To get a clearer idea of what exactly constitutes the geostrophic regime of RRBC I will partition and fill a parameter space in this section. Note that I will first assume large $Pr \geq 0.68$, and return to the changes for $Pr < 0.68$ at the end of the section.

Many studies of (geostrophic) RRBC have focused on the convective heat transfer (Nusselt number). Two different measurement strategies have been most popular: either one varies the rotation rate $\Omega$ ($\sim E^{-1}$) at constant temperature difference $\Delta T$ ($\sim Ra$) or one varies $\Delta T$ at constant $\Omega$. Both measurement strategies typically reveal a subdivision into three distinct trends, with example plots provided in figure 1:
1. At high $Ra$ and/or high $E$ values (or small $1/Ro$) there is a range where the relative influence of rotation is so minor that convection is essentially nonrotating. This means that $Nu$ values nicely merge with the nonrotating results, which can be described as a power law of the form $Nu = CRa^\beta$ with $\beta$ close to 1/3 for most fluids [27]. This range is therefore generally referred to as ‘nonrotating’ or ‘rotation-unaffected’ convection.

2. In an intermediate range we can see two distinct behaviors. Depending on the exact parameter values, and only for no-slip boundary conditions on the plates, $Nu$ may actually become larger than the nonrotating result. Correspondingly, the $Nu(E)$ graph [or $Nu(1/Ro)$] may show positive or negative slopes in this range. This ‘overshoot’ effect (addition of intermediate-level rotation can enhance heat transfer with respect to no rotation, by up to $\sim$ 50% [28,29]) is caused by so-called Ekman pumping [19]: the Ekman boundary layers formed at no-slip plates transport fluid from very close to the plates into the vortical plumes; an efficient process for heat transfer and an example of increased vertical coherence of the temperature field in systems with interaction of driving and stabilizing forces [30]. The heat transfer overshoot is typically observed for $Pr \gtrsim 1$ fluids (e.g. [28,29,31–37]), though only at moderate values of $E \gtrsim 10^{-6}$ and $Ra \lesssim 10^{10}$. For larger $Ra$ and smaller $E$ the overshoot vanishes and heat transfer is always below the corresponding nonrotating value and the $Nu(Ra)$ curve is then steeper than the nonrotating case [38–40]. In the literature this range is referred to as ‘rotation-affected’ or ‘weakly rotating’ convection. However, we will see that within this range we can already find convective flow with a principal geostrophic force balance (which implies strong rotation); it is just not explicitly reflected in the heat transfer trend.

3. For smaller $Ra/Ra_C$, correspondingly at small $Ra$ and/or $E$ values (small $1/Ro$), we find the range where $Nu$ is very sensitive to the exact values of $Ra$ and $E$. $Nu$ is strongly rising as a function of both $Ra$ and $E$. For example, simulations and experiments at $Pr = 7$ reveal a steep scaling $Nu \sim Ra^\beta$ with a large ($E$-dependent) exponent $\beta = 1.3$ at $E = 10^{-3}$ increasing to $\beta = 3.6$ at $E = 10^{-7}$ in this range [40,41]. This range is referred to as ‘rapidly rotating’, ‘rotation-dominated’ or ‘geostrophic’ convection in the literature. At large $Pr \gtrsim 2$ this range turns out to correspond to the columns flow state of rotating convection (to be discussed next), so I want to propose the terminology ‘columnar’ convection here.

These three regimes are recognized from heat transfer data in quite some studies, e.g. [33,36,39,43–49]. A third $Nu$ measurement strategy [32,50], varying parameters $Ra$ and $E$ in such a way that $Ro$ remains constant, leads to power-law $Nu(Ra)$ behaviors that are displaced from each other when different $Ro$ are compared. This strategy is significantly less likely to encounter the regime transitions as it moves nearly parallel to the transition lines, as we shall see.

The transition between heat transfer ranges 1 and 2 (see figure 1) has received a lot of attention [33,34,42,51–54]. It turns out to be a sharp transition, which is unexpected for a turbulent flow. Here, I will not go into further detail on this transition as it is out of scope for this paper. It is characterized by a transitional Rossby number $Ro_{t,1+2}$. Note that in periodic domains this transition from nonrotating to rotation-affected convection is gradual and no clear transition point can be defined (e.g. [29,55]).

In this paper we are more interested in the transition between heat transfer ranges 2 and 3, as inferred from heat transfer measurements. This transition has received less
Nu, normalized with the nonrotating value Nu(0), as a function of $1/Ro = \sqrt{Pr/Ra/\Gamma}$ at constant $Ra = 2.73 \times 10^8$, $Pr = 6.26$ and $\Gamma = 1$. Black squares: DNS results from [42]. Red circles: experimental results from [34]. There is an overshoot range where rotation enhances heat transfer compared to the nonrotating value. Figure source: [43].

Figure 1. Example results of the two main Nu measurement strategies: (a) constant $\Delta T$, (b) constant $\Omega$.

attention. What is clear, though, is that the steep scaling range 3 only manifests at low enough $E$, and grows in size as $E$ becomes smaller. A more quantitative treatment of this transition is deferred to section 3.

These two transitions, along with Chandrasekhar’s onset criterion (2b), serve as the backbone of the parameter space that is plotted in figure 2 (dashed blue lines), for both small and large $Pr$. This diagram displays $Ra/Ra_C$ as a function of $E$ following [39]. Note that formally the phase diagram should be 3D (including $Pr$ as third axis), but for clarity I will assume constant $Pr$ for the diagrams, knowing that boundaries between various flow regimes may be a function of $Pr$. The dashed blue lines give the general division according to the three heat transfer ranges identified before. It is possible to provide further detail by considering the flow phenomenology that is encountered.

First, it has been known since Rossby’s pioneering heat-flux measurements that there can be convective motions below Chandrasekhar’s onset prediction (2b). There is a small but significant rise in Nu above 1 in a region that expands in size as the rotation rate increases (smaller $E$). This was shown to be the result of a so-called wall mode [56–64], a convection mode that manifests near vertical walls with alternating lobes of up- and downflow that precesses around the circumference of the tank. The asymptotic analysis by Zhang & Liao [56] has provided the following result for the critical Rayleigh number $Ra_w^C$ for onset of wall-mode convection:

$$Ra_w^C = \pi^2 \sqrt{6\sqrt{3}E^{-1}} + 46.55E^{-2/3},$$

for the current definitions of $Ra$ and $E$, as given in [63]. This equation is for a confined domain with no-slip boundaries, as in experiments. For confined domains with stress-free boundaries, as sometimes used in DNS, wall modes can also develop but at slightly different $Ra_w^C$. Below the $Ra_w^C$ line in the diagram, there is no convective flow. In confined domains with a sidewall there will be convection as a wall mode, even below the bulk onset at $Ra/Ra_C = 1$. Obviously, for periodic domains no wall mode can
As we cross the threshold $Ra_C$ for onset of steady bulk convection the entire fluid layer exhibits convective flow. Depending on the distance to $Ra_C$, the asymptotic simulations by Julien and colleagues [23,26,65,66] have revealed a succession of various flow structures, visualized in figure 3:

- Just above onset, the convective flow takes the form of narrow cells, vortical pillars aligned with the axis of rotation and arranged in a (quasi) steady grid. We can observe ‘cold’ cells with downward transport and ‘hot’ cells transporting fluid upward. These cells are the solution for most unstable flow state from linear stability analysis [14]. At low enough $E$ the lateral dimensions of the cells are equal to $l_s^c$, as stated in equation (4b).
- For fluids with $Pr \gtrsim 2$, at larger $Ra/Ra_C$, a disordered ensemble of narrow vortical columns is observed, where each column has developed a ‘shield’ of opposite thermal contrast and opposite vertical vorticity. These structures are called con-
vective Taylor columns [23,26,68–74]. The shields prevent strong vortex–vortex interactions between columns. Thereby the columns are long-lived; the columns drift horizontally [73,75], gently ‘dancing’ around each other.

- When thermal forcing is stronger, the shields vanish and the vortex cores can interact more freely. The vertical coherence is partially lost. What remains are so-called plumes [23,26,69,72], vortical structures that form near the plates and penetrate into the bulk, where they dissolve into a turbulent environment by mutual interactions among them. These structures are not as long-lived as columns.

- At the strongest thermal input applied in the asymptotic simulations the coherent plumes also vanish and a fluctuating turbulent flow state remains, called geostrophic turbulence [23,26]. It is easier to achieve this flow state at lower Pr; so far, asymptotic simulations have only revealed this state for $Pr \leq 3$. Please take note of the terminology: geostrophic turbulence is a subrange of the geostrophic regime of RRBC. The turbulence is anisotropic. An interesting feature of this state is the occurrence of an inverse energy cascade from small to large spatial scales, resulting in the formation of large-scale vortices (LSVs) [26,76–81]. I will return to the LSVs in more detail in section 3.3.

These four flow morphologies have also been found in DNS [79]. In many experiments these flow structures have also been identified using dye [82], shadowgraphy [59], passive tracers [5,40,44,83,84] and thermochromic liquid crystals [68,73]. They could also be identified in quantitative flow measurements using particle image velocimetry (PIV) [45,69,72,74,75,85] or 3D particle tracking velocimetry (PTV) [72].

With this additional knowledge on the flow features we can add some more detail to the regime diagrams of figure 2 by placing the flow states in the appropriate locations. It is expected that each of the flow states possesses its own characteristic heat transfer scaling [86] (i.e. different scaling exponents $\beta$ in $Nu \sim Ra^\beta$), however, the exponents $\beta_i$ may be so similar between certain structures that it is hard to separate the ranges by observing the $Nu$ trends alone.

Based on the asymptotic simulations [23,26] a reasonable upper limit for the cells range is $Ra/Ra_C \approx 2$. The situation above that line is found to be dependent on the Prandtl number.

For small $Pr \lesssim 2$ (figure 2(a)) the cells give way to plumes at this transition. The steep heat transfer scaling [39] is observed throughout the plumes and geostrophic-turbulence ranges. The geostrophic regime of convection encompasses cells, plumes and geostrophic turbulence and is identified with the shaded area in figure 2(a). Above the transition to rotation-affected convection a state similar to plumes is found [50,55]. Finally, at the top right corner of the diagram we find nonrotating convection (e.g. [27]); i.e. at any fixed value of $E$ it is possible to provide strong enough thermal forcing (large enough $Ra$) that the effect of rotation becomes insignificant, and, equivalently, at constant $Ra$ we can increase $E$ to the same effect. Note that lines of constant $Ra$ are not horizontal but display a positive slope in these diagrams; the vertical axis unit is $Ra/Ra_C \sim RaE^{4/3}$.

For large $Pr \gtrsim 2$ we observe columns above the line $Ra/Ra_C \approx 2$ (figure 2(b)). It is clear from DNS [79] and experiments [40,44] that the steep $Nu(Ra)$ scaling is for columns, with exponent $\beta$ increasing as $E$ is reduced; $\beta = 3.6$ at $E = 10^{-7}$ is the steepest scaling reported so far [40]. The demarcation between regimes 2 and 3 (as inferred from heat transfer graphs) coincides with the transition between columns and plumes, hence the proposed terminology of columnar convection for regime 3. For regime 2 there is another situation where the descriptions used in the literature
cause confusion. Based on heat transfer trends alone, one might label this as ‘rotation-affected’ convection. However, we know from asymptotic simulations [23,26] (which are simulating rotationally constrained geostrophic convection by definition) that the plumes state can still be geostrophic! Thus there is a subdivision of regime 2 into a geostrophic and an ageostrophic part, which cannot be clearly recognized from heat transfer graphs alone. I thus introduce a ‘rotation-affected’ range 2a, and a ‘rotation-dominated’ range 2b. In range 2b, the flow state of geostrophic turbulence remains mostly out of reach at large $Pr$: asymptotic simulations at $Pr = 3$ have reported it; a recent DNS study [81] has observed LSVs at $Pr = 5.2$, where their generation mechanism of upscale energy transfer is a characteristic feature of geostrophic turbulence. It is not clear where to place the upper boundary of the geostrophic regime, that in fact encompasses all four flow types (cells, columns, plumes and geostrophic turbulence) and is indicated with the shaded area in figure 2(b). Above that boundary, in the rotation-affected range 2a, we expect again a flow state similar to plumes with possible heat transfer enhancement due to rotation (e.g. [47]). Moving to the top right in the diagram we again arrive at nonrotating convection.

In summary, transitions observed in heat transfer graphs are related to changes in the flow field. The transition between ranges 1 and 2, rotation-unaffected and rotation-affected convection, can be interpreted in the same fashion for all $Pr$. However, the transition between 2 and 3 is of a different origin depending on $Pr$: at low $Pr \lesssim 2$ it corresponds to the boundary of the rotation-dominated geostrophic regime of convection; at high $Pr \gtrsim 2$ the heat-transfer transition coincides with the flow change between columns and plumes, which is already within the geostrophic regime of convection. At high $Pr \gtrsim 2$ the boundary of the geostrophic regime (the blue shaded area in figure 2(b)) cannot be directly identified from heat transfer alone.

For completeness, I want to shortly indicate the situation for fluids with $Pr < 0.68$. The oscillatory mode of convection becomes active at $Ra > Ra_C$, below steady onset $Ra_s$ [14,17]. Wall modes are of course also still possible in confined domains. There is a direct transition from these oscillatory modes to plume-type turbulent flow, skipping over the steady cells [17]. It is harder to identify coherent plumes as heat diffusion is more efficient. It appears to be ‘easier’ to reach a state of geostrophic turbulence at low $Pr$ (i.e. reached at smaller $Ra/Ra_C$) [26,81].

With this conceptual image in mind, I will put studies of (geostrophic) RRBC into a set of phase diagrams for a more quantitative picture in the next section.

3. Phase diagrams: a comprehensive overview

In this section I collect the laboratory and simulation studies that have contributed to the investigation of the geostrophic regime into three phase diagrams: confined domains for Prandtl numbers $Pr \approx 0.7$ relevant to gas-phase convection (section 3.1) and for Prandtl numbers $Pr = 3 − 8$ relevant to water (section 3.2), and periodic domains (section 3.3). The differentiation of confined and periodic domains stems from the recent realization that the wall modes, that become convectively unstable at $Ra_w < Ra < Ra_C$, actually survive in the turbulent convective flow with prominent effects on the heat transfer [64,87–90]. The wall modes will be further discussed in section 4.3. Additionally, this division leaves out the experiments and confined DNS with $Pr < 0.68$ fluids again. These are only scarcely studied in the context of rotating convection. That is why I refer to the recent work by Aurnou et al. [17] for an overview of the state of the art for experiments on $Pr < 0.68$ fluids.
3.1. Gas-phase convection

The first diagram to consider is that for gas-phase convection, $Pr \approx 0.7$. The diagram is presented in figure 4. The division into the various scaling ranges is mostly given by the results of Ecke & Niemela [39]. Their heat transfer data have provided two transitions: between nonrotating and rotation-affected convection (ranges 1 and 2) at

$$\text{Ro}^{EN14}_{t,1\leftrightarrow2} = 0.35,$$  \hspace{1cm} (6a)

and between rotation-affected and rotation-dominated convection (ranges 2 and 3) at

$$\text{Ro}^{EN14}_{t,2\leftrightarrow3} = 0.25E^{-1.8}.$$  \hspace{1cm} (6b)

These are plotted in the phase diagram with dashed blue lines; the light blue shaded area identifies the geostrophic regime. The DNS by Horn & Shishkina [91] has confirmed that the reduction of heat transfer happens for $Ro < \text{Ro}^{EN14}_{t,1\leftrightarrow2}$. They could observe effects of rotation on the flow structure already at higher $Ro > \text{Ro}^{EN14}_{t,1\leftrightarrow2}$, which are not reflected in the heat transfer. By considering a poloidal–toroidal decomposition of the kinetic energy, they could identify the transition from the large-scale circulation (LSC) of nonrotating convection [27] to vortical plumes. The onset of rotational influence was found to nicely coincide with the transition relation for the sharp transition between ranges 1 and 2 as given by Weiss et al. [51]:

$$\frac{1}{\text{Ro}^{W10}_{t,1\leftrightarrow2}} = a \left(1 + \frac{b}{\Gamma}\right).$$  \hspace{1cm} (7)

This relation is the result of a Ginzburg–Landau model for vortex number density; it identifies a transitional Rossby number $\text{Ro}^{W10}_{t,1\leftrightarrow2}$ that is a function of the aspect ratio.
ratio $\Gamma$ with empirical constants $a = 0.381$ and $b = 0.061$. It is plotted in the phase diagram with dashed cyan lines for two example aspect ratios $\Gamma = 1/2$ and 1. Despite absence of enhanced heat transfer (too small $Pr$), the transition between nonrotating and rotation-affected convection can still be identified at the transition (7), but in the flow structure instead. Herewith we have described another range of convection, labeled ‘rotation-enhanced’ in the phase diagram, that displays rotational influence on the flow structure but is not reflected in the heat transfer.

There are no works in the rotation-dominated range that could help the identification of flow features with flow field measurements. Further validation of the transition to the rotation-dominated range is also not available at this time. The experiments of Zhang et al. [88] are covering the rotation-affected and rotation-enhanced ranges but only touch the rotation-dominated range of convection. Their $Nu$ data could further clarify the transition between ranges 1 and 2, but do not go low enough in $Ra/Ra_C$ to consider the transition between 2 and 3.

In terms of heat-flux scaling Ecke & Niemela [39] report $Nu \sim Ra^{0.45}$ for regime 2 (rotation-affected) and $Nu \sim Ra^{1.0-1.1}$ for regime 3 (rotation-dominated). The former has no prediction for comparison; the latter can be compared to the asymptotic prediction by Julien et al. [92] for bulk-limited heat transfer in the range of geostrophic turbulence:

$$Nu - 1 \approx 0.04 Pr^{-1/2} Ra^{3/2} E^2. \quad (8)$$

The rotation-dominated heat flux as measured does display steep scaling $Ra^{1.0-1.1}$, but not (yet) as steep as the asymptotic result $Ra^{3/2}$. Three possible reasons for the discrepancy are that $E > 2 \times 10^{-7}$ may not yet reach small enough values to display truly asymptotic behavior, the presence of no-slip plates in the experiment with their associated Ekman boundary layers could distort the scaling [66,79,80,93], or that the wall mode [64,87–90] is responsible for that (see also section 4.3). The other theoretical heat transfer prediction for rotation-dominated convection [94] is based on the concept of considering marginal convective instability of the thermal boundary layer according to Chandrasekhar’s rotation-dependent stability criterion (2b), leading to

$$Nu \approx 0.0023 Ra^3 E^4, \quad (9)$$

which is much steeper than the data of [39] and not a good match.

Overall, gas-phase geostrophic convection has not received much attention, in particular considering the flow. It is therefore hard to test the reported predictions from simulations on periodic domains here.

### 3.2. Convection in water

We then move on to Prandtl numbers $Pr = 3 – 8$ typical for experiments with water, a significantly more popular working fluid. The phase diagram is shown in figure 5. This diagram lists many studies that present results for $E > 10^{-6}$, though these only allow for a rather small rotation-dominated range. To break the $E = 10^{-6}$ ‘barrier’ requires dedicated experiments or finely resolved, long-term DNS. The parameter space is mostly filled by three studies employing tall, slender convection cells that rotate rapidly to reduce $E$ as far as possible [86]: $H \leq 0.6$ m for [90]; $H \leq 1.6$ m for [40]; $H \leq 4$ m for [44]. At the same time, the trick is to still be able to reliably provide Rayleigh numbers small enough to explore the rotation-dominated range; no simple
Figure 5. $(E, Ra/Ra_C)$ phase diagram of RRBC for $Pr = 3 − 8$. The dashed lines demarcating transitions are explained in the text. The light blue shaded area indicates the approximate extent of the geostrophic regime. References: Liu & Ecke (1997, 2009) [32, 95]; King et al. (2009, 2012) [94, 96]; Niemela et al. (2010) [38]; Weiss & Ahlers (2011) [46, 97]; Cheng et al. (2015) [40]; Cheng et al. (2020) [44]; Lu et al. (2020) [90]; Kunnen et al. (2008, 2010) [33, 45, 98]; Rajaei et al. (2017) [72]; Rajaei et al. (2018) [99]; Shi et al. (2020) [74]; Stevens et al. (2011) [100]; Horn & Shishkina (2014) [101]; de Wit et al. (2020) [87]; Yang et al. (2020) [29]; Weiss et al. (2010) [51].
task given that $Ra \sim H^3$ \cite{86}. That is why the earlier attempt by Niemela et al. (2010) \cite{38} only measured minor changes to $Nu$ while rotating their setup: $Ra$ remained so large that effects of rotation were barely noticeable despite $E \sim 10^{-8}$.

In terms of the various ranges, we can start off in the top right of the diagram. Rotation-unaffected convection is observed above the criterion (7) introduced by Weiss et al. (2010) \cite{51}. It is well-validated for $\Gamma = O(1)$ and convection in water. I include the line for $\Gamma = 1/10$, the smallest aspect ratio of the Eindhoven setup TROCONVEX \cite{44,86}, to illustrate the strong dependence on $\Gamma$ of this transition relation.

Below the Weiss et al. transition we find the range of rotation-affected convection. Towards the bottom right is the range of moderate $Ra$ and moderate $E$ where heat transfer may increase by addition of rotation; it is labeled rotation-enhanced. It is not clear yet whether the Weiss et al. transition marks the boundary of rotational influence on the flow. As for $Pr \approx 0.7$ the flow structure may be altered below this line while $Nu$ remains mostly unaffected. Cheng et al. (2020) \cite{44} report that rotation reduces heat transfer with respect to the nonrotating scaling below the (dashed blue) line

$$Ro = 0.06.$$ \hfill (10)

Lu et al. (2020) \cite{90} have identified that the transition is dependent on aspect ratio $\Gamma$. At $\Gamma = 0.4$ they identify reduced heat transfer below the (dashed yellow) line

$$Ra = 3.4E^{-1.7}.$$ \hfill (11)

These lines are reasonably close together in the parameter space.

For the transition to the steepest scaling range (columnar convection), the three studies provide the following relations. For $\Gamma = 0.4$, Lu et al. \cite{90} find the (dashed yellow) line

$$Ra = 0.17E^{-1.7}.$$ \hfill (12)

Cheng et al. (2015) \cite{40} report the (dashed red) line

$$Ra = 5.4E^{-1.47},$$ \hfill (13)

which is the result of combining experimental data with DNS data on a periodic domain. Finally, Cheng et al. (2020) \cite{44} report close agreement with the (dashed blue) boundary between columns and plumes from asymptotic simulations \cite{65}:

$$Ra = 55E^{-4/3}.$$ \hfill (14)

The three lines are remarkably close together, particularly when one takes into account the ranges of operation of each experiment. A previous transition prediction between ranges 2 and 3 based on the crossing of thermal and kinetic boundary layers, conceptually introduced in \cite{96} and refined in \cite{48,94}, is given by the (dashed cyan) line

$$Ra = 10E^{-3/2},$$ \hfill (15)

is found to be too high for the three large-scale experiments. Equation (15) is in large part based on DNS on a periodic domain; the wall mode may once again be the cause
of the disagreement.

I have indicated the approximate extent of the geostrophic regime in figure 5 using the light blue shading. Note that the upper bound of this regime is not known exactly; see also section 2.

Cheng et al. (2020) [44] find a further subdivision of the heat-transfer range 2, discussed in section 2, where plumes (and possibly geostrophic turbulence) are expected to be the foremost flow features. It is partially based on the consideration of the magnitude of the vertical gradient in the mean temperature. Originating from enhanced lateral mixing due to interacting plumes [50], the temperature gradient has shown to possess diagnostic properties concerning the flow structure [26,65,80]. Cheng et al. (2020) noticed a change in the gradient trend exactly at the relation that is given as the theoretical upper bound for the validity of the asymptotically reduced equations [92] (plotted with a dashed blue line)

$$Ra = Pr^{3/5}E^{-8/5}. \quad (16)$$

Thus a change in flow structure is anticipated at this transition. It also turns out to be accompanied by a change in the heat-flux trend (to be discussed later). Cheng et al. postulate that the ‘asymptotic’ flow structures exist below the transition (16) and that above that line a different flow could be anticipated, labeled ‘rotationally-influenced turbulence’ (RIT) which is currently only described by trends in heat transfer and mean temperature gradient.

Concerning the heat transfer, the three studies provide evidence of a division into three ranges, qualitatively similar to the gas-phase results [39] but of a different origin in terms of flow structure:

- Above onset, there is a steep scaling in the ranges where cells and columns are the dominant flow features (regime 3 in figure 1; panels (a) and (b) in figure 3). The only study reporting small-$E (< 10^{-6}$) experimental data on the heat transfer scaling in this range is by Lu et al. [90]. They report $Nu \sim Ra^\beta$ with $\beta = 1.48$ at constant $\Gamma = 0.4$ over a range $2 \times 10^{-7} \leq E \leq 7.4 \times 10^{-7}$. They also report a strong dependence of $\beta$ on $\Gamma$ in this range, with $\beta$ increasing to 2.04 at $\Gamma = 2$ and $E = 1.9 \times 10^{-6}$. We can contrast these findings with periodic DNS results reported by Cheng et al. [40], interpreting a periodic domain as $\Gamma \rightarrow \infty$. Cheng et al. [40] report exponents $\beta = 2.8, 3.0, 3.56$ at $E = 10^{-5}, 10^{-6}, 10^{-7}$, respectively. We can conclude that confined domains clearly display a smaller $\beta$ in this range (but rising as $\Gamma$ is increased) than periodic domains (i.e. $\Gamma \rightarrow \infty$). A plausible explanation of the $\Gamma$-dependent $\beta$ trend is based on the wall mode: its thickness is proportional to $E^{1/3}$ and is independent of $\Gamma [56,60,90,102]$, thus it covers a larger fraction of the cross-sectional area at small $\Gamma [90]$. If we then assume that in this range the bulk heat flux displays a steeper scaling (suggested by the periodic DNS) than the heat flux in the wall mode, we could qualitatively explain the observed dependence on $\Gamma$ of the composite net full-domain (i.e. bulk and wall mode) scaling exponent $\beta$.

- After columns have receded in favor of plumes (figure 3(c)) the heat transfer exhibits a shallower slope (regime 2 in figure 1). Lu et al. [90] report that the dependence of the scaling exponent $\beta$ on aspect ratio $\Gamma$ has basically vanished. Cheng et al. [44] confirmed this in points where their three $\Gamma = 1/10, 1/5, 1/2$ gave overlapping $Nu$ at otherwise equal $Ra$, $E$ and $Pr$. The local exponent $\beta$ as reported by [90] takes on values $\beta \approx 0.5 – 0.7$ in this regime. The two
subregimes identified by [44] display different heat transfer characteristics. Below the transition (16) an $E$-dependent exponent is found, with $\beta = 0.55, 0.65, 0.75$ respectively at $E = 3 \times 10^{-7}, 5 \times 10^{-8}, 1 \times 10^{-8}$. As $E$ goes down, the exponent is rising towards the asymptotic value $\beta = 1.5$ of equation (8), expected to be valid for geostrophic turbulence (figure 3(d)). Above transition (16) an $E$-independent collapse is found with $\beta = 0.51$. The results of the two studies [44,90] are quantitatively similar with $\beta \approx 0.5$ at similar $E \sim 10^{-7}$.

- At even higher $Ra$ the heat transfer scales as in nonrotating convection. Exponents slightly smaller than the classical prediction $\beta = 1/3$ [103] are found, with $\beta = 0.322$ [40], 0.308 [44] and 0.317 [90]. These small differences can originate from the subtle dependence on both $\Gamma$ and $Pr$ in this regime [104,105]. Even though large $Ra$ values are achieved, no signs of a transition to a steeper scaling is observed. This ‘ultimate’ convection regime is anticipated to display a larger scaling exponent $\beta = 1/2$, triggered by the boundary layers becoming turbulent and heat transfer independent of molecular diffusive properties of the fluid [27,106,107]. Absence of ultimate scaling is in line with recent DNS in a nonrotating slender cylinder [108], suggesting that slender cylinders are not the best geometry to detect the (disputed) transition to ultimate convection [109,110].

In terms of fluids with $Pr > 8$ there is almost no heat transfer data for rotating convection. One exception is the work by Weiss et al. [37], who report data for $Pr = 12.34, 23.9, 28.9$ and 35.6. Combined with data for water published earlier [36], they find that for $Pr > 3$ the upper boundary of the range of heat transfer enhancement by rotation (Ekman pumping), i.e. the transition between ranges 1 and 2a of figure 2(b), is a function of $Pr$:

$$1/Ro = 0.75Pr^{-0.41}. \quad (17)$$

At the highest $Pr \geq 12.34$ the ranges where rotation becomes dominant (ranges 2b or 3 in figure 2(b)) could not be reached in their experiments. Furthermore, Stevens et al. [28] focus on maximizing the heat-transfer overshoot in a cylindrical domain at moderate rotation, using Prandtl numbers up to $Pr = 55$ at $Ra = 10^8$. Yang et al. [29] extend this investigation to periodic domains at $Pr = 25$ and 100 with $Ra = 10^7, 10^8$ and $10^9$, finding a condition for optimal heat transfer as

$$1/Ro_{opt} \approx 0.12Pr^{1/2}Ra^{1/6}. \quad (18)$$

In terms of flow velocity statistics there are significantly fewer results reported, and all well above $E = 10^{-6}$, leading to a quite short rotation-dominated scaling range. Kunnen et al. [45,98] have used both PIV and DNS to show that velocity fluctuations, both horizontal and vertical, in the rotation-affected range show an approximate scaling $Re \sim Ro^{0.2} \sim E^{0.2}$ at $Ra = 6 \times 10^8$ and $10^9$ and $Pr = 6.4$. There is strong anisotropy: vertical velocity fluctuations are larger than horizontal in the center. Closer to the plate at $z/H = 0.8$ a more isotropic situation is observed. Moving down in $Ro$ (or $E$) to the rotation-dominated range at $Ro \approx 0.1$, $Re$ is rapidly reduced. These results have been confirmed by Rajaei et al. [99] using PTV, identifying the ratio $Re_{hor}/Re_{ver}$, i.e. ratio of horizontal over vertical velocity fluctuations, as a good indicator of entering the rotation-dominated columnar range. In that range vertical velocity becomes even more prominent over horizontal in the center, while near the plates it is opposite. Lagrangian acceleration measurements show that in the
Figure 6. \((E, \frac{Ra}{Ra_C})\) phase diagram of RRBC on periodic domains. References: Stress-free conditions: Schmitz & Tilgner (2009) [111]; Julien et al. (2012) [26,92]; Favier et al. (2014) [77]; Guervilly et al. (2014) [78]; Maffei et al. (2020) [112]. No-slip conditions: King et al. (2009,2012,2013) [48,94,96]; Plumley et al. (2016) [66]; Aguirre Guzmán et al. (2020) [81]. Studies comparing both conditions: Julien et al. (1996) [50]; Stellmach et al. (2014); Kunnen et al. (2016) [80].

center acceleration remains nicely isotropic throughout the three ranges of nonrotating, rotation-affected and rotation-dominated convection, but horizontal acceleration becomes prominent over vertical in rotation-affected convection, and even more so in rotation-dominated convection.

Rajaei et al. [72] have used PIV and PTV to compare flow fields to the flow structures as interpreted from the asymptotic simulations [65]. Using horizontal correlations of vertical vorticity, they found decent agreement with the temperature correlations of [65] in terms of length scales and spatial distribution of coherent structures.

Shi et al. [74] have applied PIV to consider the interior structure of the vortical plumes in the rotation-affected and rotation-dominated regime. They found good agreement with the theoretical model for convective Taylor columns that was tested for the asymptotic simulations [70].

Outside of crude visualizations [40,44] there are no data on flow structure for \(E < 10^{-6}\). Input in that regard is more than welcome, to disentangle and interpret the various heat transfer scaling ranges.

3.3. Convection in periodic domains

The final phase diagram to consider is that for simulations on periodic domains. In absence of sidewalls and the inevitable wall modes, these periodic domains are conceptually simpler. The phase diagram is shown in figure 6. No regimes or regime boundaries are plotted in this figure since they are typically dependent on \(Pr\); a wide range of \(Pr\) values is included in the plot.

It turns out that the mechanical boundary conditions on the plates (either no-slip or stress-free) affect the scaling behaviors of flow statistics in unforeseen ways. The Ekman boundary layers, with a thickness \(\delta_E \sim E^{1/2}H\) [19,20], shrink rapidly as the
Ekman number goes down. The associated effects of Ekman pumping were expected to be negligibly small for \( E \lesssim 10^{-6} \) and \( Pr = O(1) \) fluids [18,23,26]. Then, DNS comparing the two boundary conditions side-by-side at Ekman numbers as small as \( 10^{-7} \) [79,80] have shown that, counterintuitively, simulations with no-slip boundary conditions (with more friction) could actually reach significantly higher heat transfer (up to a factor 8 at \( Pr = 7 \)) than stress-free simulations at the same parameter values, an effect already alluded to by earlier numerical work at higher \( E \geq 10^{-4} \) [35]. The coupling of the convective Taylor columns, with their axial vorticity and their layer-spanning height, to the Ekman boundary layers, with Ekman pumping, provide for a highly efficient mode of heat transport. This mode benefits from higher \( Pr \) as this further reduces heat diffusing out of the hot columns (or, equivalently, heat diffusing into cold columns). Steep scaling \( Nu \sim Ra^\beta \) with exponents \( \beta \) as large as 3.6 have been reported for \( E = 10^{-7} \) and \( Pr = 7 \) [40]. The enhanced heat transfer remains active throughout the plumes range, though differences between no-slip and stress-free are smaller [79] as the plumes do not retain vertical coherence throughout the fluid layer and live shorter.

The large persisting influence of the Ekman boundary layers has prompted the extension of the asymptotic model with Ekman pumping boundary conditions [66,67,93]. Introducing \( E \) as an additional control parameter, the effects of finite rotation rates and the corresponding parametrized Ekman pumping could now be assessed. At \( E = 10^{-7} \) good agreement with DNS is observed for smaller \( Pr \leq 1 \), though differences grow at higher \( Pr \). Nonetheless, now the effects of Ekman pumping could be assessed at way smaller \( E \) than in DNS. Plumley et al. [93] have done this for \( E = 10^{-11} \) and \( 10^{-9} \) at \( Pr = 1 \). I did not explicitly plot these points in the regime diagram of figure 6 as they would add significant white space to the diagram (no other studies come close) and press together the other datapoints. Their results point at the development of a gestrophic-turbulence regime at high \( \tilde{\text{Ra}} = RaE^{4/3} \) with the heat transfer scaling as \( Nu \sim \tilde{\text{Ra}}^{3/2} \) but settling at higher \( Nu \) than for stress-free conditions:

\[
Nu - 1 = 0.04(1 + 5.97E^{1/8})Ra^{3/2}E^2,
\]

which, compared to the stress-free relation (8), adds a multiplicative factor \( 1 + 5.97E^{1/8} \).

Kunnen et al. [80] have compared no-slip and stress-free DNS at \( Pr = 1 \) and two values of \( Ra \) = \( 1 \times 10^{10} \) and \( 5 \times 10^{10} \), while varying \( E \) as \( 1.3 \times 10^{-7} < E < 2 \times 10^{-6} \). On their constant-\( Ra \) path through parameter space they found the upper limit of the rotation-dominated range at \( E \approx 9 \times 10^{-7} \) for \( Ra = 10^{10} \) and \( E \approx 3 \times 10^{-7} \) for \( Ra = 5 \times 10^{10} \), irrespective of the boundary conditions. These numbers are in decent agreement with equation (16), the theoretical upper limit of validity of the asymptotic model: \( RaPr^{-3/5}E^{8/5} \approx 2 \) in both cases. It also became clear that different statistical quantities displayed different transition values \( E \), so that a transition range is a more appropriate description. Below the transition a decent agreement with the geostrophic-turbulence heat-transfer scaling as \( Nu \sim E^2 \) [as per equation (8)] was observed. For no-slip plates the approximate scaling exponent \( \gamma \) in \( Nu \sim E^\gamma \) was significantly lower, \( \gamma \approx 1.1 \approx 1.3 \).

The stress-free studies shown in the phase diagram of figure 6 share one common interest: the occurrence of upscale energy transfer or an inverse energy cascade, leading to the formation of large-scale vortices (LSVs). A well-known feature of 3D homogeneous and isotropic turbulence is the direct energy cascade [113,114]: energy is injected
by some forcing mechanism at large spatial scale, then is transferred step-by-step to structures of ever smaller scale, until it is converted into heat by viscous action at the smallest flow scales. In 2D turbulence [115] the transfer is actually upscale: energy injected at smaller scales is transferred upscale and tends to accumulate at the largest scales allowed by the domain. Under special conditions, 3D turbulence can also have upscale energy transfer [116]: rotation, stratification, magnetic fields or spatial confinement may lead to energy transfers to the largest scales in the domain. In RRBC simulations rotation may lead to energy accumulation at the box scale, resulting in domain-filling LSVs [26,76–78,80]. The use of the word ‘cascade’ is perhaps not most appropriate here: energy transfer computations based on Fourier decompositions of the flow field [76,77,80] have revealed a direct energy transfer from a wide range of spatial scales to the largest scale (the box size) without participation of intermediate scales. This transfer is nonlocal in Fourier space, where the usual direct cascade favors transfers between subsequent modes in a step-by-step fashion (e.g. from Fourier wavenumber mode \( k \) to \( k + 1 \)).

For stress-free boundary conditions LSVs have been observed mostly at \( Pr = 1 \) [26,77,78,80], but recently also at larger \( Pr \) up to 7 [112]. LSV formation requires stronger thermal forcing at higher \( Pr \), i.e. larger values of \( \bar{Ra} \sim Ra/Ra_C \). At \( Pr = 1 \) the range where LSVs form in DNS is bounded by:

1. Strong enough rotational constraint, \( E \lesssim 10^{-4} \) [77].
2. Large enough supercriticality: \( \bar{Ra} \gtrsim 20 \) [78] or \( Ra/Ra_C \gtrsim 5 \) [77].
3. The ratio between thermal forcing and rotation remains small enough. Favier et al. [77] expect a maximal convective Rossby number; Guervilly et al. [78] propose a maximal value of a local Rossby number \( Ro_{l_z} = u_z'/2\Omega l_h \lesssim 0.15 \), where \( u_z' \) is the vertical root-mean-square velocity and \( l_h \) the horizontal lengthscale at which the largest fraction of vertical kinetic energy is found.

Asymptotic simulations [112] have formulated the lower limit of LSV growth as a Reynolds number \( \bar{Re} = Re_z E^{1/3} \gtrsim 6 \), where \( Re_z \) is the Reynolds number based on \( u_z' \) and horizontal length scale \( \ell/H = E^{1/3} \).

However, studies comparing no-slip and stress-free conditions could not see LSVs for no-slip plates at the same parameters where the stress-free case did display LSV formation [79,80], or only transiently would LSVs grow to be destroyed again shortly after [66]. By using higher \( Ra/Ra_C \) values than before at small \( E \), Aguirre Guzmán et al. [81] have found stable LSVs in domains with no-slip plates at \( Pr = 5.2, E = 10^{-7}, Ra/Ra_C > 40 \), and at \( Pr = 0.1, 2 \times 10^{-7} \leq E \leq 5 \times 10^{-7}, Ra = 10^{10} \). Indeed, stronger buoyant forcing is required for the upscale energy transfer to withstand the Ekman-plume perturbations to the bulk from the no-slip boundaries and achieve LSV formation.

By changing the shape of the periodic domain, i.e. making one horizontal side of the periodic box longer than the other, the LSVs could be replaced by patterns of alternating jets moving in opposite directions [117,118]. In fact, jets form in favor of LSVs for a change as small as making one horizontal side 8% longer than the other [117].

Additionally, the development of a flow with inverse energy transfer and a sustained LSV can be triggered under nominally subcritical conditions by the initialization of the flow field with a vortex-like initial condition [119], an example of bistability in turbulent flows.

LSVs or jets are prominent features in the periodic simulations. An interesting
open question is whether LSVs can also develop in confined domains. I will discuss some important issues that experimentalists (and researchers simulating convection in confined domains) are faced with in the next section.

4. Complications in laboratory studies of geostrophic convection

It is quite recent that experimentalists have been able to decisively enter into the geostrophic regime of convection. That is mostly due to the specific demands of achieving very small Ekman numbers $E$ along with a Rayleigh number $Ra$ that is large, but not too large. These limits have been illustrated in section 2 and made quantitative in section 3. The topics that I want to discuss in this section are: domain size, effects of centrifugal buoyancy, confinement and wall modes, non-Oberbeck–Boussinesq effects and inertial wave resonance.

4.1. Domain size

Geostrophic convection requires simultaneously large $Ra$ and small $E$. Looking at the definitions of $Ra$ and $E$ in equation (1), the obvious first variable to focus on is the cell height $H$. Modern set-ups aimed at exploring the geostrophic regime are tall [39,40,44,86,90], $H > 0.5 \text{ m}$. However, bigger is not always better [86]. There is a trade-off. Given that the critical $Ra_C \sim E^{-4/3} \sim H^{8/3}$ but the actual $Ra \sim H^3$, ever increasing $H$ will lead to a situation where the thermal forcing is too strong, that no large enough rotational constraint can be provided. At a given height $H$ there will be a lower limit on $Ra$ that can be accurately and reasonably achieved, which directly links to the minimal temperature difference $\Delta T_{\text{min}}$ that can be sustained, with water-based setups limited to $\Delta T_{\text{min}} \approx 1^\circ \text{C}$ [86]. The upper limit is set by the maximum heating and cooling power at bottom and top, respectively. What is more realistic, though, is that we wander out of the geostrophic regime into nonrotating-style convection at the highest power.

The choice of horizontal size is related to the choice of height. Technically, the setup should be as narrow as possible; smaller volumes of fluid are more easily temperature-controlled, less material costs for the setup, and (treated in detail next) less centrifugal effects. However, the domain should still be wide enough that the effects of proximity to the sidewall do not dominate the entire flow field. The argument employed here is one that is also used in simulations to determine the horizontal size of periodic domains. The onset scale of convection, $\ell_C$ of equation (4), is an accurate estimate of the size of convective features, even in the turbulent range [23,26,50,79]. The horizontal widths $L$ of simulation domains are chosen to be several times larger than $\ell_C$, i.e. $L = m\ell_C$ with $m = 20$ or 40 being popular choices [26]. The same argument is used to set the diameter $D = mL_C$ in experiments [86]. Given that $\ell_C \sim E^{1/3}$, the domain aspect ratio $\Gamma = D/H$ (as the equivalent $\Gamma = L/H$ for periodic simulation domains) becomes small, $\Gamma < 1$. Tall, slender setups are most capable of exploring the geostrophic regime, though they are not optimal to study nonrotating convection, where sidewall-proximity effects are likely.
4.2. Effects of centrifugal buoyancy

Next to the Coriolis acceleration, the second dynamical effect of rotation on a flow is the centrifugal acceleration. It can be combined with gravitational acceleration into an effective gravity:

$$ g_{\text{eff}} = -g\hat{z} + \Omega^2 r \hat{r}, $$

where $\hat{z}$ is the vertical unit vector and $\hat{r}$ the unit vector pointing radially outward; $r$ is the radial coordinate, the shortest distance from a given point to the rotation axis. In isothermal fluids this effect leads to parabolic isobars; a free surface of a liquid takes on a parabolic shape. For confined layers and incompressible single-phase fluids the static pressure distribution is inconsequential for the dynamics. However, as soon as there are density differences in the fluid, the centrifugal acceleration will lead to centrifugal buoyancy: cold fluid is pressed outward and hot fluid moves inward.

The parameter usually invoked to express the magnitude of centrifugal effects is the Froude number (e.g. [120])

$$ Fr = \frac{\Omega^2 R}{g}, \quad (20) $$

to directly compare the centrifugal and gravitational accelerations at characteristic radius $R$. In most geophysical and astrophysical flows $Fr \ll 1$; for example, Earth’s atmosphere ($\Omega = 2\pi \text{ rad/day} = 7.3 \times 10^{-5} \text{ rad/s}, R_{\text{max}} = 6.4 \times 10^8 \text{ m}$) experiences at most $Fr = 0.003$ near the equator. That is why one wants to exclude centrifugal buoyancy as much as possible in studies of RRBC; simple enough in DNS by excluding the acceleration term altogether, always present in experiments. Typically, a numerical limit $Fr \leq 0.05 - 0.1$ was suggested to be sufficient for centrifugal buoyancy to be negligibly small [34,86]. This amounts to an upper limit on the physical rotation rate $\Omega$.

Recent cylinder DNS by Horn & Aurnou [121,122] have explicitly invoked centrifugal buoyancy, a force term that in DNS can be varied independently of $E$. They considered Froude numbers $0 \leq Fr \leq 10$ at $Ra = 10^7$ and $10^8$, $Pr = 6.52$ and $\Gamma = 0.73$, for various Rossby numbers $0.0125 \leq Ro (\text{Ekman numbers } E \geq 10^{-5}$ for $Ra = 10^7$ and $E \geq 3.2 \times 10^{-6}$ for $Ra = 10^8$). They report that larger $Fr$ leads to a transition to (quasi)cyclostrophic behavior, a principal balance between pressure gradient and centrifugal effects with tornado-like flow structures. They use the free-fall velocity $U = \sqrt{g\alpha \Delta T H}$ for the scaling of gravitational buoyant flow and centrifugal velocity scale $U_C = \sqrt{\Omega^2 R^2 \alpha \Delta T}$ for centrifugal buoyant flow, then use convective Rossby number $Ro_{\parallel} = U/2\Omega H$ and centrifugal Rossby number $Ro_{\bot} = U_C/2\Omega R = Ro_{\parallel}\sqrt{2Fr/\Gamma}$ to assess effects of rotation. They propose that the transition argument $Pr^{5/4}Ra^{1/4}Ro_{\parallel}^{3/2} = RaE^{3/2} \sim O(10)$ of [94] also applies for $Ro_{\bot}$. Taken together, this amounts to a transition to centrifugally dominated flow at $Ro_{\parallel,t} \approx Ro_{\bot,t}$, or

$$ Fr \simeq \frac{\Gamma}{2}. $$

For $Fr$ above this limit, centrifugation can affect the heat transfer. When $Fr > \Gamma/2$ and $Ro_{\parallel} > Ro_{\bot,t}$ $Nu$ is enhanced with respect to the nonrotating case. If $Fr > \Gamma/2$ and $Ro_{\parallel} < Ro_{\bot,t}$ then $Nu$ is lower. The circulation set up by centrifugal buoyancy
leads to strong asymmetry in the vertical mean temperature profiles [122,123]. The fluid near the cylinder axis is warmer on average while near the sidewall it is colder.

The aspect-ratio dependent limit (21) of the Froude number is in direct competition with the trend of using tall, shallow convection cells. Vertical temperature profile measurements in the sidewall by Cheng et al. [44] have provided vertically quite symmetrical profiles, even for $\Gamma = 1/10$ and $Fr$ up to 0.115. A validation of the role of centrifugal effects in these extreme setups is welcome, but technically complex and not readily accessible by DNS.

4.3. Confinement and wall modes

It has been known for a long time that in confined domains convection may onset below Chandrasekhar’s criterion (2) as a so-called wall mode [56–64]: a wave that travels along the sidewall of the container with alternating up- and downflowing lobes. The wall mode has a characteristic thickness $E^{1/3}$ that is independent of the aspect ratio $\Gamma$ [56,60,62]. Above steady onset, towards the turbulent regime, the wall mode signatures were found to disappear in experiments.

However, very recently, as studies focused more on low Ekman numbers, it was found that wall modes persist well into the turbulent range [64,87–90]. An $m = 1$ mode with anticyclonic precession is found in DNS at $\Gamma = 1/2$, $Ra = 10^9$, $Pr = 0.8$ and $E \leq 1.4 \times 10^{-5}$ [88] and at $\Gamma = 1/5$, $E = 10^{-7}$, $Pr = 5.2$ and $5 \times 10^{10} \leq Ra \leq 4.3 \times 10^{11}$ [87]. Earlier DNS results based on azimuthally averaged flow fields in a cylinder [43] have revealed a mean circulation that is in line with [88] and is probably also due to wall modes given the similar parameter values. The azimuthally averaged circulation could be well-explained using the linear theory of Stewartson boundary layers [124].

A remarkable finding of the recent works is that the wall-mode region actually contributes significantly to the overall heat transfer. While wall modes could elevate heat flux by a bit above conduction below steady onset (e.g. $Nu < 2$ in water [31]), a significant fraction of the heat transfer (up to 60% in [88]) is carried entirely within the near-wall region under rapid rotation. The strong dependence of the $Nu(Ra)$ scaling exponent $\beta$ on aspect ratio $\Gamma$ [90] is another evident clue: variation of $\Gamma$ changes the fraction of the cross-sectional area taken by the wall mode, given that its thickness is determined by $E$ alone [56,60,62]. By comparing periodic and cylinder DNS, de Wit et al. [87] have shown that the heat flux in the bulk of the cylinder (i.e. outside of the wall mode) actually coincides with the heat flux in periodic domains, and is thus largely unaffected by the wall mode. Additionally, the scaling of total heat transfer is similar between periodic and cylinder DNS, as long as they consider the range of parameter space where plumes and geostrophic turbulence are expected [87]; scaling in the columns range certainly differs more between cylinders of various aspect ratio $\Gamma$ [90] and by extension also in comparison to periodic domains.

The contribution of the wall mode to the overall convective heat transfer and its effect on the interior flow statistics are important questions that are largely unanswered now. These issues are crucial for the interpretation of results of experiments and simulations on confined domains and their relevance to unconfined systems. At the same time, the survival of the wall mode in highly turbulent flow is an intriguing feature in itself.
4.4. Non-Oberbeck–Boussinesq effects

The vast majority of studies on Rayleigh–Bénard convection assume a so-called Oberbeck–Boussinesq fluid [125,126], meaning that the fluid properties $\alpha$, $\nu$ and $\kappa$ are taken to be constant and independent of temperature over the entire domain, and that density variations are only significant when coupled with gravity. The problem is vertically symmetric in this approach. In the limit of small temperature differences $\Delta T$ this is also true for realistic systems. However, the search for high Rayleigh numbers in particular has led to the use of higher $\Delta T$ in experiments, by which the consideration of non-Oberbeck–Boussinesq (NOB) effects is warranted [127]. Without rotation the principal NOB effects are the asymmetry of the lower (hot) and upper (cold) thermal and kinetic boundary layers. The mean temperature in the bulk also shifts as the temperature drops over the boundary layers are unequal. In convection in gases (like helium) the temperature drop over the top boundary layer is smaller than for the bottom leading to a reduced bulk temperature [128]. The opposite is true for typical liquids (like glycerol or water) [127]: a larger temperature drop over the top boundary layer and increased bulk temperature. The change in bulk temperature could be shown to be the principal reason why the global response parameters $Nu$ and $Re$ only change by a few percent when the bulk temperature and boundary layer scales vary by as much as 20% [127].

Horn & Shishkina [101] have considered NOB effects in DNS of RRBC. For large $Ro \gtrsim 0.29$ the principal findings of nonrotating NOB convection as discussed before are confirmed. However, for $Ro \lesssim 0.29$ the mean bulk temperature rapidly reduces and approaches the mean temperature $T_m = \frac{1}{2}(T_{\text{bottom}} - T_{\text{top}})$ expected for Oberbeck–Boussinesq fluids. However, this is not a sign of reduced NOB effects as the boundary layer thicknesses at top and bottom remain different. The heat transfer $Nu$ is again only weakly affected by NOB effects, although the changes in the boundary layer thicknesses with rotation do cause some variation in the relative magnitude of NOB effects. Hence, for a true appraisal of NOB effects in RRBC it is best to consider measurement of the boundary layers, which is not an easy task in experiments.

4.5. Inertial wave resonance

Rajaei et al. [99] have performed 3D PTV measurements in RRBC. In PTV one tracks the motion of individual tracer particles in time; by using multiple cameras 3D trajectories can be obtained. Under the strongest rotational constraint, with considerably reduced convective flow, they observed that the trajectories displayed wavy patterns on top of the overall displacement. The waves were even more prominent in the velocity and acceleration autocorrelations. The (angular) frequency of the wavy perturbation perfectly matched the applied rotation rate $\Omega$. The cause of the oscillations was found in near-resonant inertial waves.

It is well-known that a rotating fluid is a medium that is suitable for the propagation of so-called inertial waves [19]. In fact, many effects of rotation on the flow, such as the vertical organization and the Taylor–Proudman theorem, are a direct result of these inertial waves. Waves generated in confined domains (like a cylindrical convection cell) can resonate and inertial waves are no exception, e.g. [129]. The question was: what could drive the inertial waves, that were clearly forced at frequency $\Omega$? Many forcing mechanisms for inertial waves are known: differential rotation of boundaries, oscillating objects, libration (periodic variation of the rotation rate) and precession (periodic variation of the axis of rotation). They arrived at the precession of the total
rotation vector of the cylinder, a vector sum of the rotation of the table on which
the experiment is placed and Earth’s rotation. Such precession forcing in cylindrical
domains has been studied by Liao & Zhang [130]. They argued that the dominant term
of precession could be written as a time-dependent extra body force in axial direction,
proportional to the Poincaré number

\[ Po = \frac{|\Omega_P|}{|\Omega|} , \]

where \( \Omega \) is the applied rotation (of the rotating table) and \( \Omega_P \) the precession angular
velocity, i.e. Earth’s rotation vector. Of course, in this case the Poincaré number is
small, \( Po \lesssim 4.4 \times 10^{-5} \), but large enough to explain the effect: DNS including tracer
particles and the dominant term of the precessional motion could quantitatively replicate
the effect on velocity and acceleration autocorrelations. Resonance of precession-
driven waves in cylindrical domains was studied by Meunier et al. [131]. They found
a principal resonance for a \( \Gamma = 1 \) cylinder at \( \omega = 0.996 \), where \( \omega = |\Omega| / (|\Omega| + \Omega_P \cdot \hat{k}) \)
and \( \hat{k} = \Omega / |\Omega| \), i.e. the applied rotation rate is very close to the total rotation rate
(applied and precession). The combination of strongly suppressed convection and a
near-resonant condition due to the choice of aspect ratio made Rajaei et al. experi-
ence the effects of Earth’s rotation rate in their small (\( D = H = 0.2 \) m) convection
experiment. This effect should be kept in mind in particular for velocimetry involving
tracer particles. A smart choice of the domain aspect ratio away from resonance goes
a long way in suppressing the wave amplitude.

5. Conclusion and perspectives

The geostrophic regime of rotating Rayleigh–Bénard convection is an interesting state
of turbulent convective flow. It is characterized by the principal geostrophic force bal-
ance between the Coriolis force and the pressure gradient. It displays a wide variety
of flow structures (cells, convective Taylor columns, plumes and geostrophic turbu-
lence) and scaling properties of statistical quantities, like the heat transfer expressed
as the Nusselt number \( Nu \). The exploration of the geostrophic regime has started
only rather recently due to the challenges faced by experimentalists and numericians
alike to achieve the extremely small Ekman numbers \( E < 10^{-6} \) required to decisively
enter this new regime [86]. Additionally, the lower \( E \) can be the larger the range of
geostrophic behavior becomes [39], allowing the consideration of subranges based on
flow structure and associated scalings.

Our current knowledge of this geostrophic regime stems mostly from numerical sim-
ulations of a set of asymptotically reduced equations, taking the limit of rapid rotation,
i.e. \( E \to 0 \) and \( Ra \to 0 \) [23,26]. More and more quantitative comparisons with DNS
and experiments can be made, e.g. [72,79], indicating that the methods are showing
overlap in terms of parameter space that can be covered. Most experimental work is
aimed at the convective heat transfer, a quantity that is comparatively straightforward
to measure. DNS, but also future velocimetry, is needed to complete the picture,
to further elucidate where we can expect transitions in flow structure and why these
transitions occur there. Additionally, there are extra challenges which are inherent to
experiments but absent from (periodic) simulations: wall modes, centrifugation and
other effects.

In our continued exploration of geostrophic convection, the following issues demand
1. There is a strong need for flow field information from DNS and experiments at small $E < 10^{-6}$. We have seen that only at these small $E$ values it is possible to distinguish the various subregimes with corresponding flow structures. The link of heat-transfer scaling with the prevailing flow structure is a very helpful input for modeling.

2. The role of the wall mode in confined rotating convection must be better quantified. How does it interact with the bulk? Can we consider heat transfer in bulk and sidewall layer separately? And what is the parameter range of existence of the wall modes?

3. The exact role of the Ekman boundary layers remains a puzzle. On the one hand it is clear that, as $E$ goes down, the Ekman layers become thinner and thinner, but on the other hand a vanishingly thin Ekman layer would give rise to a singular boundary condition on the plates. Some effect of Ekman boundary layers is bound to remain, but whether they can still be actively influencing the bulk flow is an open question.

4. Related to the three problems mentioned above, there is an open question whether the confined domains of experiments can facilitate large-scale vortex (LSV) growth. The long-time evolution of these LSVs could in principle be more efficiently monitored in experiments than in computationally expensive simulations. What is the role of the wall mode here? And when can the disturbances due to Ekman pumping be overcome?

5. In all these problems the Prandtl number $Pr$ plays an important role. Overall, the geostrophic regime is underexplored for low-$Pr$ fluids ($Pr < 0.68$). Low-$Pr$ fluids pose considerable difficulties in both experiments and numerical simulations, but they are present in many of the geophysical and astrophysical flows that we want to understand better.

The field of rotating convection is in the privileged position that different approaches can be directly compared (e.g. experiments and cylinder DNS, periodic DNS and asymptotic studies). We can thus benefit from the complementarity of the methods; to cover larger parts of the phase diagram, as well as to improve our understanding by combining inputs from the different methods.

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