MASTER

Hybrid integrator-gain based notch filter design

Hebers, Kyrola S.

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K.S. Hebers
0809714

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Supervisors:
Prof. Dr. Ir. M.F. Heertjes
Ir. S.J.A.M. van den Eijnden

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Abstract

Linear feedback control has been studied extensively and is generally used in industry due to its simplicity, transparency and intuitive design application. However, linear control has its fundamental limitations such as Bode's gain-phase relationship and Bode's sensitivity integral. Nonlinear control is proposed to possibly overcome these fundamental limitations. In particular, the Hybrid Integrator with Gain-Switching functionality (HIGS) is presented as a possible solution. Using a describing function analysis, the HIGS shows the behaviour of an integrator with significantly less phase lag. This phase advantage can lead towards an increase in bandwidth as well as improved low-frequency disturbance suppression. In this thesis, HIGS is exploited in a notch filter design. Three HIGS-based notch filters are designed and compared in a quasi-linear loop shaping procedure. The designs show the potential for increasing the expected bandwidth and low-frequency disturbance suppression properties. Subsequently, the quasi-linear optimal design is tested for its stability in the resulting nonlinear closed loop system. The stability analysis has been conducted with the use of piecewise quadratic Lyapunov functions. As a result, it appears that a quasi-linear Nyquist stability criterion is a useful tool to check stability of the true nonlinear closed loop system. To show the true behaviour of the HIGS-based notch filters, simulations in time-domain are conducted, which show the performance improving potential.
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1 Introduction

In the high tech industry, a demand for improved motion control regarding speed and precision is significant. Hereto, linear feedback control has been widely applied in the industry due to its simplicity and effectiveness. However, with the increasing demand linear feedback control is pushing towards its limitations. Feedforward control has been able to improve high-precision motion control systems by enhanced tracking behaviour. Though, the demand for faster and more precise motion control is still present.

1.1 Limitations linear control

The limitations of linear feedback control are fundamental [6, 27, 28]. One limitation is the Bode’s gain-phase relation [6, 28], which dictates the direct relation between gain and phase. This relation limits the control design due to necessary control filters that induce phase lag. Another limitation is the Bode’s sensitivity integral (waterbed effect) [6, 28] expressed by

\[ \int_0^\infty \ln |S(j\omega)|d\omega = 0, \]

where \( S(j\omega) \) is the sensitivity function. This integral holds when the sensitivity function contains an open loop with a relative degree of two or higher. The integral often expresses a trade off between disturbance rejection at low frequencies and amplification of noise at high frequencies i.e., improvements in one aspect comes at the cost of deterioration of the other. Due to the fact that improvement regarding speed and precision in motion control is desired, nonlinear control could offer a possibility to better deal with these fundamental limitations.

1.2 Reset control

An example of a nonlinear controller seemingly without the aforementioned limitations, is a reset integrator such as proposed by Clegg [8]. This so-called Clegg integrator resets its buffer when the input crosses zero. In this manner, the input and output are always equal of sign. These dynamics are described in state-space as

\[
\mathcal{R} := \begin{cases} 
\dot{x}_r(t) = \omega_r e(t), & \text{if } e(t) \neq 0 \\
x_r(t^+) = 0, & \text{if } e(t) = 0 \\
u_r(t) = x_r(t), & \text{if } e(t) = 0
\end{cases}
\]

with state \( x_r \in \mathbb{R} \), integrator frequency \( \omega_r \in \mathbb{R}_{>0} \), input \( e \in \mathbb{R} \), \( t^+ \) the time instance after the reset, and \( u_r \in \mathbb{R} \) the output. Through the describing function analysis, the Clegg integrator induces a phase lag of 38.15 degrees instead of 90 degrees for a linear integrator. Because
of the reduction in phase lag, the bandwidth could potentially be increased and a better performance could be obtained. These advantages of the Clegg integrator are shown in for example [3, 4]. This strategy has been generalized to higher order filters such as first order reset elements (FORE), second order reset elements (SORE) and other generalizations, which results in significant improvements due to the absence of the linear control limitations [15, 16, 17, 25]. However, the resetting behaviour results in a discontinuous output signal which could possibly excite higher harmonics of the system and as such result in deterioration of the performance. A possible solution to this problem is the recently developed Hybrid Integrator with Gain-Switching functionality (HIGS) as presented in [9, 10]. Due to the specific switching strategy of the HIGS between an integrator and gain mode, the input and output are always of equal sign thus adhering to the philosophy underlying reset control, while its output is continuous (non-smooth). The latter seems to provide a distinct advantage over reset control.

1.3 Problem formulation

It has been shown through experimental studies that HIGS accomplishes significant improvement in the performance of a scanning wafer stage [9, 10] through an add-on integrator. In addition, the HIGS has been applied in other common linear control filters such as a proportional-integrator-derivative (PID) filter in [11] and a second-order low pass filter as in [13] to further benefit from its advantages. Another common linear control filter is the notch filter, which causes unwanted local phase lag. To address this phase lag, the following step would be to implement a HIGS into a notch filter, which has already been partly investigated by [5] in continuous time. In this thesis, the HIGS will be implemented in a normal notch filter to address resonances in the mid frequency domain. In addition, an inverse notch filter will be studied for extra local low-frequency suppression as well as a skew notch filter for extra high frequency suppression. These HIGS-based notch filters should contain less phase lag due to HIGS’s behaviour and could therefore make use of the benefits. The objective of this thesis can be formulated as:

The development of HIGS-based notch filters in the form of a normal, an inverse, and a skew notch filter to achieve increased bandwidths, improved low-frequency disturbance suppressions and enhanced performances while meeting robustness margins.

To achieve this objective, a roadmap is made in terms of sub-goals as defined below.

- Design HIGS-based notch filters for a normal, an inverse and skew notch filter and evaluated their behaviour in frequency domain through describing function analysis.
- Obtain improved bandwidths during a quasi-linear loop shaping procedure using the describing function analysis.
- Make conclusions about the stability of the nonlinear closed loop systems using piece-wise quadratic Lyapunov functions.
• Convert the HIGS-based closed loop systems into discrete-time to show possible improvements in performance thought time-domain simulations.

1.4 Outline

In the remainder of this thesis, Chapter 2 elaborates on the background of HIGS in time and frequency domain. Furthermore, the HIGS-based notch filter designs are introduced and their frequency domain behaviour is discussed. In Chapter 3, quasi-linear loop shaping is conducted to obtain increased bandwidths for the HIGS-based notch filters. Subsequently, a Lyapunov stability analysis for the closed loop systems containing these HIGS-based notch filters is conducted in continuous-time. Chapter 4 shows the conversion of the nonlinear closed loop systems from continuous into discrete-time. Thereafter, time-domain simulations are conducted to show the possible performance improvements. Finally, this thesis concludes with a summary of the findings and gives recommendations for future research in Chapter 5.
2 Hybrid Integrator-Gain System

2.1 Introduction

Linear feedback control inherently suffers from fundamental design limitations. As a possible means to better deal with such limitations, nonlinear control is often proposed. Although the class of possible nonlinear control designs is potentially infinite, a particular design is discussed which is known as the Hybrid Integrator Gain System (HIGS). This hybrid system switches between gain and integrator dynamics. The switching occurs in a manner that the output is a continuous (non-smooth) signal and that the input and output are equal of sign. The behaviour of this nonlinear switching system represents a possibility to overcome the aforementioned limitations of linear control. The Hybrid Integrator Gain System is elaborated on in this chapter to show its possibilities in more detail.

In section 2.2 the HIGS is presented in state-space formulation with its switching conditions. Furthermore, the time domain behaviour of the system is described. In addition, using the describing function method an analysis with resulting conclusions in frequency domain is provided. Section 2.3 addresses three notch filter designs with integrated HIGS behaviour. These designs show the benefits of a HIGS notch filter in comparison to a linear notch filter in frequency domain. Finally, section 2.4 summarizes this chapter.

2.2 Hybrid Integrator Gain-Based Switching

The dynamics of the HIGS are described in state-space form using the following Differential Algebraic Equations (DAEs):

\[
\mathcal{H} := \begin{cases} 
\dot{x}_h = \omega_h e, & \text{when } (e, u, \dot{e}) \in \mathcal{F}_1 \\
x_h = k_h e, & \text{when } (e, u, \dot{e}) \in \mathcal{F}_2 \\
u = x_h, & 
\end{cases}
\]  

(2.1)

with state \( x_h \in \mathbb{R} \), input \( e \in \mathbb{R} \) with corresponding time derivative \( \dot{e} \in \mathbb{R} \), control output \( u \in \mathbb{R} \), integrator frequency \( \omega_h > 0 \), gain \( k_h > 0 \) and with \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) representing the subspaces in \( \mathbb{R}^3 \) for the integrator and gain mode, respectively. The state-space model as described here including its subspaces was first introduced in [9]. To guarantee that \( \dot{e} \) exists, it is assumed that the input signal \( e \) is at least one-time differentiable.
The state subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as represented in (2.1) are defined as

\[
\mathcal{F}_1 := \left\{(e, u, \dot{e}) \in \mathbb{R}^3 \mid eu \geq \frac{1}{k_h} u^2 \land (e, u, \dot{e}) \notin \mathcal{F}_2\right\},
\]
\[
\mathcal{F}_2 := \left\{(e, u, \dot{e}) \in \mathbb{R}^3 \mid u = k_h e \land \omega_h e^2 > k_h \dot{e} \right\}.
\]

These subspaces are contained in the complete flow set $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2$. It is desired to be in integrator mode $\mathcal{F}_1$ until the scaled output is reached which is defined by the constraint $eu \geq \frac{1}{k_h} u^2$. In that case, the system switches to gain mode $\mathcal{F}_2$. It leaves the gain mode when its constraint $\omega_h e^2 > k_h \dot{e} e$ is no longer satisfied which, for example, occurs at zero crossings. The subspaces are illustrated in Figure 2.1 in the $(e, u)$-plane and the $(e, u, \dot{e})$-space, which shows the volume of integrator mode and the plane of gain mode.

Figure 2.1: The schematic presentations of the subspaces $\mathcal{F}_1$ and $\mathcal{F}_2$ in (a) in the $(e, u)$-plane and in (b) in the three-dimensional $(e, u, \dot{e})$-space.

### 2.2.1 Time Domain

To better understand and appreciate the mechanism underlying HIGS, its behaviour is demonstrated through open loop time-domain simulations. The behaviour is shown in Figure 2.2, where a multi-sine function is used as the input $e$ of the system. This input is defined as $e(t) = \sin(3\pi t) + 0.75 \sin(8\pi t)$. Furthermore, the parameters of the HIGS are set to $\omega_h = 2\pi$ rad/s and $k_h = 1$. The system starts in integrator mode, as can be observed in Figure 2.2, generating the output $u_{\mathcal{F}_1}$. By the time that the maximum allowed output is reached, the system switches into gain mode $u_{\mathcal{F}_2}$. The figure shows that the gain mode satisfies the sector boundary $k_h e$. The system switches back to the integrator mode when this does not result in a violation of the integrator mode constraints. By switching between these modes constantly the output and input remain of equal sign. In this manner, the system is always pushed in the direction toward zero error.
2.2.2 Frequency Domain

Although generally not defined for nonlinear systems, to better understand the benefits coming from HIGS, its behaviour is evaluated in frequency domain. Hereto, a describing function analysis has been applied to obtain a proper approximation of the relation between input and output of the time-invariant nonlinearities. The output of the HIGS contains multiple harmonics as a result of its nonlinearity. It is chosen to use the first harmonic to obtain the approximation. The describing function analysis investigates the relation between a sinusoidal input signal and a harmonic of the output. To apply a frequency domain conversion, a Fourier expansion is constructed for the output, using $e(t) = A\sin(\omega t)$ as input signal. The input depends on the amplitude $A$ and the angular frequency $\omega$ in rad/s. The output is approximated as
\[ u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)), \] (2.3)

with the coefficients defined as

\[ a_n = \frac{2}{T} \int_0^{T/2} u(t) \cos(n\omega t) dt, \] (2.4)
\[ b_n = \frac{2}{T} \int_0^{T/2} u(t) \sin(n\omega t) dt, \] (2.5)

and \( n \) an integer denoting the order of the harmonics. \( T \) is the corresponding period. Since the nonlinearity is of the odd-type, \( a_0 \) is considered to be equal to zero. As mentioned previously, only the first harmonic of the output is investigated and therefore the higher order harmonics are neglected. This first-order harmonic results in the describing function

\[ \Psi_h(\omega, A) = b_1 + j a_1 A. \] (2.6)

To be able to implement the coefficients, the corresponding output of the HIGS in time-domain is given by

\[ u(t) = \begin{cases} \frac{\omega h}{\omega} A(1 - \cos(\omega t)) & \text{if } 0 \leq t < \frac{\gamma}{\omega}, \\ k_h A \sin(\omega t) & \text{if } \frac{\gamma}{\omega} \leq t < \frac{\pi}{\omega}, \\ \frac{\omega h}{\omega} A(-1 - \cos(\omega t)) & \text{if } \frac{\pi}{\omega} \leq t < \frac{\pi + \gamma}{\omega}, \\ k_h A \sin(\omega t) & \text{if } \frac{\pi + \gamma}{\omega} \leq t < \frac{2\pi}{\omega}. \end{cases} \] (2.7)

The switching time \( \gamma \), where the output switches from integrator to gain mode, is obtained by taking the intersection of the outputs at \( t = \frac{\pi}{\omega} \). Using trigonometric identities this results in

\[ \gamma = 2 \arctan \left( \frac{k_h \omega}{\omega h} \right). \] (2.8)

The coefficients of the Fourier expansion are derived using Equation (2.7) as

\[ a_1 = \frac{2\omega}{\pi} \left( \int_0^{\gamma/\pi} \frac{\omega h}{\omega} A(1 - \cos(\omega t)) \cos(\omega t) dt + \int_{\gamma/\pi}^{\pi/\omega} k_h A \sin(\omega t) \cos(\omega t) dt \right) \]
\[ = \frac{A}{2\pi} \left( \frac{\omega h}{\omega} (4 \sin(\gamma) - \sin(2\gamma) - 2\gamma) - k_h (1 - \cos(2\gamma)) \right), \] (2.9)
\[ b_1 = \frac{2\omega}{\pi} \left( \int_0^{\pi/\gamma} \frac{\omega_h}{\omega} A(1 - \cos(\omega t)) \sin(\omega t) \, dt + \int_{\pi/\gamma}^{\pi} k_h A \sin(\omega t) \sin(\omega t) \, dt \right) \]
\[ = \frac{A}{2\pi} \left( \frac{\omega_h}{\omega} (3 - 4 \cos(\gamma) + \cos(2\gamma)) + k_h(2(\pi - \gamma) + \sin(2\gamma)) \right). \]  

(2.10)

Subsequently, the coefficients \( a_1 \) en \( b_1 \) are implemented in (2.6). Rewriting this equation and using Euler’s formula, the following describing function is obtained

\[ \Psi_h(j\omega) = \frac{\omega_h}{j\omega} \left( \frac{\gamma}{\pi} + j \frac{e^{-\gamma^{2\gamma}} - 1}{2\pi} - 4j \frac{e^{-\gamma} - 1}{2\pi} \right) + k_h \left( \frac{\pi - \gamma}{\pi} + j \left( \frac{e^{-\gamma^{2\gamma}} - 1}{2\pi} \right) \right) \].  

(2.11)

As can be seen in Equations (2.8) and (2.11), the tuning parameters for the describing function are \( \omega_h \) and \( k_h \). To address the influence of the tuning of these parameters, Figure 2.3 shows the describing function for different values of \( k_h \) and \( \omega_h \).

![Figure 2.3](image)

(a)  
(b)

Figure 2.3: A Bode representation of the describing function with (a) a varying parameter \( k_h \) for \( \omega_h = 2\pi \) and (b) a varying parameter \( \omega_h \) with \( k_h = 1 \).

First, the effect of tuning the parameter \( k_h \) is considered. Figure 2.3a shows that for lower frequencies, the describing function tends towards a static gain. This observation is stated as

\[ \lim_{\omega \to 0} \Psi_h(j\omega) = k_h. \]  

(2.12)

At high frequencies, \( k_h \) tends towards integrator behaviour with significantly less phase lag in comparison to a linear integrator. This behaviour is defined by using \( \lim_{\omega \to \infty} \gamma = \pi \) as
which is equal to the describing function of a Clegg integrator [8]. The behaviour of the Clegg integrator is seen in Figure 2.3a for \( k_h \to \infty \). The magnitude and phase of this describing function have been calculated as

\[
\lim_{\omega \to \infty} |\Psi_h(j\omega)| = \frac{\omega_h}{\omega} \sqrt{1 + \frac{16}{\pi^2}} \approx 1.619 \frac{\omega_h}{\omega},
\]

(2.14)

\[
\lim_{\omega \to \infty} \angle \Psi_h(j\omega) = \arctan\left(-\frac{4}{\pi}\right) \approx -38.1^\circ,
\]

(2.15)

respectively. The magnitude represents that of an integrator with \( \omega_h \) the cross-over frequency and a factor of 1.619 increase of the gain. So the cross-over frequency of the describing function is formulated as

\[
\omega_c = 1.619 \frac{\omega_h}{k_h}
\]

(2.16)

using Equations (2.12) and (2.14). As a result, the describing function has a phase lag of approximately 38.1\(^\circ\) at high frequencies which is a significant increase in comparison to a linear integrator with a phase lag of 90\(^\circ\). This phase advantage of 51.9 degrees has significant benefits among which the ability to increase the bandwidth of the open loop system.

In the case of the tuning parameter \( \omega_h \), the value determines the breaking point between gain and integrator mode. For lower values of \( \omega_h \), the describing function leads towards integrator behaviour and for higher values of \( \omega_h \) the function tends more towards a gain as shown in Figure 2.3b.

As stated before, only the first harmonic is considered for the computation of the describing function. Therefore, it is necessary to examine whether higher harmonics are of influence in the approximation and how this associates to the tuning parameters. To investigate this relation, the square root of the cumulative power spectral density (cPSD) of the output of HIGS is given in Figure 2.4. Herein, using a sinusoidal input signal the energy distribution of the output is shown. Figure 2.4a shows the cPSD for various \( k_h \) with \( \omega_h = 2\pi \) rad/s. For \( k_h = 0.1 \) the energy of the excitation frequency is directly transferred to the output. This occurrence is explained using Figure 2.3a, where it shows that the describing function acts as a gain at low frequencies. In the case that \( k_h \) increases, more higher harmonics are present. For \( k_h \to \infty \) the most harmonics are present, which coincides with HIGS starting to behave as a Clegg integrator. For various \( \omega_h \) with \( k_h = 1 \) the energy distribution is shown in Figure 2.4b, where it is observed that this parameter also affects the presence of higher harmonics. Lower values of \( \omega_h \) show that a significant amount of higher harmonics are
present, which correlates with its integrator behaviour as shown in Figure 2.3b. If $\omega_h \to \infty$ no higher harmonics seem present since the describing function acts as a gain.

To conclude, it is shown that the tuning of $k_h$ and $\omega_h$ is of significant importance for the behaviour of the HIGS and should be chosen with great care. Furthermore, it is stated that with careful tuning of the parameters the presence of the higher order harmonics can be limited. As a result, the describing function of HIGS is a proper approximation of its actual behaviour.

![Graphs showing normalized cumulative power spectral density](image)

*Figure 2.4: Normalized cumulative power spectral density of the output of HIGS in situation (a) with different values of $k_h$, $\omega_h = 2\pi$, and an input signal $e(t) = \sin(2\pi t)$. In situation (b) $\omega_h$ varies with $k_h = 1$ and using an input signal $e(t) = \sin(20\pi t)$.*

Previously in this section, the HIGS has been explained, the describing function has been computed, and its effectiveness has been demonstrated. Subsequently, the HIGS is used to obtain advantages in comparison to linear control filters. In [9], the HIGS was successfully implemented as an add-on to the closed loop. In addition, in [11] and [13] the HIGS was successfully integrated into a PID and a second-order low pass filter, respectively. These implementations of the HIGS showed the advantages of nonlinear control by the improvement of performance. Because of the common use of PID controllers, second-order low pass filters and notch filters in linear control, the logical next step is to implement HIGS into a notch filter design to be able to make further use of its advantages.
2.3 Notch Filter Design

The common use of a notch filter in linear control is due to the ability of compensating specific unwanted frequencies in the system. Another use is an inverse notch filter to obtain extra local low-frequency suppression. In addition, a skew notch filter can be used to compensate unwanted frequencies while establishing additional high-frequency suppression. If the HIGS is implemented in a notch filter design, the desired behaviour is that of similar magnitude with a decrease in the local phase lag. As a result, the bandwidth of the control system could be increased. This increase could be desired for the performance of the system as a whole.

Three HIGS-based notch filter designs are developed using the describing function analysis in the continuation of this section. First, a HIGS notch filter design is conducted using the low pass characteristics of the describing function. Next, a design using a gain with phase lead is investigated. Thereafter, a HIGS integrator and differentiator are used to implement into a HIGS notch filter design.

2.3.1 Notch with improper phase lead

The first design is based on the low pass characteristics of the describing function as discussed in [22]. For \( k_h = 1 \) the describing function in (2.11) shows a similar magnitude as a linear first order low pass filter, the latter being defined as

\[
C_{lp}(s) = \frac{\omega_{lp}}{s + \omega_{lp}},
\]

(2.17)

where \( \omega_{lp} = 1.619 \cdot \omega_h \) and \( \omega_h = 25 \cdot 2\pi \). In Figure 2.5 this similarity in magnitude can be observed, as well as the difference in phase lag. At high frequencies the phase lag increases with 51.9 degrees for the describing function. This phase advantage is also desired for implementation in a notch filter. Using the fact that a notch filter contains low pass characteristics, it can be implemented to create a HIGS notch filter design.
Figure 2.5: A Bode representation of a linear low pass filter and the describing function of HIGS with \( k_h = 1 \) and \( \omega_h = 25 \cdot 2\pi \).

At first, a linear notch filter is defined as

\[
C_n(s) = \frac{\omega_n^2}{\omega_n^2} \cdot \frac{s^2 + 2\omega_n \beta_n s + \omega_n^2}{s^2 + 2\omega_n \beta_n s + \omega_n^2},
\]

(2.18)

with the cut-off frequencies \( \omega_n^1 \) and \( \omega_n^2 \) and the damping coefficients \( \beta_n^1 \) and \( \beta_n^2 \). The transfer function of the notch filter can be split into a low pass and a high pass part. Subsequently, the low pass part can be divided into two first-order low pass filters provided that \( \beta_n^2 = 1 \). One of these first-order low pass filters can be replaced by the describing function in (2.11) with the consequence of limiting the choice of the damping coefficients. To eliminate this limitation an additional part is introduced to compensate for the decrease in magnitude at high frequencies of the describing function. The additional part is defined as \( C_{add}(s) = \frac{s + \omega_c}{\omega_c} \).

This reasoning results in the following describing function of a HIGS notch filter

\[
\Psi_n(j\omega) = \Psi_h(j\omega)C_{add}(j\omega)C_n(j\omega).
\]

(2.19)

The additional function \( C_{add} \) is improper, which produces an improper HIGS notch filter design. The zero in (2.19) is used to cancel the approximated pole of the describing function in (2.11) which compensates for the decrease in magnitude. To resolve the unwanted improper HIGS notch filter design, a linear control filter such as a second order low pass filter can be
added into the control loop. Due to the fact that this is a commonly used control filter, it poses no further problems.

Figure 2.6 shows a linear notch filter compared to the HIGS-based notch filter according to Equations (2.18) and (2.19) with the following values: $k_h = 1$, $\omega_{n1} = \omega_{n2} = 50 \cdot 2\pi \text{ rad/s}$, $\omega_h = 25 \cdot 2\pi \text{ rad/s}$, $\beta_{n1} = 0.1$, and $\beta_{n2} = 0.6$. The figure shows that the magnitude of the filters are approximately the same, however, the HIGS-based notch filter has a clear phase advantage. In the low-frequency range, the phase advantage starts with less phase lag. At higher frequencies, it results in additional phase lead. Eventually, the phase lead is 51.9 degrees in comparison to the linear notch filter.

![Bode representation of filters](image)

Figure 2.6: A Bode representation of a linear notch filter and the describing function of the HIGS notch filter of design A.

The benefit of this phase advantage is that it can give an increased bandwidth. To show this advantage, a quasi-linear loop shaping procedure is conducted in Section 3.3. In the remainder of this thesis, this designed HIGS notch filter is referred to as design A.

### 2.3.2 Notch with phase lead

Due to the fact that the formation of design A is improper, another approach is investigated. This approach is inspired by the gain with phase lead as conducted by [26]. This “Constant in gain Lead in phase” (CgLp) element is considered because a notch filter is a narrow band-stop filter. Therefore, outside of this stop-band, the filter has gain characteristics. Around the notch frequency, phase lag and lead are present. The goal is to eliminate the phase lag
of the notch filter using the CgLp element. In [26], the generalized first order reset element (GFORE) [14] and the generalized second order reset element (GSORE) [26] have been used in combination with a corresponding order lead filter to accomplish a gain with phase lead. In the previous section, it has been demonstrated that the describing function of HIGS has a low pass structure for $k_h = 1$. Therefore, this low pass structure is applied instead of a GFORE in combination with a lead filter. The lead filter is given as

$$C_{\text{lead}}(s) = \frac{s/\omega_l + 1}{s/\omega_l + 1},$$

with frequencies $\omega_{l2} \gg \omega_{l1}$. Remark that when $\omega_{l2}$ goes to infinity, the lead filter becomes the additional transfer function $C_{\text{add}}$ of design A. The describing function of the gain with phase lead results in

$$\Psi_k(j\omega) = \Psi_h(j\omega)C_{\text{lead}}(j\omega),$$

which is shown in Figure 2.7. In addition, the describing function of (2.11) and the lead filter are depicted. As expected, the gain with phase lead shows gain characteristics with an increase in phase. However, at $\omega > \omega_{l2}$, the gain decays to a -1 slope with corresponding phase lag. Therefore, $\omega_{l2}$ should be considered at a significantly high frequency.

![Figure 2.7: Constant gain with phase lead using the describing function of a HIGS and a lead filter as depicted here using the values $k_h = 1$, $\omega_h = 10 \cdot 2\pi$, $\omega_{l1} = \omega_c$ and $\omega_{l2} = 100 \cdot \omega_{l1}$.](image-url)
The obtained local phase advantages are desired for the notch filter design. Therefore, the describing function of this HIGS notch filter is derived using Equations (2.18) and (2.21) giving

$$
\Psi_{n,k}(j\omega) = \Psi_k(j\omega)C_n(j\omega).
$$

(2.22)

The parameters have been adjusted to achieve nearly no phase lag in the band-stop region, as well as in the high frequency range. This modified notch filter design (hereinafter referred to as design B) is compared to a linear notch filter in Figure 2.8. The absence of almost all phase lag can be beneficial in the quasi-linear loop shaping procedure, which is discussed in Section 3.3.

Figure 2.8: A Bode representation of a linear notch filter in comparison to the describing function of HIGS with added lead filter as implemented in a notch filter with the parameters valued as $k_h = 1$, \(\omega_h = 10 \cdot 2\pi\).

### 2.3.3 Notch with local phase lead

Already two promising HIGS notch filter designs are proposed. However, both designs have unwanted aspects at high frequencies. Design A has undesired phase lead and design B tends towards a -1 slope with corresponding phase lag at the high frequencies. To solve these issues, the accomplished phase lead should be eliminated. Therefore, it is discussed here to create phase lead and lag at the relevant frequencies only. This behaviour is achieved by converting an integrator and differentiator into HIGS filters. The HIGS integrator has been
introduced in [11]. Using a linear integrator $C_i = \omega_i / s$ and Equation (2.11), the describing function for a HIGS integrator is given by

$$
\Psi_i(s) = \Psi_h \left( \frac{s + \omega_c}{\omega_c} \left( \frac{\omega_i}{s} \right) \right). \tag{2.23}
$$

The frequency $\omega_c$ is chosen according to Equation (2.14) and $\omega_i$ is the crossover frequency of the linear integrator. In Equation (2.23) the function $C_{add}$ is added to compensate for the decrease in gain from the describing function as executed in Equation (2.19). However, in this case the improper aspect is not an issue because of the series configuration with the linear integrator. In Figure 2.9a the comparison between this linear and HIGS integrator can be observed. To be able to eliminate the achieved phase at high frequencies, it is decided to decrease the phase. Therefore, a differentiator with less phase lead is designed using $\Psi_d = 1 / \Psi_i$. This filter is compared to a linear differentiator $C_d = s / \omega_d$ in Figure 2.9b where $\omega_d$ is the crossover frequency.

![Figure 2.9](image.png)

Figure 2.9: Bode representation of linear filters in comparison to a describing function in (a) for an integrator and in (b) for a differentiator.

Figure 2.9 shows that the HIGS integrator achieves an increase in phase of 51.9 degrees and the HIGS differentiator obtained a phase decrease of 51.9 degrees. If the frequency $\omega_h$, which is the breaking point between gain and integrator mode, is chosen different for the HIGS integrator ($\omega_h,i$) and differentiator ($\omega_h,d$), a gain with local phase advantage occurs as can be seen in Figure 2.10a. The benefit of this gain with phase lead in comparison to the $C_{gLp}$ element is that at high frequencies both the magnitude and phase return back to zero. The downside of this design is that less phase advantage is present at the desired frequency. To show the use of this proposed element, the describing function of this HIGS-based notch filter design is given by

$$
\Psi_{n,i}(j\omega, \omega_{h,i}, \omega_{h,d}) = \Psi_i(j\omega, \omega_{h,i}) \Psi_d(j\omega, \omega_{h,d}) C_n(j\omega). \tag{2.24}
$$
This HIGS notch filter (appointed as design C) is shown in Figure 2.10b.

Figure 2.10: (a) represents a gain with phase lead using a HIGS integrator and differentiator and (b) shows this structure in a notch filter design.

This design shows the desired behaviour, namely, a similar magnitude as a linear notch filter with a local phase increase around the notch frequency. However, this design also becomes more complex by utilizing two describing functions as shown in Equation (2.24). In addition, the describing function analysis relies on the use of a sinusoidal input which is not the case here. These concerns should be taken into account during the quasi-linear design procedure.

2.4 Summary

The HIGS has been introduced in this chapter. Furthermore, the behaviour is evaluated in time and frequency domain. The conversion into frequency domain has been executed using the describing function method. From the describing function, it has been concluded that the phase lag at high frequencies decreases with 51.9 degrees. This advantage has been implemented into three HIGS notch filter designs. Design A is developed using the low pass characteristics of the describing function for $k_h = 1$. The design has acquired the expected phase increase, however, it also obtained infeasible phase lead because of the design being improper. Therefore, design B has been conducted. This design makes use of the CgLp element, which resulted in a gain with local phase increase. This behaviour was desired, however, at high frequencies the gain tended towards a -1 slope with corresponding phase lag. This necessitated the development of design C. Design C uses a HIGS integrator and differentiator to accomplish the wanted phase lead and lag at specific frequencies. The use of two describing functions has resulted in the desired behaviour, however, describing functions are based on a sinusoidal input. Therefore, this concern should be taken into account regarding the obtained approximation of the describing function. The potential advantages of the obtained HIGS notch filter designs are discussed in Section 3.3.
3 Quasi-linear robust design procedure

3.1 Introduction

As discussed in Chapter 2, the HIGS-based notch filters can result in designs with substantially less phase lag which can lead into an increase of the open loop bandwidth. To investigate this property further, a (quasi-)linear loop-shaping design procedure is performed. This procedure is elaborated on in the remainder of this chapter. In addition, a conclusion is made on the stability of the obtained nonlinear closed loop systems by means of a Lyapunov analysis.

In section 3.2 the linear closed loop system is presented. A parametric model of the system is derived from a fourth-order dynamical system representation. The parameters of the model are obtained by fitting it to the frequency response function (FRF) of the system. Thereafter, linear controllers are designed to achieve maximum bandwidth while meeting the introduced robustness constraints. In Section 3.3 a quasi-linear loop-shaping procedure is discussed. This procedure is conducted with the parametric model and various nonlinear controllers. The nonlinear controllers are designed wherein for the purpose of comparison the linear notch filters are substituted with the HIGS-based notch filter designs A, B, and C. Section 3.4 shows a rigorous stability analysis using piecewise quadratic Lyapunov functions for the obtained nonlinear systems. Finally, Section 3.5 summarizes this chapter.

3.2 System description

Figure 3.1 shows a schematic representation of the closed loop system containing a linear controller $C_{lin}$ and a plant $P$. The inputs of the closed loop system are the setpoint $r$, an external disturbance $d$ and measurement noise $\eta$. The closed loop system contains one output $y$, which is the output of the plant. The remaining signals are the error $e$ and the input of the plant $u$.

![Figure 3.1: Schematic representation of the linear closed loop system.](image)
3.2.1 Plant dynamics

The considered plant is represented by a motion system containing two masses, which are connected by a flexible shaft. One mass is attached to a rotary motor and an encoder. The other mass is only attached to an encoder. Figure 3.2a shows this set-up, which is further referred to as the PATO system. The PATO system is approximated by a fourth-order dynamical system as shown in Figure 3.2b. This dynamical system is favored because it gives the ability to acquire a fairly simple parametric model which is representative for a large class of motion control systems and which will be used to perform continuous (quasi-)linear loop shaping.

![Figure 3.2: The PATO system: (a) the experimental set-up and (b) a simplified fourth-order system representation.](image)

The dynamical system of Figure 3.2b contains various parameters. The torque $\tau$ of the motor is the input of the system, $J_1$ and $J_2$ are the moments of inertia of the corresponding masses, the rotations $\theta_1$ and $\theta_2$ are the outputs, $d$ is the damping coefficient and $k$ is the torsional spring constant of the shaft connecting the two masses. The rotation $\theta_2$ of the non-collocated inertia $J_2$ is considered. The relation from the input torque $\tau$ to the output $\theta_2$ is given by the transfer function

$$P(s) = \frac{ds + k}{J_1 J_2 s^4 + (J_1 + J_2) s^3 + (J_1 + J_2) k s}.$$  \hspace{1cm} (3.1)

To identify the parameters of the dynamical model, the transfer function is rewritten into its rigid body mode and its flexible mode as

$$P(s) = \frac{1}{J_{12} s^2} + \frac{-1}{J_{12} (s^2 + 2 \omega \beta s + \omega^2)},$$  \hspace{1cm} (3.2)
with $J_{12} = J_1 + J_2$, $\beta = \frac{J_{12}d}{2J_1J_2\omega}$, and $\omega = \sqrt{\frac{J_{12}k}{J_1J_2}}$.

The values of these parameters are obtained by fitting the model to the measured FRF. Both the measured FRF and the fitted model are shown in Figure 3.3. The input/output delay (of three samples) present in the data is corrected before fitting the model to the FRF. The obtained parameters are $J_{12} = 0.39 \text{ g/m}^2$, $\beta = 0.0185$, and $\omega = 54.03 \cdot 2\pi \text{ rad/s}$.

![Figure 3.3: Frequency Response Function of the measured plant and the parametric model fit.](image)

3.2.2 Linear controller

To control the continuous parametric model, linear (continuous) controllers are designed. The controllers consist of a proportional-integrator-derivative (PID) filter, a second-order low pass (SOLP) filter, and a notch filter. The transfer functions of these filters are given respectively by

\begin{align}
C_{\text{pid}}(s) &= k_{\text{pid}} \left( 1 + \frac{\omega_i}{s} + \frac{s}{\omega_d} \right), \quad (3.3) \\
C_{\text{solv}}(s) &= \frac{\omega_{lp}^2}{s^2 + 2\omega_{lp}\beta_{lp}s + \omega_{lp}^2}, \quad (3.4)
\end{align}
resulting in linear controllers formulated as \( C_{\text{lin}}(s) = C_{\text{pid}}(s)C_n(s)C_{\text{solp}}(s) \). By varying the parameters of the notch filter, linear controllers utilizing a normal notch filter, an inverse notch filter, and a skew notch filter are obtained to make use of their different applications. The first designed linear controller uses a normal notch filter to compensate the resonance frequency of the system. To this end loop shaping is performed to determine the tuning parameters using the parametric model as given in (3.2). Loop shaping is conducted with a robustness constraint of a maximum allowable peak in the (closed loop) input sensitivity function at 6 dB while maximizing the bandwidth \( \omega_\beta \). The sensitivity function is obtained from the linear closed loop system of Figure 3.1 by taking input signals \( d \) and \( \eta \) to be zero and derive the relation between the Fourier transforms of setpoint \( r \) and of the error \( e \) denoted by \( \hat{e} \) and \( \hat{r} \). This results in the linear closed loop sensitivity function

\[
\frac{\hat{e}}{\hat{r}} = \frac{1}{1 + PC_{\text{lin}}} = S.
\]

The correlated robustness constraints are the gain margin for a minimum of 6 dB and the phase margin with a minimum of 30 degrees. The robust loop shaping procedure has resulted in the values for the controller parameters conducting a normal notch filter as shown in Table 3.1.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>PID</td>
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<tr>
<td></td>
<td>( \omega_i )</td>
<td>2.21 \cdot 2\pi</td>
</tr>
<tr>
<td></td>
<td>( \omega_d )</td>
<td>5.45 \cdot 2\pi</td>
</tr>
<tr>
<td>SOLP</td>
<td>( \omega_p )</td>
<td>100 \cdot 2\pi</td>
</tr>
<tr>
<td></td>
<td>( \beta_p )</td>
<td>0.50</td>
</tr>
<tr>
<td>Notch</td>
<td>( \omega_{n1} )</td>
<td>54.03 \cdot 2\pi</td>
</tr>
<tr>
<td></td>
<td>( \omega_{n2} )</td>
<td>54.03 \cdot 2\pi</td>
</tr>
<tr>
<td></td>
<td>( \beta_{n1} )</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>( \beta_{n2} )</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>( \omega_\beta )</td>
<td>16.24 \cdot 2\pi</td>
</tr>
</tbody>
</table>

The acquired controller and parametric model of the plant result in the open loop system \( PC_{\text{lin}} \) as depicted in Figure 3.4. It is observed that the resonance frequency of the plant is indeed compensated by the normal notch filter of the controller. This compensation has allowed the bandwidth to increase to a maximum of \( 16.24 \cdot 2\pi \) rad/s with a gain margin of 7.10 dB and a phase margin of 40.22°.
Second, a linear controller using an inverse notch filter is designed. An inverse notch filter can achieve extra local low-frequency suppression in closed loop. Subsequently, the values for the linear controller parameters are obtained in a loop shaping procedure while meeting the robustness constraints. The obtained values are given in Table 3.2. Knowing the controller values and the plant, the open loop system is acquired as shown in Figure 3.5a. The figure shows the inverse notch filter as well as the resonance frequency of the system. The exclusion of the resonance compensation has resulted in a significant smaller maximum bandwidth of $11.12 \cdot 2\pi$, while meeting the robustness constraints with $9.95$ dB and $31.73^\circ$ for the gain and phase margin respectively.

Table 3.2: Values of controller parameters using an inverse notch filter.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>$k_{pid}$</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>$\omega_i$</td>
<td>$1.35 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_d$</td>
<td>$3.55 \cdot 2\pi$</td>
</tr>
<tr>
<td>SOLP</td>
<td>$\omega_{lp}$</td>
<td>$39 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{lp}$</td>
<td>1.00</td>
</tr>
<tr>
<td>Notch</td>
<td>$\omega_{n1}$</td>
<td>$1.00 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{n2}$</td>
<td>$1.00 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n1}$</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n2}$</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>$\omega_j$</td>
<td>$11.12 \cdot 2\pi$</td>
</tr>
</tbody>
</table>

Table 3.3: Values of controller parameters using a skew notch filter.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>$k_{pid}$</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>$\omega_i$</td>
<td>$2.27 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_d$</td>
<td>$5.31 \cdot 2\pi$</td>
</tr>
<tr>
<td>SOLP</td>
<td>$\omega_{lp}$</td>
<td>$199 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{lp}$</td>
<td>0.30</td>
</tr>
<tr>
<td>Notch</td>
<td>$\omega_{n1}$</td>
<td>$57.0 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{n2}$</td>
<td>$38.0 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n1}$</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n2}$</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>$\omega_j$</td>
<td>$15.17 \cdot 2\pi$</td>
</tr>
</tbody>
</table>
Hereafter, a linear controller with a skew notch filter is designed to partly address the resonance frequency while obtained additional high frequency suppression. The values of the controller parameters are given in Table 3.3, which are acquired through a loop shaping procedure. Figure 3.5b shows the resulting open loop system. Since the resonance frequency of the system is not completely compensated, the maximum allowable bandwidth is smaller in comparison to the normal notch filter. The maximum open loop bandwidth has become $15.17 \cdot 2\pi$ with a gain margin of $8.93$ dB and a phase margin of $37.47^\circ$.

![Figure 3.5](image)

(a) shows the open loop using a controller that includes an inverse notch filter and (b) depicts the open loop for a skew notch filter implemented in the controller.

3.3 Quasi-linear loopshaping

In Section 3.2.2, linear controllers have been designed in a loop shaping procedure using a notch filter, an inverse notch filter, and a skew notch filter. In the continuation of this section, these various notch filters are substituted with HIGS-based notch filters according to the developed designs A, B, and C. The describing function of these designs, which have been derived in Equations (2.19), (2.22), and (2.24), are used to be able to perform a quasi-linear loop shaping procedure. The objective of this section is to increase bandwidth using the obtained decrease in phase lag of the describing functions for the HIGS-based notch filter designs.

3.3.1 Design A

HIGS-based notch filter design A has obtained desired reduced phase lag as well as phase lead at high frequencies based on its describing function of (2.19). A possible solution for the infeasible phase lead has been proposed. Therefore, the advantages of this design are evaluated through a quasi-linear loop shaping procedure. Quasi-linear loop shaping has been performed using the parametric plant of (3.2), the linear controller filters $C_{\text{pid}}$ and $C_{\text{solp}}$, and the describing function of design A as a substitution of the linear notch filter $C_n$. The
HIGS-based controller is rewritten in its linear components as $C_{\text{nom,A}} = C_{\text{add}}C_{\text{lin}}$ and its nonlinear component defined as $\Psi_h$. During the quasi-linear loop shaping procedure, the bandwidth is increased to its maximum extent while still meeting the robustness constraints. To meet the robustness constraints, the sensitivity of the quasi-linear closed loop system is given by

$$S_h = \frac{1}{1 + P(j\omega)C_{\text{nom,A}}(j\omega)\Psi_h(j\omega)}.$$ (3.7)

In the same manner as for linear loop shaping, different variations of the notch filter are investigated by changing the controller parameters. The obtained values are given in Table 3.4.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Normal</th>
<th>Inverse</th>
<th>Skew</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>$k_{\text{pid}}$</td>
<td>2.85</td>
<td>2.07</td>
<td>2.73</td>
</tr>
<tr>
<td></td>
<td>$\omega_i$</td>
<td>3.58·2\pi</td>
<td>4.02·2\pi</td>
<td>4.60·2\pi</td>
</tr>
<tr>
<td></td>
<td>$\omega_d$</td>
<td>8.04·2\pi</td>
<td>5.98·2\pi</td>
<td>9.70·2\pi</td>
</tr>
<tr>
<td>SOLP</td>
<td>$\omega_{lp}$</td>
<td>85·2\pi</td>
<td>19·2\pi</td>
<td>130·2\pi</td>
</tr>
<tr>
<td></td>
<td>$\beta_{lp}$</td>
<td>0.50</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Notch</td>
<td>$\omega_{n1}$</td>
<td>54.03·2\pi</td>
<td>1.0·2\pi</td>
<td>57.0·2\pi</td>
</tr>
<tr>
<td></td>
<td>$\omega_{n2}$</td>
<td>54.03·2\pi</td>
<td>1.0·2\pi</td>
<td>38.0·2\pi</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n1}$</td>
<td>0.02</td>
<td>0.79</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n2}$</td>
<td>0.60</td>
<td>0.01</td>
<td>0.68</td>
</tr>
<tr>
<td>Add</td>
<td>$\omega_c$</td>
<td>1.619·\omega_h</td>
<td>1.619·\omega_h</td>
<td>1.619·\omega_h</td>
</tr>
<tr>
<td>HIGS</td>
<td>$k_h$</td>
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<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\omega_h$</td>
<td>10·2\pi</td>
<td>0.5·2\pi</td>
<td>10·2\pi</td>
</tr>
<tr>
<td></td>
<td>$\omega_\beta$</td>
<td>23.64·2\pi</td>
<td>14.78·2\pi</td>
<td>17.64·2\pi</td>
</tr>
</tbody>
</table>

The table shows that for all variants of the HIGS-based controller using design A, the bandwidths of the nonlinear open loop systems have increased compared to their linear counterparts. The increase in bandwidth for using the normal, inverse, and skew HIGS notch in the open loop systems are 45.57%, 32.92%, and 16.27% respectively. In addition, it is noted that the gains of the controllers are increased by more than twofold in comparison to the controller gains in Tables 3.1, 3.2, and 3.3. This increase results in improved disturbance suppression at low frequencies. Figure 3.6 shows this improvement by means of the quasi-linear sensitivity functions in comparison to their corresponding linear sensitivity functions.
Figure 3.6: Quasi-linear sensitivity functions using the HIGS-based notch filter design A for (a) a normal notch filter, (b) an inverse notch filter and (c) a skew notch filter in comparison with their linear counterparts.

To show that the linear and nonlinear controllers meet the robustness constrains, their Nyquist-like plots are given in Figure 3.7 for the normal (3.7a), inverse (3.7b), and skew notch filter (3.7c). The gain and phase margin for the nonlinear system of the normal notch filter are 7.00 dB and 52.77 degrees, respectively. For the nonlinear system of the inverse notch filter the margins are 12.77 dB and 33.61 degrees. The gain and phase margin for the nonlinear system conducting the skew notch filter are 9.75 dB and 33.72 degrees.
Figure 3.7: Quasi-linear Nyquist plot using the HIGS-based notch filter design A for (a) a normal notch filter, (b) an inverse notch filter and (c) a skew notch filter in comparison with their linear counterparts.

### 3.3.2 Design B

The quasi-linear loop shaping procedure using the describing function of design A has shown promising results in terms of low-frequency disturbance rejection properties. Here, the HIGS-based notch filter of design B has been used which has achieved a decrease in phase lag utilizing a CgLp element without infeasible phase lead as for design A. The describing function of design B as in (2.22) is implemented into the controller as an alternative for the control filter $C_n$. The linear part of the controller becomes $C_{nom,B} = C_{lead}C_{lin}$ and the nonlinear part is defined as $\Psi_h$. The procedure using the HIGS-based controller of design B results in the controller parameters and bandwidths of Table 3.5. Again the bandwidths of the open loop systems are increased in comparison to their linear counterparts with 41.01%, 30.10%, and 14.12% for the systems conducting a normal, inverse, and skew HIGS notch filter respectively. However, the increased bandwidths are lower than the obtained bandwidths using design A. Apparently, the additional phase lead at high frequencies for design A due to the absence of
a pole has its advantages regarding the maximum achievable bandwidth. Furthermore, an increase in gain is observed which leads to an improvement of low-frequency disturbance suppression. However, the gains have increased less in comparison to the HIGS-based controller using design A.

### Table 3.5: Values of the parameters for the HIGS controller of design B.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Normal</th>
<th>Inverse</th>
<th>Skew</th>
</tr>
</thead>
<tbody>
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<td>$\omega_i$</td>
<td>4.59·2π</td>
<td>4.02·2π</td>
<td>4.40·2π</td>
</tr>
<tr>
<td></td>
<td>$\omega_d$</td>
<td>8.05·2π</td>
<td>5.78·2π</td>
<td>9.67·2π</td>
</tr>
<tr>
<td>SOLP</td>
<td>$\omega_{lp}$</td>
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<td>19·2π</td>
<td>130·2π</td>
</tr>
<tr>
<td></td>
<td>$\beta_{lp}$</td>
<td>0.50</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Notch</td>
<td>$\omega_{n1}$</td>
<td>54.03·2π</td>
<td>1.0·2π</td>
<td>57.0·2π</td>
</tr>
<tr>
<td></td>
<td>$\omega_{n2}$</td>
<td>54.03·2π</td>
<td>1.0·2π</td>
<td>38.0·2π</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n1}$</td>
<td>0.02</td>
<td>0.79</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n2}$</td>
<td>0.60</td>
<td>0.01</td>
<td>0.68</td>
</tr>
<tr>
<td>Lead</td>
<td>$\omega_1$</td>
<td>$\omega_c$</td>
<td>$\omega_c$</td>
<td>$\omega_c$</td>
</tr>
<tr>
<td></td>
<td>$\omega_2$</td>
<td>100·$\omega_c$</td>
<td>1000·$\omega_c$</td>
<td>100·$\omega_c$</td>
</tr>
<tr>
<td>HIGS</td>
<td>$k_h$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\omega_h$</td>
<td>10·2π</td>
<td>0.5·2π</td>
<td>10·2π</td>
</tr>
<tr>
<td></td>
<td>$\omega_3$</td>
<td>22.90·2π</td>
<td>14.47·2π</td>
<td>17.31·2π</td>
</tr>
</tbody>
</table>

#### 3.3.3 Design C

The HIGS-based controller utilizing design A has been able to obtain a higher bandwidth in comparison to the controller using design B. Design C has been conducted using a HIGS integrator and differentiator with different crossover frequencies to aim at the desired local phase advantages. However, it seems that for the open loop bandwidth local phase advantages may not be the most beneficial. Quasi-linear loop shaping is performed for design C to further investigate this observation. The HIGS-based controller of design C is divided into its linear and nonlinear part. The linear part is written as $C_{nom,C} = C_{add}C_iC_{lp}C_{d}C_{lin}$ and the nonlinear part is $\Psi_{h,C} = \Psi_h(\omega_{h,i})\Psi_h(\omega_{h,d})$. The quasi-linear loop shaping results in the control parameters of Table 3.6. As expected, the bandwidths are higher in comparison to their linear counterparts, but lower than the HIGS-based controllers of design A and design B. However, the open loop bandwidths have still increased with 26.48%, 30.31%, and 2.26% for the HIGS-based controller using a normal, inverse, and skew HIGS notch filter respectively and also an improvement in low-frequency suppression is obtained.
### Table 3.6: Values of the parameters for the HIGS-based controller using design C.

<table>
<thead>
<tr>
<th>Filter</th>
<th>Parameter</th>
<th>Normal</th>
<th>Inverse</th>
<th>Skew</th>
</tr>
</thead>
<tbody>
<tr>
<td>PID</td>
<td>$k_{pid}$</td>
<td>2.76</td>
<td>1.22</td>
<td>2.13</td>
</tr>
<tr>
<td></td>
<td>$\omega_i$</td>
<td>$3.95 \cdot 2\pi$</td>
<td>$5.08 \cdot 2\pi$</td>
<td>$4.74 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_d$</td>
<td>$8.89 \cdot 2\pi$</td>
<td>$5.27 \cdot 2\pi$</td>
<td>$8.52 \cdot 2\pi$</td>
</tr>
<tr>
<td>SOLP</td>
<td>$\omega_{lp}$</td>
<td>$85 \cdot 2\pi$</td>
<td>$30 \cdot 2\pi$</td>
<td>$130 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{lp}$</td>
<td>0.50</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Notch</td>
<td>$\omega_{n1}$</td>
<td>$54.03 \cdot 2\pi$</td>
<td>$1.0 \cdot 2\pi$</td>
<td>$57.0 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{n2}$</td>
<td>$54.03 \cdot 2\pi$</td>
<td>$1.0 \cdot 2\pi$</td>
<td>$38.0 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n1}$</td>
<td>0.02</td>
<td>0.79</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>$\beta_{n2}$</td>
<td>0.60</td>
<td>0.01</td>
<td>0.68</td>
</tr>
<tr>
<td>HIGS</td>
<td>$k_{h,i}$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\omega_{h,i}$</td>
<td>$10 \cdot 2\pi$</td>
<td>$0.5 \cdot 2\pi$</td>
<td>$10 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$k_{h,d}$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\omega_{h,d}$</td>
<td>$100 \cdot 2\pi$</td>
<td>$10 \cdot 2\pi$</td>
<td>$100 \cdot 2\pi$</td>
</tr>
<tr>
<td></td>
<td>$\omega_3$</td>
<td>$20.54 \cdot 2\pi$</td>
<td>$12.94 \cdot 2\pi$</td>
<td>$15.51 \cdot 2\pi$</td>
</tr>
</tbody>
</table>

### 3.3.4 Discussion

For HIGS-based notch filter designs A, B, and C a quasi-linear loop shaping procedure is performed using describing functions. All HIGS-based controllers have shown an increase in open loop bandwidth in comparison to the bandwidth of the corresponding linear controllers. In addition, an increase in gain to improve low-frequency disturbance suppression is obtained for all the HIGS-based controllers. As discussed in Chapter 2, a local phase advantage for a HIGS notch filter may be favourable. However, it shows that in the quasi-linear loop shaping procedure the HIGS notch filter of design A, with the additional high-frequency phase lead, is the most beneficial. This statement is made using the fact that the HIGS-based controller of design A has resulted in the highest gains of the controllers and the maximum open loop bandwidths of the systems. The aforementioned limitation of the improperness of the filter has shown to be solvable with the use of a SOLP filter. The SOLP filter also ensures the properness of the used PID filter. Therefore, the HIGS-based controller of design A results in a control filter with a relative degree of zero. In conclusion, design A gives the highest gains and the maximum open loop bandwidths. Therefore, this design is chosen to use in the continuation of this thesis. However, it is necessary to take into account that this nonlinear design is an approximation in frequency domain and does not guarantee performance improvements.
3.4 Rigorous stability analysis

The quasi-linear loop shaping procedure has resulted in the desired increase of the open loop bandwidth. In this procedure, the HIGS-based controller using design A has been shown to be the most beneficial in terms of quasi-linear predictions. To be able to implement this designed controller, a conclusion on its stability has to be made. The closed loop with the implemented HIGS notch filter cannot be guaranteed to be stable by using the Nyquist criterion and the describing function due to its different nonlinear description that is only exact in time-domain. Therefore, another method is considered to prove the stability of the closed loop system. The chosen method is the use of Linear Matrix Inequalities (LMIs) in a piecewise fashion such as developed in the works of [1, 12, 21]. This method proves stability based on Lyapunov analysis.

3.4.1 State-space representation of the nonlinear closed loop system

![Schematic representation of the nonlinear closed loop system.](image)

The nonlinear closed loop system is depicted in Figure 3.8. The linear part of the describing function of design A is implemented as the nominal controller $C_{nom} = C_{nom,A}$. The plant of the closed loop system remains the parametric model of (3.2). To be able to formulate the closed-loop stability problem through LMIs, the closed loop system is defined in a state-space representation. The state-space realization of the obtained parametric model of the plant is

$$\mathcal{P} := \begin{cases} \dot{x}_p = A_p x_p + B_{p,c} u_c + B_{p,d} d, \\ y_p = C_p x_p, \end{cases} \quad (3.8)$$

with $x_p \in \mathbb{R}^n$ the state of the plant, $u_c$ the output of the controller, $d$ the disturbance, and $y_p$ the output of the plant and of the closed loop system. The plant is strictly proper with a relative degree of three.
The nominal controller $C_{\text{nom}}$ is represented by the following state-space model:

$$C_{\text{nom}} := \begin{cases} \dot{x}_c = A_c x_c + B_c u_h, \\ u_c = C_c x_c + D_c u_h, \end{cases} \quad (3.9)$$

where $x_c \in \mathbb{R}^m$ are the states of the controller and $u_h$ is the input of the controller. The state-space matrices of the plant and the controller are both a minimal realization. The nonlinear part of the closed loop system is the HIGS as defined in (2.1). In state-space form, HIGS is given by

$$\mathcal{H} := \begin{cases} E_{h,k} \dot{x}_h = A_{h,k} x_h + B_{h,k} e, \quad \text{for } [e, u_h, \dot{e}]^\top \in \mathcal{F}_k \\ u_h = C_h x_h, \end{cases} \quad (3.10)$$

with $x_h \in \mathbb{R}$ as the state and $e$ is the input of the HIGS. The two domains of the HIGS are $k = 1$ for integrator mode and $k = 2$ for gain mode. The matrices are defined as $([E_{h,1}, E_{h,2}], [A_{h,1}, A_{h,2}], [B_{h,1}, B_{h,2}], C_h) = ([1, 0], [0, -1], [\omega_h, k_h], 1)$. To include the switching dynamics of the HIGS, the sub-domains of (2.2) are rewritten as in [12]. This results in

$$\mathcal{F}_1 := \left\{ v \in \mathbb{R}^3 | v^\top C_1^\top M_1 C_1 v \geq 0 \wedge v \notin \mathcal{F}_2 \right\},$$

$$\mathcal{F}_2 := \left\{ v \in \mathbb{R}^3 | v^\top C_2^\top M_2 C_2 v > 0 \wedge Lv = 0 \right\},$$

with matrices

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \frac{1}{2} \\
\frac{1}{2} & -k_h \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & -\frac{1}{2} \\
-\frac{1}{2} & \frac{\omega_h}{k_h} \end{bmatrix},$$

$$L = \begin{bmatrix} k_h & -1 & 0 \end{bmatrix}.$$ \quad (3.12)

Using the configuration of the closed loop system in Figure 3.8, the state-space model of the closed loop can now be derived as

$$T := \begin{cases} \dot{x} = A_k x + B_k u, \quad \text{if } [e, u_h, \dot{e}]^\top \in \mathcal{F}_k \\ y = C x + D u, \end{cases} \quad (3.13)$$

with $x = [x_p, x_c, x_h]^\top$ as the states, and $u = [d, r]^\top$ the inputs. The closed loop matrices are given by
\[ A_k = \Pi_k \begin{bmatrix} A_p & B_p C_p & B_p D_c \\ 0 & A_e & B_e \\ -B_{h,k} C_p & 0 & A_{h,k} \end{bmatrix}, \]
\[ B_k = \Pi_k \begin{bmatrix} B_p & 0 \\ 0 & 0 \\ 0 & B_{h,k} \end{bmatrix}, \]
\[ C = [C_p \ 0 \ 0], \]
\[ D = [0 \ 0], \]

with \( \Pi_1 = I_q, \quad \Pi_2 = \begin{bmatrix} I_{m+n} & 0 \\ -k_h C & 0 \end{bmatrix}, \) for \( q = m + n + 1. \)

(3.14)

Since the closed loop is defined in state-space, its stability can be determined by finding a Lyapunov function that is positive definite and that has a derivative that is monotonically decreasing. For the construction of the LMIs, for simplicity, it is temporarily assumed that the setpoint is equal to zero. This assumption is fairly mild given the fact that the effect of the setpoint, which is considered the largest 'known' disturbance to the system, can be cancelled with advanced feedforward control. Therefore, the remaining input signal of the closed loop system is the unknown disturbance \( d. \)

### 3.4.2 Piecewise quadratic Lyapunov function

To ensure the stability of the obtained state-space representation of the nonlinear closed loop system, the LMIs are constructed as follows. As a first constraint, a quadratic Lyapunov function candidate is proposed

\[ V(x) = x^\top P x, \]

(3.15)

where \( P = P^\top \succ 0 \) is a positive definite matrix. The closed loop system with switching is stated to be globally uniformly asymptotically stable if

\[ \dot{V}(x) = x^\top (A_k^\top P + PA_k) x < 0 \quad \forall \ x \in \mathbb{R}^q \setminus \{0\}, \quad k = 1, 2. \]

(3.16)

These conditions can be relaxed by including the known switching law using the S-procedure [7]. In this manner, the stability analysis is potentially less conservative. In addition, piecewise quadratic Lyapunov functions are considered to further reduce the conservatism of the LMI stability analysis [21]. Therefore, the \((e,u)\)-plane is divided into multiple sectors as shown in Figure 3.9. The sectors are enclosed by their boundaries, which depend on the corresponding angles

\[ 0 = \theta_0 < \theta_1 < ... < \theta_N = \arctan(k_h), \quad i = 1, ..., N. \]

(3.17)
Figure 3.9: The (e,u)-plane divided in \( i = N \) sectors.

The vectors that define the boundaries of the sections are normalized as

\[
    r_i = \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix}, \quad i = 1, \ldots, N. \tag{3.18}
\]

The sectors are enclosed by two boundaries which are formulated as \( p_i = R_i^{-1}[e, u]^\top \) with \( R_i = [r_{i-1}, r_i] \). For each sector \( S_i \), the constraints of the integrator mode should hold. The constraints are converted into the inequalities \( u \geq 0 \) and \( k_h e - u \geq 0 \) or \( u \leq 0 \) and \( k_h e - u \leq 0 \). These inequalities are element-wise described as

\[
    S_i x := R_i^{-1} C_1 C_v x \geq 0 \lor S_i x \leq 0,
\]

where \[
    \begin{bmatrix} e \\ u \end{bmatrix} = C_1 v \quad \text{and} \quad v = C_v x,
\]

with \( C_v = \begin{bmatrix} -C_p & 0 & 0 \\ 0 & 0 & 1 \\ -C_p A_1 & 0 & 0 \end{bmatrix} \),

with \( C_1 \) as given in Equation (3.12). Furthermore, the inequalities for the gain mode are needed to construct the LMIs. Using the constraints of the gain mode results in the inequalities \( e \geq 0 \) and \( \omega_h e - k_h \dot{e} \geq 0 \), or \( e \leq 0 \) and \( \omega_h e - k_h \dot{e} \leq 0 \), and the constraint \( k_h e - u = 0 \), yielding element-wise as

\[
    Z x := \begin{bmatrix} -k_h & \omega_h \\ 0 & 1 \end{bmatrix} C_2 C_v x \geq 0 \lor Z x \leq 0 \land G x = L C_v x. \tag{3.20}
\]
The inequalities of both modes are defined, which results in the ability to construct their relaxed LMI conditions using the S-procedure. Another necessary condition is to obtain a continuous Lyapunov function. Therefore, obtaining continuity along the common boundaries of the sectors is necessary. To provide this, a continuity matrix \( F_i \) is introduced in

\[
P_i = F_i^T T F_i,
\]

as formulated in [20]. Here \( T = T^T \) is a symmetric matrix containing the decision variables. For this continuity matrix, it holds that \( F_i x = F_j x \) for all \( x \in S_i \cap S_j \). To satisfy this property, the matrix \( E_i \in \mathbb{R}^{(N+1) \times 2} \) is introduced. This matrix consists of all zeros and an identity matrix \( I_2 \), which is shifted according to the corresponding sectors. A full rank continuity matrix is obtained by adding an identity matrix, which results in

\[
F_i = \begin{bmatrix} E_i S_i & I_q \end{bmatrix}.
\]

The LMI conditions are constructed using the Lyapunov function with the obtained inequalities in (3.19) and (3.20), and the continuity condition of (3.22). By applying the S-procedure and Finsler’s lemma, the LMI conditions are given by

\[
\begin{align*}
A_1^T P_i + P_i A_1 + S_i^T U_i S_i &< 0, \quad i = 1, ..., N \\
A_2^T P_N + P_N A_2 + Q^T G + G^T Q + Z^T Y Z &< 0, \\
P_i - S_i^T W_i S_i &> 0, \quad i = 1, ..., N.
\end{align*}
\]

These conditions ensure that a positive definite matrix \( P_i \) is searched for within its sector. The matrices \( S_i, G, \) and \( Z_i \) are given in Equations (3.19) and (3.20). The nonlinear closed loop stability is obtained for finding the positive definite matrix \( P_i \), the matrices \( U_i, V, W_i \) with positive elements, and \( Q \) which satisfy the LMI conditions for

\[
\dot{V}(x) = x^T (A_{ik} P_i + P_i A_k) x \leq -\gamma \| x \|^2,
\]

with \( k = 1, 2 \) representing its corresponding mode and \( \gamma > 0 \). To achieve input-to-state stability (ISS) as well, the unknown input disturbance \( d \) should be considered in the derivative of the Lyapunov function as follows

\[
\dot{V}(x) = x^T (A_{ik} P_i + P_i A_k) x + 2 x^T P_i \Pi_k B_k u,
\]

where \( u = [d, r]^T \). It has been assumed that \( r = 0 \) due to its cancellation by advanced feedforward. Furthermore, using the assumption that \( d \) is bounded results an the upper-bound for \( \dot{V} \) as
\[
\dot{V}(x(t)) \leq -(\gamma - \delta)\|x\|^2 - \delta\|x\|^2 + \rho\|x\|\|d\|, \tag{3.26}
\]

where \(0 < \delta < \gamma\) and \(\rho = \max_{i,k}\|2P_i\Pi_k B_k\|\) for \(i = 1, \ldots, N\) and \(k = 1, 2\). Resulting in a decreasing derivative of the Lyapunov function for \(\|x\| > \rho/\delta\|d\|\).

The ISS stability of the system is shown by

\[
\|x(t)\| \leq \alpha e^{-\lambda(t-t_0)}\|x(t_0)\| + \zeta \sup_{0 \leq \tau \leq T} \|d(\tau)\| \tag{3.27}
\]

where \(\alpha, \zeta, \lambda > 0\) which shows the independence of the initial states and input for the nonlinear closed loop stability.

### 3.4.3 Numerical study

The LMI conditions are derived to check stability of the nonlinear closed loop systems. The controller parameters of Table 3.4 are implemented to check if the designed systems based on (approximate) describing functions are stable for the true nonlinearity too. The LMIs for the closed loop systems with the integrated HIGS-based notch filter of design A are infeasible. Therefore, a case study is conducted to analyse this infeasibility. It is said that a closed loop system is infeasible if a maximum of 100 partitions has not been proven solvable. The controller for the case study is the nonlinear controller for the normal notch filter of design A, however, some values of the controller parameters are adjusted. The adjusted values are \(k_{\text{pid}} = 2.51\) and \(\omega_h = 20 \cdot 2\pi\), which results in the controller \(C_{h,1}\). The Nyquist-like plot of this controller is shown in Figure 3.10a using its describing function. According to the linear Nyquist stability criterion, the controller should be stable. This is in agreement with the feasibility of the LMIs with the use of two partitions. The behaviour of this true nonlinear system is shown in the step response of Figure 3.10b, which depicts a corresponding stable system. In this manner, the feasibility of the LMIs seems accurate. To test the accuracy of the LMIs, the gain of the controller is varied to obtain non-robust and unstable controllers. This results in the controllers \(C_{h,2}, C_{h,3}\), and \(C_{h,4}\) with their corresponding gains of 3.54, 4.80, and 6.00. Controller \(C_{h,2}\) is feasible with eight partitions as it comes closer to the minus 1 point. However, controller \(C_{h,3}\) is infeasible even though the Nyquist using the describing function and the step response show that the controller should be stable. Controller \(C_{h,4}\) is unstable in the Nyquist plot as well as in its step response and its LMIs are infeasible. Therefore, it seems that the describing function is an accurate representation of the stability of the true nonlinear system as shown in the step response. So, it is expected that the LMIs are too conservative. This statement is further examined by taken a look at the Nyquist plot of the gain mode as shown in Figure 3.11.
The figure shows that the gain mode is unstable for the controllers $C_{h,3}$ and $C_{h,4}$, which relates to the infeasibility of the LMIs. Since $\omega_h$ has been chosen to be relatively small, the nonlinear system is mostly located in integrator mode. Therefore, the stability of the gain mode should not be an issue due to relaxation in the LMIs. However, this relaxation seems to be inaccurate, which results in unsolvable LMIs. This bottleneck could be resolved by extending the LMIs into a three-dimensional space to implement the plane of the gain mode. This is a recommendation for further research. In this research, the nonlinear systems for design A has been checked using the Nyquist-like plot and the step response. Using these tools, the designed nonlinear systems are said to be stable.

Figure 3.11: Nyquist plot for the gain modus of the nonlinear controller with varying $k_{pid}$. 
3.5 Summary

This chapter started with a linear closed loop system consisting of a simple plant model of the PATO setup together with a linear controller. A parametric model of the PATO system has been derived and validated. The linear controllers for this model have been designed using a straightforward loop shaping procedure. The controllers have been varied with a normal, an inverse, and a skew notch filter. Thereafter, quasi-linear loop shaping has been conducted for the designed HIGS-based notch filters A, B, and C with the same variations in notch filters. The HIGS-based controller using design A has been shown to obtain the maximum gain and open loop bandwidth, while meeting frequency-domain robustness constraints. Since linear stability analysis methods such as the Nyquist criterion are generally not sufficient for nonlinear systems, stability as resulting from the loop shaping procedure has been checked by finding piecewise quadratic Lyapunov functions. Stability of the nonlinear closed loop system using design A has been checked under these conditions, which resulted in an infeasible solution. Therefore, a numerical case study has been conducted to check the accurateness of the LMIs. It seemed that the LMIs were too conservative and an extension into a three-dimensional space has been recommended. The stability of the nonlinear systems has been checked according to their Nyquist using their describing function and the step response of the true nonlinear system, which has shown the stability of the nonlinear systems. Since the stable closed loop systems using a HIGS notch filter has been obtained, in Chapter 4, the design will be implemented and tested by time-domain simulations. These studies possibly shows the benefits for performance and the usefulness of the describing function based design in providing estimates for nonlinear control performance in time-domain.
4 Nonlinear performance analysis - a discrete-time perspective

4.1 Introduction

In Chapter 2 HIGS has been introduced and three HIGS notch filter designs have been developed. Subsequently, in Chapter 3 the designs have been analysed in the frequency domain specially from the perspective of quasi-linear frequency response functions. As a result, design A has been appointed to be the most beneficial in terms of its approximated frequency characteristics. Furthermore, a conclusion on the stability of the true nonlinear closed loop system has been made. In this chapter, the digital aspects of all the components of this nonlinear closed loop system are discussed. Thereafter, the discrete nonlinear closed loop systems are analysed in frequency and time-domain via simulations.

In Section 4.2 the conversion of HIGS into discrete-time and its implementation is discussed as well as the discretization of the linear components in frequency-domain. In addition, the discrete nonlinear open and closed loop system are obtained and analysed in frequency domain. Thereafter, in Section 4.3 a simulation study is conducted, starting with the effect of exogenous input signals on the nonlinear closed loop in time-domain. Acquainted with the behaviour of the nonlinear closed loop and the influence of its inputs, a simulated experiment is conducted. This simulated experiment shows possible benefits of the nonlinear designs regarding time-domain performance. Section 4.4 finalises this chapter with a summary.

4.2 Digital aspects

Hitherto, the continuous nonlinear closed loop systems have been analysed by frequency-domain and time-domain evaluations. However, for implementation purposes the systems need to be converted (and evaluated) into discrete-time. Therefore, in Section 4.2.1 the time-domain HIGS is converted into a discrete-time element and its implementation is discussed. Thereafter, the continuous linear controllers are discretized in frequency domain in Section 4.2.2. Finally, the discrete open and closed loop are discussed and analysed in Sections 4.2.3 and 4.2.4 respectively, resulting in a discrete HIGS-based closed loop system. The philosophy used in this thesis is to conduct the control design with continuous system representations while evaluating stability and performance taking into account discrete-time aspects. Note that this does not necessarily leads to optimal design choices when compared to full discrete control design.
4.2.1 HIGS filter design

Since the frequency domain representation of the HIGS is a describing function, which is only an approximation, it cannot be converted into a discrete filter in a common manner. Therefore, the continuous-time HIGS is converted into a discrete-time HIGS first. The implementation of this discrete-time HIGS is discussed and a frequency domain approximation in the sense of a discrete-time describing function representation is obtained. The continuous-time HIGS has been given in Equation (2.1). The HIGS contains integrator and gain dynamics. Therefore, the integrator is converted into discrete-time using the Backward Euler method. This method is defined by the difference equation

$$y(k) = y(k-1) + T_s u(k),$$

(4.1)

where $y(k)$ is the current output, $y(k-1)$ the previous output, and $u(k)$ the current input at time instance $k$, which is assumed constant over the sampling period $T_s$. Taking the Z-transform, Equation (4.1) converts into

$$\frac{y(z)}{u(z)} = \frac{T_s}{1 - z^{-1}},$$

(4.2)

for a discrete integrator. This conversion results in the discrete-time representation of HIGS as

$$\mathcal{H}_d := \begin{cases} u_{\mathcal{F}_1}(k) = \omega_h T_s e(k) + u(k-1), & \text{when } (e, u, \dot{e}) \in \mathcal{F}_{1,d}, \\ u_{\mathcal{F}_2}(k) = k_h e(k), & \text{when } (e, u, \dot{e}) \in \mathcal{F}_{2,d}. \end{cases}$$

(4.3)

The sector constraints are converted into discrete-time as well. Using Equations (2.2) and (4.3), the following discrete sector constraints are obtained

$$\mathcal{F}_{1,d} := \left\{ (e(k), u(k), u_{\mathcal{F}_1}(k)) \in \mathbb{R}^3 \mid e(k)u(k) \geq \frac{1}{k_h^2} u(k)^2 \wedge (e(k), u(k), u_{\mathcal{F}_1}(k)) \notin \mathcal{F}_{2,d} \right\},$$

$$\mathcal{F}_{2,d} := \left\{ (e(k), u(k), u_{\mathcal{F}_1}(k)) \in \mathbb{R}^3 \mid u(k) = k_h e(k) \wedge u_{\mathcal{F}_1}(k)^2 > k_h u_{\mathcal{F}_1}(k)e(k) \right\}.$$  

(4.4)

Figure 4.1 shows the implementation of this discrete HIGS filter, which is adopted from [9]. It is observed that the HIGS filter generates both gain and integrator mode dynamics and switches in between according to sector constraints formulated in time-domain. Remark that a zero crossing detection is present in Figure 4.1. This zero crossing detects when the input crosses zero to ensure that the input and output are equal of sign by resetting the buffer of the integrator. This detection is necessary as it is highly unlikely that the zero crossing is observed due to sampling, i.e., the zero crossing generally occurs somewhere during the sample.
It is desired to compare the continuous-time and the discrete-time HIGS in frequency domain. The continuous-time HIGS is approximated by the describing function $\Psi_h$ in frequency domain. For the discrete-time HIGS an approximation in frequency domain is derived as well. This approximation is obtained by applying a sine wave with varying frequencies from 0.1 to 1024 Hz as input $e(k)$. Thereafter, the output $u(k)$ is measured and a Discrete Fourier Transform (DFT) analysis is conducted on the signals. The DFT of the corresponding frequencies for the input and the output are divided by each other within the considered frequency range to obtain a mapping of a discrete describing function of the HIGS referred to as $\Psi_{h,d}$. Figure 4.2 shows the comparison between the continuous describing function of the HIGS from Equation (2.11) and the discrete mapping of the HIGS for a sample frequency of 2048 Hz. The used Backward Euler method implies that the phase goes back to zero at the Nyquist frequency. This can be observed in the figure, as well as a proper match in magnitude and phase between the continuous and discrete representation.
Figure 4.2: Bode representation of the continuous describing HIGS filter in comparison to
the discrete HIGS filter with \(k_h = 1\), \(\omega_h = 20\pi\) rad/s, and sampling frequency \(F_s = 2048\) Hz.

4.2.2 Linear components

Since a discrete frequency domain mapping of the HIGS has been obtained, the linear control filters are discretized in frequency domain as well. The continuous linear controller of the nonlinear closed loop system is given by \(C_{\text{nom}}\). This controller consists of the continuous control filters \(C_{\text{pid}}, C_{\text{add}}, C_{\text{n}},\) and \(C_{\text{solp}}\). All the components of the linear controller are merged and discretized to potentially limit the phase loss. The complete controller is discretized using the Backward Euler method for a sampling frequency of 2048 Hz, resulting in the obtained discrete linear controller given by

\[
C_{\text{nom,d}}(z) = C_{\text{pid,d}}(z) C_{\text{add,d}}(z) C_{\text{n,d}}(z) C_{\text{solp,d}}(z).
\] (4.5)

The linear control filters of the linear closed loop system are discretized as well to show the potential advantages of the nonlinear closed loop system. The control filters are discretized in the same manner and it results in discrete linear controller \(C_{\text{lin,d}}(z) = C_{\text{pid,d}}(z) C_{\text{n,d}}(z) C_{\text{solp,d}}(z)\).

4.2.3 Open loop system

All components of the HIGS-based controller are discretized, to obtain the discrete nonlinear open loop system only the discrete plant is required. The discrete representation of the plant \(P_d\) is obtained by using the zero-order hold method on the parametric plant of (3.2) without delay correction. The discrete nonlinear open loop system is now defined as \(P_d C_{\text{nom,d}} \Psi_{h,d}\)
and the parameters can be acquired from Table 3.4. This open loop system has been obtained in the same manner as for the discrete mapping of the HIGS. Meaning, the output of the nonlinear open loop system is divided by the input (a sine wave) of the system for the corresponding frequencies using the DFT. Figure 4.3 shows this obtained open loop system in comparison to the continuous nonlinear open loop system $PC_{nom}\Psi_h$ for the normal HIGS notch filter. It is observed that the discrete representation of the nonlinear open loop system is a fairly accurate match to the continuous nonlinear open loop system.

![Figure 4.3: Bode representation of the comparison between the continuous and discrete nonlinear open loop system conducting the normal HIGS notch filter.](image)

4.2.4 Closed loop system

To show the possible benefits of low-frequency disturbance suppression as has been acquired for the continuous nonlinear closed loop system in Section 3.3.1, it is desired to obtain and analyse the discrete nonlinear closed loop system in frequency domain as well. It seems that conducting the describing function analysis results in accurate matches for the continuous and discrete representations of the nonlinear open loop system. For closed loop systems, however, the controller input becomes a function of the controller output. Since HIGS introduces higher order harmonics, this may be of influence on the closed loop system performance. It is examined whether these higher order harmonics influence the frequency domain description through first-order describing functions. Therefore, the nonlinear closed loop sensitivity functions are obtained by dividing the DFT signals of the error $e$ with the input $r = \sin(2\pi ft)$ for their corresponding frequencies $f$, provided that the input signals $d$ and $\eta$ are zero. The obtained discrete nonlinear sensitivity functions $S_{h,d}$ are shown in comparison to the continuous nonlinear sensitivities $S_h$ of (3.7) in Figure 4.4 for the controller designs of the normal (4.4a), the inverse (4.4b), and the skew notch filter (4.4c). This figure also shows the continuous linear sensitivities $S$ from (3.6) and the obtained discrete linear sensitivity
functions $S_d$ using the schematic closed loop representation of Figure 3.1. In this manner, the possible advantages of the discrete HIGS-based notch filters can be seen in comparison to the discrete linear notch filters in frequency domain.

Figure 4.4: Sensitivity functions using the HIGS-based notch filter design A for (a) a normal notch filter, (b) an inverse notch filter and (c) a skew notch filter in comparison with their linear counterparts in continuous and discrete representations.

The figures show that the discrete linear sensitivities are an exact match for the continuous linear sensitivity functions. Furthermore, the discrete nonlinear sensitivity with the use of a HIGS skew notch filter is an accurate match with its quasi-linear continuous sensitivity function. However, the continuous quasi-linear sensitivities show differences with the obtained nonlinear sensitivities for the systems containing a normal and inverse HIGS notch filter. It is presumed that this variation could be caused by the higher harmonics present in the controller input. To investigate this assumption, a sine wave of 2 Hz is applied as setpoint into the system with a HIGS inverse notch filter. For a linear closed loop system, it is assumed that the error of the system contains a sine wave of 2 Hz with a reduced gain and a difference in phase. However, Figure 4.5 shows that the error of the nonlinear system mainly contains higher harmonics. This dominant content results that the HIGS switches according to these harmonics instead of the desired switching according to a sine wave of 2 Hz. In this
manner, the HIGS is not able to reach its full potential regarding its integrator characteristics. As a result the expected gain increase according to the quasi-linear sensitivity can not be reached, which is in correspondence with the obtained discrete nonlinear sensitivity for the HIGS inverse notch filter.

Figure 4.5: Error signals of the linear and nonlinear closed loop systems and the output of the HIGS filter with an input of \( r = \sin(4\pi t) \).

In the case of a nonlinear system with a normal HIGS notch filter, the harmonics seem to be less present according its sensitivity. The difference between this system and the inverse notch system is that an inverse notch filter excites local frequency in constrast to a normal notch filter which locally reduces frequency content. Therefore, more harmonics could have been filtered. For the nonlinear system with the HIGS skew notch filter, the high-frequency content is filtered additionally due to its low pass characteristics. A final note could be that for the describing function it has been shown in Figure 2.4b that with the use of lower values of \( \omega_h \) more higher order harmonics are present. Since the HIGS inverse notch filter is obtained with \( \omega_h = 0.5 \cdot 2\pi \) this could add to the distinctly present higher order harmonics.

For the nonlinear continuous closed loop systems, a conclusion on the stability has been made through their quasi-linear Nyquist and step responses. In discrete-time, this conclusion can be made as well. However, it is desired to prove the discrete-time stability. To assure the stability on all the discrete nonlinear systems, a discrete-time three-dimensional Lyapunov analysis should be conducted which is a proposition for further research.

### 4.3 Simulation study

Thus far, the discrete nonlinear closed loop systems have been mostly analyzed in frequency domain. In this section, the HIGS-based control designs will be analyzed through time-domain simulations. The test plan for these time-domain simulations is explained in Section 4.3.1. The analysis starts with the transient performance in Section 4.3.2. Thereafter, the steady-state behaviour depending on the input signals is analyzed in Sections 4.3.3 and 4.3.4. As a result, a simulation is conducted with multiple inputs onto the nonlinear closed
loop system in Section 4.3.5.

4.3.1 Test plan

For the time-domain simulations the linear and nonlinear closed loop systems of Figures 3.1 and 3.8, respectively, are implemented in discrete-time. The transient behaviour is tested by applying a step response as a setpoint. Thereafter, the steady-state error is analysed by a third-order setpoint and a stochastic signal as input of the closed loop systems. Since the discrete nonlinear system with the HIGS skew notch filter shows the best disturbance suppression in frequency domain, this nonlinear controller is chosen for the analysis of these input signals. Finally, simulations are conducted using several input signals to mimic an experiment for all the HIGS-based notch filter designs. In this manner, conclusions on the time-domain HIGS-based controller can be made in addition to the conclusions in frequency domain.

4.3.2 Step response

A step response is applied to the closed loop systems to show possible improvements in the nonlinear transient performance in terms of overshoot and settling time. The notch filter of the linear closed loop system has been replaced by a HIGS notch filter to show the influence of the HIGS on the transient behaviour. It is expected that the tuning of $\omega_h$ can results in increased phase margins, which in linear terms would result in a decrease of overshoot. Figure 4.6a shows the step response for a closed loop system with a normal HIGS notch filter. The step response shows that for $\omega_h = 1000 \cdot 2\pi$ the nonlinear transient behaviour is equal to the linear behaviour. This is explained by the fact that for a high value of $\omega_h$, the HIGS is mainly active in gain mode. Furthermore, it is observed that for lower values of $\omega_h$ the overshoot decreases, which is in correspondence with the obtained increase in phase margin as shown in the Nyquist-like plot in Figure 4.6b.
Figure 4.6: (a) shows the output of the closed loop system with a normal HIGS notch filter for varying values of $\omega_h$ and (b) shows the Nyquist-like plot for the nonlinear system also for different values of $\omega_h$.

It has to be noted that the settling time increases for the decrease in overshoot, however, the decrease in overshoot is more significant. For example, the overshoot for $\omega_h = 20 \cdot 2\pi$ rad/s has decreased with 51.60% while the settling time has increased with 7.26% in comparison to the linear closed loop system. The nonlinear transient behaviour for the closed loop system with a HIGS inverse notch filter is shown in Figure 4.7a. It can be seen that for the HIGS inverse notch filter a resonance is present in its transient response, which deteriorates the desired decrease in overshoot. This resonance is the uncompensated resonance of the plant, which rotates towards the minus one point due to reduced phase lag as shown in the Nyquist of Figure 4.7b. From this observation, it can be concluded that the quasi-linear Nyquist is an appropriate prediction of the nonlinear behaviour. Therefore, the decrease in $\omega_h$ should be followed by a quasi-linear loop shaping procedure to obtain the desired decrease in overshoot. The closed loop system with the HIGS skew notch filter shows improvement in a reduced overshoot as shown in Figure 4.8a, however, the resonance of the plant is still present here as well as can be seen in the Nyquist of Figure 4.8b. The HIGS skew notch filter seems more robust, but the resonance frequency should still be taken into consideration in a quasi-linear loop shaping procedure while obtaining improved transient behaviour.
Figure 4.7: (a) shows the output of the step response for an inverse HIGS notch filter, (b) shows the corresponding quasi-linear Nyquist plot, both for different values of $\omega_h$.

Figure 4.8: (a) shows the output of the step response for a skew HIGS notch filter and (b) shows the corresponding quasi-linear Nyquist plot with different values for $\omega_h$.

4.3.3 Third-order setpoint

The first analysed input signal for the steady-state behaviour is a third-order setpoint with a velocity of 10 rad/s, an acceleration of 200 rad/s$^2$, and a jerk of 30000 rad/s$^3$. The other signals $d$ and $\eta$ are taken to be equal to zero to analyse the effect of this setpoint on the (non)linear closed loop system. In general, feedforward is used to compensate (to a large extent) for the effect of third-order setpoint on the closed loop system. However, in focusing on the properties of the HIGS feedback controller, feedforward is omitted here. To presume a conclusion based on the frequency domain analysis in Section 4.2.4, the third-order setpoint is examined in frequency domain by obtaining the cPSD of the signal. The normalized cPSD of the input signal is shown in Figure 4.9. It is observed that the setpoint mainly has content at low frequencies. For the HIGS-based skew notch filter it has been shown in Figure 4.4
that the nonlinear closed loop sensitivity functions have a better low-frequency suppression in comparison to their linear counterparts. Therefore, it is expected that the nonlinear closed loop system tracks the third-order setpoint better.

Figure 4.9: Normalized cumulative power spectral density of the third-order setpoint.

Figure 4.10a shows the error of the (non)linear closed loop system using the third-order setpoint. It is observed that the error for the nonlinear system has decreased according to expectation. However, higher harmonics from the HIGS seems present. To better analyse the suppressed error, its cPSD is shown in Figure 4.10b. It can be observed that the low frequency contribution of the third-order setpoint has been suppressed better for the nonlinear closed loop system. Furthermore, it is observed that higher harmonics are present for the nonlinear system from approximately 20 up to 50 Hz. However, these contributions are significantly suppressed as was expected from the accurate match in sensitivities in Figure 4.4c. To calculate the decrease in error using a HIGS-based skew notch filter, the root mean square (RMS) of the error is obtained. The RMS value of the error for the nonlinear closed loop system has decreased with **43.87%** in comparison to the error of the linear closed loop system.
Figure 4.10: (a) shows the (non)linear error with a third-order setpoint as input, (b) depicts the cPSD for the (non)linear error conducting a (HIGS) skew notch filter.

4.3.4 Stochastic signal

A white noise signal with zero mean, finite variance, and a power of 0.005 is applied at the input $\eta$ to investigate the influence of the higher harmonics on the nonlinear closed loop system. The noise is a direct input of the HIGS in contrast to $d$ which gets filtered by the plant first as shown in Figure 3.8. Therefore, higher harmonics as a result of the nonlinearity of the HIGS are expected. Figure 4.11a shows the error for the linear and nonlinear closed loop systems. It seems that the error of the nonlinear system exceeds the linear systems error while both contain high frequency noise. The energy of the errors are shown in Figure 4.11b which indeed shows the increase in the error signal for the nonlinear system in comparison to the linear system. Furthermore, a significant disturbance around 6 Hz is observed which is unexpected since neither the discrete nonlinear sensitivity nor the white noise signal have a significant contribution here. It is expected that the white noise signal has amplified a harmonics of the HIGS at this frequency. Following these observations, it can be concluded that noise can deteriorate the performance improvements of a HIGS-based closed loop system (with 40.93% in this case). This conclusion should be taken into account regarding measurement noise for an experiment.
4.3.5 Simulation

To mimic an experiment, a simulation is conducted using the previously specified third-order setpoint and white noise (representing measurement noise) signal as inputs for the linear and nonlinear closed loop systems as presented in Figures 3.1 and 3.8 respectively. For the analysis of the input signals, only the nonlinear closed loop system containing a HIGS skew notch filter has been used. Here, simulations for all the HIGS-based notch filters in closed loop are conducted. In this manner, the possible performance improvements can be compared to the frequency domain advantages. According to the discrete nonlinear sensitivities of Figure 4.4, the system with the normal HIGS notch filter should have improvement in performance. The inverse HIGS notch, however, probably shows no improvements at low frequencies while deteriorating at high frequency due to the presence higher order harmonics. For the HIGS skew notch, it has been shown that the performance improves in agreement with its frequency domain improvements. However, now simulations are conducted with two input signals. The errors resulting these simulations for all the HIGS-based notch filters closed loop systems are depicted in Figure 4.12.

For the error using the normal HIGS notch filter in Figure 4.12a, it seems that the linear and nonlinear error are similar. However, for the inverse HIGS notch filter design a deteriorating of the error is seen. Furthermore, the skew HIGS notch filter of Figure 4.12c shows significant improvement in the nonlinear error. To analyse these error signal properly, the cPSDs of these signal are given in Figure 4.13.

Figure 4.11: (a) shows the linear and HIGS error with a white noise as input, (b) depicts the cPSD for these errors.
Figure 4.12: Errors using the HIGS-based notch filter design A for (a) a normal notch filter, (b) an inverse notch filter and (c) a skew notch filter in comparison with their linear counterparts.

Figure 4.13a shows that the error of the nonlinear system using a normal HIGS-based notch filter has improved in comparison to the linear error. Mainly low-frequency suppression is the contribution to this decrease in error. At approximately 50 Hz an increase in the nonlinear error occurs, this observation concatenates with the peak of the discrete nonlinear sensitivity of Figure 4.4a. Therefore, higher harmonics could be amplified here which is observed in this case. For the nonlinear system using an inverse HIGS notch filter, the cPSD of the error is shown in Figure 4.13b. As seen in Figure 4.12b the nonlinear error exceeds the linear error which was also expected according to its discrete nonlinear sensitivity. The nonlinear sensitivity as well as the cPSD of the nonlinear error show that inbetween 3 and 20 Hz the inverse HIGS notch filter shows an improvement in comparison to the linear error, and after 20 Hz the nonlinear error exceeds the linear error. It is expected that the increase in the nonlinear error is due to the increase of harmonics for a small $\omega_h$ value as well as the local amplification from the inverse notch filter characteristics. Figure 4.13c shows the cPSD of the linear and nonlinear error concerning the nonlinear system using a HIGS skew notch filter. It is observed that the nonlinear error has significantly decreased in comparison to the linear error. This decrease is obtained by the low-frequency suppression as shown in the
nonlinear sensitivity function of Figure 4.4c. Furthermore, nearly no contributions at high frequencies are detected. As discussed in Section 4.2.4 for the frequency domain analysis, the low pass characteristics of the skew notch filter reduces the contributions of the higher order harmonics which is also observed here.

To compare all the linear and nonlinear systems, the RMS values of the errors are calculated and given in Table 4.1. It is noted that for the linear errors, the lowest and highest error correspond with the highest to the lowest bandwidth, respectively, which is expected for a linear system. However, this does not hold for the nonlinear systems due to the presence of higher order harmonics as discussed previously. Although, the normal HIGS notch systems error has decreased with 21.40%. For the inverse HIGS notch filter, the error has increased with 13.38% in comparison to the linear system. The HIGS skew notch filter has obtained the highest decrease in error of 35.64%. Therefore, the HIGS-based normal and skew notch filter implemented in a closed loop system result in low-frequency disturbance improvements and increased setpoint tracking.
Table 4.1: RMS values for simulated experiment and the improvement for the nonlinear error.

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>Inverse</th>
<th>Skew</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{RMS}(e) \cdot 10^{-2}$</td>
<td>3.56</td>
<td>6.30</td>
<td>3.90</td>
</tr>
<tr>
<td>$\text{RMS}(e_h) \cdot 10^{-2}$</td>
<td>2.80</td>
<td>7.14</td>
<td>2.51</td>
</tr>
<tr>
<td>Improvement</td>
<td>21.40%</td>
<td>-13.38%</td>
<td>35.64%</td>
</tr>
</tbody>
</table>

4.3.6 Discussion

In this section, the true behaviour of the nonlinear systems have been shown. It has been noted that the discrete nonlinear sensitivies are an accurate prediction of their discrete-time behaviour. Furthermore, if the higher harmonics resulting from the switching of the HIGS are filtered correctly the continuous quasi-linear sensitivity is an accurate prediction as well. The HIGS-based skew notch filter suppresses its harmonics correctly, therefore, this nonlinear system can be tuned using its describing function according to desired specifications. During the tuning, it has to be taken into account that $\omega_h$ should be lower than the notch frequency to reduce the generated phase lag of the skew notch filter. However, $\omega_h$ should not be chosen too low because it will generate phase lead instead of reduced phase lag. In addition, a low value of $\omega_h$ could increase the presence of higher order harmonics. These design specifications also apply to the use of the HIGS-based normal notch filter. However, no additional high-frequency suppression is present which increases the influence of higher order harmonics. Therefore, it is necessary to acquire a discrete sensitivity function to observe whether the nonlinear system has improvements in low-frequency suppression. Since an inverse notch filter is desired at low frequencies to obtain local low-frequency suppression, an HIGS-based inverse notch filter is applied at low frequencies as well. However, in that case a significantly low value of $\omega_h$ is needed to reduce the phase lag of the inverse notch filter, which is undesired due to the expected increase in higher order harmonics. The increased amount of higher order harmonics in combination with the inverse notch filter characteristic to amplify certain frequencies, makes that the use of an HIGS-based inverse notch filter in this form is not likely to gain any performance. Therefore, the HIGS-based inverse notch filter as presented, is not recommended to be implemented.

4.4 Summary

In this chapter, the HIGS has been converted into discrete-time and its implementation has been shown. For a comparison in frequency domain for the continuous-time and discrete-time HIGS, the describing function and a discrete mapping of a describing function are conducted. Furthermore, the discretization of the linear control filters have been discussed in frequency domain. Thereafter, the nonlinear open and closed loop systems have been analysed and discrete nonlinear sensitivities are obtained. The obtained sensitivities differ from the continuous quasi-linear sensitivities due to higher order harmonics present in the controller input. The HIGS switches according to these harmonics instead of the input signal of the closed loop system, which has resulted in the inability to reach the expected increase in
gain. Hereafter, the transient behaviour of the nonlinear systems have been analyzed using a step response. It has been shown that overshoot can be reduced for different values of $\omega_h$. Time-domain simulations for the HIGS-based skew notch filter have been performed with varying input signals to investigate their behaviour. It has been shown that an improved setpoint tracking can be achieved. Although, a stochastic signal distorts the tracking at high frequencies due to the increase of higher harmonics on account of the nonlinearity of the HIGS. At last, an experiment is simulated by applying a third-order setpoint and a white noise. The obtained nonlinear error signals behaved in correspondence to their obtained discrete nonlinear sensitivity functions. Therefore, these functions are an important indication on the nonlinear performance behaviour. In the case of the HIGS-based skew notch filter, the continuous describing function is also a proper approximation. Using this describing function, quasi-linear loop shaping can be performed with improvement in bandwidth and performance taken some design specifications into account. The errors of all the linear and nonlinear systems are compared by obtaining their RMS values. It has been noted that the HIGS-based normal and skew notch filter have decreased in error with 21.40% and 35.64% respectively, which has resulted in improved low-frequency suppression and setpoint tracking. The inverse HIGS notch filter has increased 13.38% in error, therefore, it is not desired to implement this filter.
5 Conclusions and recommendations

5.1 Conclusions

In this thesis, the objective was to obtain increased bandwidths, improved low-frequency suppression and a better performance for HIGS-based notch filters while meeting the robustness margins. The HIGS-based notch filters that have been investigated are a normal notch filter to address mid frequency resonances, an inverse notch filter to achieve additional low-frequency suppression, and a skew notch filter for extra suppression of high frequencies.

To achieve this objective, the HIGS has been introduced including an analysis in time and frequency domain. By performing a describing function analysis, the frequency domain approximation of HIGS has shown a reduction in phase lag of 51.9 degrees. To make use of this advantage, HIGS-based notch filters have been designed using the describing function. In this manner, three designs have been developed namely, design A, design B, and design C. Design A has been based on the low pass characteristics of the describing function of HIGS for \( k_h = 1 \). This design reduced the local phase lag, however, it also obtained infeasible phase lead at high frequencies. Therefore, design B has been developed using the CgLp element, which has resulted in an increase in phase. However, at higher frequencies the gain tended towards a -1 slope with corresponding phase lag. This behaviour has resulted in the development of design C for HIGS-based notch filters. Design C obtained the desired local phase increase by using two describing function for a HIGS integrator and differentiator, however, this has introduced unwanted complexity as well as the concern regarding the approximation of the describing functions.

These designs have been further examined in a quasi-linear loop shaping procedure in comparison to their linear counterparts. The loop shaping procedure started with the creation of a parametric model for the PATO system. By conducting loop shaping, linear controllers have been obtained regarding a normal notch, an inverse, and a skew notch filter. Thereafter, quasi-linear loop shaping has been conducted for the designed HIGS-based notch filters A, B, and C with the above-mentioned modification in notch filters. The HIGS-based controller of design A has been the most benificial by obtaining the maximum gain and bandwidth while meeting the robustness constraints. Due to the fact that this was an approximation, improved performance has not been guaranteed. To be able to achieve possible improved performance, the stability of the nonlinear has been checked by LMI conditions. These conditions have been obtained by piecewise quadratic Lyapunov functions and resulted in an infeasible solution due to conservatism in the LMIs. A conclusion on the nonlinear stability is made according to its describing function using the Nyquist criterion and the step response of the nonlinear system.

Since the HIGS-based notch filter of design A has shown improvements in bandwidth, better low-frequency suppression, and a conclusion on its stability has been made, the design has been implemented to obtain possible performance improvements. The implementation started with the discretization of the nonlinear closed loop systems in frequency domain.
First, an approximation of the discrete-time HIGS in frequency domain has been obtained. Thereafter, all linear control filters and the parametric model of the plant have been discretized in frequency domain. In this manner, the discrete nonlinear open loop and closed loop are analysed. This analysis has resulted in discrete nonlinear sensitivities, which have been compared to their corresponding describing functions. The discrete sensitivities of the closed loop systems of the normal and inverse HIGS notch filters are dissimilar as compared to their describing functions. It has been shown that higher harmonics are present in the controller input, resulting the HIGS to switch according to these dominant harmonics. The nonlinear system conducting a HIGS skew notch filter has shown an accurate match with its describing function due to its additional high-frequency low pass characteristics. To show the improvements in transient behaviour, the notch filter in the linear closed loop system has been replaced by the HIGS notch filter. This substitution has resulted in a decrease of overshoot for a decrease in the value of $\omega_h$. In steady-state simulations, the input signals have been investigated for the nonlinear system of the HIGS-based skew notch filter, which has shown improved setpoint tracking for a third-order setpoint. The influence of higher harmonics have been analysed by applying a stochastic signal which has distorted the tracking performance as expected. These input signals have been combined to mimic an experiment. In this simulation, the nonlinear error signals behaved in correspondence to their obtained discrete nonlinear sensitivity functions which shows their importance. In the case of the HIGS-based skew notch filter, the continuous quasi-linear sensitivity function is also a good approximation. Quasi-linear loop shaping can be performed for this nonlinear system with improvement in bandwidth and performance while some design specifications are taken into account. Furthermore, the obtained nonlinear errors for the HIGS-based normal and skew notch filter have decreased with 21.40% and 35.64% respectively while the inverse HIGS notch filter has increased with 13.38% in comparison to the linear notch filter configurations. The application of the HIGS-based normal and skew notch filter in the closed loop system have resulted in the desired improved low-frequency suppression and setpoint tracking in correspondence with their increased bandwidths.

5.2 Recommendations

A first recommendation for this research would be to conduct an experiment using the implementation of the HIGS-based controller using design A. In this manner, the behaviour of the discrete-time HIGS can be examined further where the improved low-frequency suppression and setpoint tracking could be observed in real time.

Another recommendation would be to extent the stability analysis in continuous-time conducting LMIs. This extension should be in a three-dimensional space to reduce the conservatism of the LMIs. Furthermore, a stability analysis in discrete-time is desired conducting three-dimensional LMIs as well.

Furthermore, it would be recommended to tune the time domain performance along with maximizing the bandwidth in frequency domain in a data-driven design procedure as conducted in [18]. In this manner, the improvement in performance using a HIGS notch filter can be exploited further. In addition, this procedure allows to eliminate the human factor in loop
shaping, minimizing the effect of tuning on the comparison between a linear and a nonlinear system.

For more complex systems, typically multiple resonances are present. In that case, the use of multiple HIGS-based notch filters can be desired. However, this adds up to the complexity of the controller. In previous work, multiple HIGS filters have been implemented for a PID and SOLP filter in series. Therefore, further research is desired regarding multiple HIGS-based notch filters.
References


Declaration concerning the TU/e Code of Scientific Conduct for the Master's thesis

I have read the TU/e Code of Scientific Conduct¹.

I hereby declare that my Master's thesis has been carried out in accordance with the rules of the TU/e Code of Scientific Conduct.

Date
20-07-2020

Name
K.S. Hebers

ID-number
0809714

Signature

Submit the signed declaration to the student administration of your department.

¹ See: http://www.tue.nl/en/university/about-the-university/integrity/scientific-integrity/
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January 15 2016