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# A polyhedral approach to the delivery man problem

C.A. van Eijl\*

## Abstract

We propose a mixed integer programming formulation for the delivery man problem and derive additional classes of valid inequalities. Computational results are presented for instances of the delivery man problem with time windows. In particular, the quality of the lower bounds obtained from the linear programming relaxation and the effectiveness of the additional inequalities in improving these bounds are studied.

## 1 Introduction

In this paper we study the delivery man problem (DMP) from a polyhedral point of view. This problem is a variant of the well-known traveling salesman problem (TSP) in which the objective is to find a tour starting from a given depot that minimizes the sum of the waiting times of the customers. The DMP can also be interpreted as a single-machine scheduling problem with sequence-dependent processing times in which the total flow time of the jobs has to be minimized.

Polyhedral methods have been proven to be very successful for the TSP and many of its extensions. Most of these extensions deal with extra conditions on the graph structure, e.g. precedence constraints, such that the problem can still be formulated as a 0–1 model. However, we will formulate the DMP as a mixed integer programming problem by introducing time variables. This formulation can easily be adapted to the problem with time windows, i.e., when each customer has to be visited within a specified interval. Only a few papers deal with this kind of extension. Ascheuer [1] has developed a branch-and-cut code for the TSP with time windows. Escudero and Sciomachen [3] and Escudero, Guignard, and Malik [4] study the sequential ordering problem with time windows, where the sequential ordering problem is the problem of finding a minimum weight Hamiltonian path subject to precedence constraints. For this problem Maffioli and Sciomachen [8] propose a formulation that is similar to our formulation of the DMP.

In order to strengthen the linear programming relaxation we will derive additional classes of valid inequalities. These can be considered as a generalization of the inequalities derived by Queyranne [11] for the single-machine scheduling problem with sequence-independent processing times. The latter were proven to form a complete description of the polyhedron associated with this problem. For a comprehensive survey on polyhedral results for machine scheduling problems we refer to [12].

Optimization algorithms for the DMP are provided by Simchi-Levi and Berman [13], Lucena [7], and Fischetti, Laporte and Martello [5]. The first describe a branch-and-bound

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scheme based on a shortest spanning tree relaxation. Lucena proposes an enumerative algorithm based on a nonlinear integer programming formulation in which lower bounds are obtained from a Lagrangean relaxation. Computational results are reported for problems with up to 30 nodes. Fischetti et al. provide a branch-and-bound algorithm based on an integer programming formulation. The paper contains results on the cumulative matroid problem which are used to derive lower bounds. Problems with up to 60 nodes are solved to optimality.

The outline of this paper is as follows. In Section 2 we formally introduce the problem and present a mixed integer programming formulation. In Section 3 additional classes of valid inequalities are derived. A cutting plane procedure is described in Section 4. In Section 5 we report computational results.

## 2 Problem formulation

The DMP is formally stated as follows. We consider the complete directed graph  $G = (V \cup \{0\}, A)$ , where  $V = \{1, \dots, n\}$ . With each arc  $(i, j)$  we associate a travel time  $p_{ij} \in \mathbb{Z}_0^+$ . It is assumed that visiting time is included in the travel time, hence, the arrival time at node  $i$  equals the departure time at node  $i$ . Node 0 is the depot, i.e., every tour starts and finishes in node 0. Furthermore, we assume that every tour starts at time 0. Then the waiting time of the customer located at node  $i$  is equal to the departure time at node  $i$ . Now the problem is to find a tour that minimizes the sum of the departure times at the nodes.

The DMP can be modeled using the following two types of variables. For every arc  $(i, j)$  there is a binary variable  $x_{ij}$  that indicates whether the arc  $(i, j)$  is included in the tour or not, and a time variable  $t_{ij}$  defined as follows:

$$t_{ij} = \begin{cases} \text{departure time at node } i, & \text{if } x_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since we assumed that every tour starts at time 0, the variables  $t_{0j}$  can be omitted from the model. This gives rise to the following formulation, in which  $C$  denotes a large constant:

$$\text{(DMP) minimize } \sum_{i=1}^n \sum_{j=0, j \neq i}^n t_{ij}$$

$$\text{s.t. } \sum_{j=0, j \neq i}^n x_{ij} = 1, \quad i = 0, \dots, n, \quad (1)$$

$$\sum_{i=0, i \neq j}^n x_{ij} = 1, \quad j = 0, \dots, n, \quad (2)$$

$$\sum_{i=1, i \neq j}^n t_{ij} + \sum_{i=0, i \neq j}^n p_{ij} x_{ij} = \sum_{k=0, k \neq j}^n t_{jk}, \quad j = 1, \dots, n, \quad (3)$$

$$0 \leq t_{ij} \leq C x_{ij}, \quad i, j = 0, \dots, n, \quad i \neq j, \quad i \neq 0, \quad (4)$$

$$x_{ij} \in \{0, 1\}, \quad i, j = 0, \dots, n, \quad i \neq j. \quad (5)$$

Constraints (1) and (2) ensure that every node, including the depot, is visited exactly once. Constraints (3) guarantee that if  $x_{ij} = 1$ , then the departure time at node  $j$  equals the departure time at node  $i$  plus the travel time  $p_{ij}$ . It is readily checked that these constraints also prevent subtours. If  $C$  is an upper bound on the departure time at node  $i$ , e.g.

$C = n \cdot \max_{i,j} p_{ij}$ , then (4) is valid when  $x_{ij} = 1$ . Furthermore, these constraints force that  $t_{ij} = 0$  if  $x_{ij} = 0$ .

This model can easily be adapted to the problem with time windows (DMPTW), i.e., when each node  $i$  has to be visited within a specified interval  $[e_i, l_i]$ . The delivery man may arrive at node  $i$  before  $e_i$ , but cannot deliver before the opening of the time window. In this case, the departure time at node  $i$  is strictly larger than the arrival time at node  $i$ . To model this, equalities (3) must be replaced by

$$\sum_{i=1, i \neq j}^n t_{ij} + \sum_{i=0, i \neq j}^n p_{ij} x_{ij} \leq \sum_{k=0, k \neq j}^n t_{jk}, \quad j = 1, \dots, n. \quad (6)$$

Furthermore, substituting

$$e_i x_{ij} \leq t_{ij} \leq l_i x_{ij}, \quad i, j = 0, \dots, n, i \neq j, i \neq 0. \quad (7)$$

for (4) yields that  $t_{ij} = 0$  if  $x_{ij} = 0$  and that the departure time at node  $i$  is in the time window  $[e_i, l_i]$  otherwise. The sum of the waiting times is now  $\sum_i (\sum_j t_{ij} - e_i)$ .

It is evident that, apart from the objective function, the formulation for the DMPTW models the TSP with time windows as well. Compared to the well-known model with variables  $x_{ij}$  and  $t_i$  (cf. [2]), an important feature of the formulation presented here is that it does not involve big- $M$  coefficients in the constraints.

As mentioned before, the DMP can also be considered as a model for the single-machine scheduling problem with sequence-dependent processing times. In this case we have  $n$  jobs, a dummy job 0 that is scheduled twice, once at the beginning and once at the end of the schedule, and sequence-dependent processing times  $p_{ij}$ . Furthermore, in the presence of time windows, every job  $i$  is released at time  $e_i$  and has to be started not later than  $l_i$ .

A frequently used objective in scheduling problems is minimizing the completion time of the last job in the schedule. In our formulation this can easily be expressed as  $\sum_i (t_{i0} + p_{i0} x_{i0})$ . For this problem Maffioli and Sciomachen [8] have proposed a similar formulation in which, in addition to the  $t_{ij}$ , time variables  $y_{ij}$  and  $u_{ij}$  are introduced to denote the departure time at node  $j$  and the time the delivery man has to wait at node  $j$ , respectively, when  $i$  is visited before  $j$ . Thus,  $\sum_i y_{ij} = \sum_k t_{jk}$  for every  $j \neq 0$  and  $u_{ij} = y_{ij} - p_{ij} x_{ij} - t_{ij}$ . The latter can be fixed to zero when time windows are not involved. We will also consider this model and the one with variables  $x_{ij}$  and  $t_i$  in our computational experiments.

### 3 Valid inequalities

In this section we derive valid inequalities for both the DMP and the DMPTW. These inequalities contain both  $x$ - and  $t$ -variables which may lead to a stronger connection between the two types of variables.

#### 3.1 Inequalities for the delivery man problem

The inequalities presented in this section can be considered as a generalization of the inequalities derived by Queyranne [11] for the single-machine scheduling problem with sequence-independent processing times. We will briefly discuss the latter, using the terminology of the

DMP. Recall that for every solution to this problem the departure time equals the arrival time at any node but the depot.

Sequence-independent processing times correspond to travel times  $p_{ij}$  that only depend on node  $i$ , i.e.,  $p_{ij} = p_i$ . Furthermore,  $p_0 = 0$ . For the model that only involves departure times  $t_i$  for  $i \in V$ , Queyranne derived the following inequalities:

$$\sum_{i \in S} p_i t_i \geq \sum_{i, j \in S, i \neq j} p_i p_j, \quad (8)$$

where  $S$  is a nonempty subset of  $V$ . He proves that the set of all inequalities of the form (8) defines the convex hull of the set of all feasible tours.

The validity of (8) is easily shown. Let  $S = \{\pi(1), \dots, \pi(s)\}$  and suppose without loss of generality that the nodes in  $S$  appear in the order  $\pi(1), \dots, \pi(s)$  in the tour, thus  $t_{\pi(i)} \leq t_{\pi(i+1)}$  for  $1 \leq i < s$ . Then  $t_{\pi(i)} \geq \sum_{j=1}^{i-1} p_{\pi(j)}$ , hence

$$\sum_{i \in S} p_i t_i = \sum_{i=1}^s p_{\pi(i)} t_{\pi(i)} \geq \sum_{i=1}^s \sum_{j=1}^{i-1} p_{\pi(i)} p_{\pi(j)} = \sum_{i, j \in S, i \neq j} p_i p_j.$$

Thus, (8) is valid and equality holds if and only if every node in  $S$  is visited before any node in  $V \setminus S$ .

Let us first indicate how the above inequalities can be transformed into valid inequalities for our model. The terms  $p_i t_i$  in (8) will be split into terms  $p_{ij} t_{ij}$ . At the righthand side  $p_i$  will be replaced by  $\sum_j p_{ij} x_{ij}$ , since a travel time  $p_{ij}$  occurs only if  $x_{ij} = 1$ . Unfortunately, this yields quadratic terms at the righthand side. These have to be linearized, as we can only deal with *linear* inequalities. However, we will first present the quadratic inequalities obtained in this way and show their validity.

In the sequel, a tour is identified with a solution  $(x, t)$  to the DMP, i.e.,  $(x, t)$  satisfies the constraints (1)–(5). Furthermore, the following notation is used. As usual,  $x(S) = \sum_{i, j \in S} x_{ij}$  and  $x(S_1, S_2) = \sum_{i \in S_1, j \in S_2} x_{ij}$ . If  $S$  is a subset of  $V$ , then  $S_0 = S \cup \{0\}$ . We abbreviate  $\sum_{j \neq i} t_{ij}$  by  $t_i$ .

**Proposition 1** *For all  $S \subseteq V$ ,  $S \neq \emptyset$ ,*

$$\sum_{i, j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \left( \sum_{i \in S_0, j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i \in S_0, j \in S} p_{ij}^2 x_{ij} \quad (9)$$

*is a valid (quadratic) inequality.*

**Proof.** Let  $S = \{\pi(1), \pi(2), \dots, \pi(s)\} \subseteq V$  and define  $\bar{S} = V \setminus S$ . The validity of (9) is first established for tours for which every node in  $S$  is visited before any node in  $\bar{S}$ , i.e.,  $t_i \leq t_j$  for all  $i \in S$ ,  $j \in \bar{S}$ . Let  $(x, t)$  be a tour satisfying this restriction and assume that the elements of  $S$  are visited in the order  $\pi(1), \pi(2), \dots, \pi(s)$ . Thus  $x_{\pi(i), \pi(i+1)} = 1$ ,  $0 \leq i < s$ , where  $\pi(0) = 0$ . Since we assumed that  $(x, t)$  satisfies (3), we have

$$t_{\pi(1), \pi(2)} = p_{\pi(0), \pi(1)}, \quad t_{\pi(i), \pi(i+1)} = t_{\pi(i-1), \pi(i)} + p_{\pi(i-1), \pi(i)} = \sum_{j=0}^{i-1} p_{\pi(j), \pi(j+1)}, \quad 2 \leq i < s.$$

Obviously,  $t_{\pi(i),j} = x_{\pi(i),j} = 0$  for  $j \neq \pi(i+1), 0 \leq i < s$ , and  $t_{\pi(s),j} = x_{\pi(s),j} = 0$  for  $j \in S$ . Hence,

$$\begin{aligned} \sum_{i,j \in S} p_{ij} t_{ij} &= \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} t_{\pi(i),\pi(i+1)} = \sum_{i=1}^{s-1} p_{\pi(i),\pi(i+1)} \cdot \left( \sum_{j=0}^{i-1} p_{\pi(j),\pi(j+1)} \right) \\ &= \frac{1}{2} \left( \sum_{i=0}^{s-1} p_{\pi(i),\pi(i+1)} \right)^2 - \frac{1}{2} \sum_{i=0}^{s-1} p_{\pi(i),\pi(i+1)}^2 = \frac{1}{2} \left( \sum_{i \in S_0, j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i \in S_0, j \in S} p_{ij}^2 x_{ij}. \end{aligned}$$

Thus (9) is satisfied at equality by all tours for which  $t_i \leq t_j$  for all  $i \in S, j \in \bar{S}$  holds.

To establish the validity of (9) for all tours, we introduce the notion of a *block*  $S'$ , which is defined as a maximal set of nodes of  $S$  that are visited successively in the tour, i.e., if  $|S'| > 1$ , then  $x(S') = |S'| - 1$ , and  $x(S \setminus S', S') = x(S', S \setminus S') = 0$ . Obviously, if  $x(S) = |S| - k^*$  for some  $k^* \geq 1$ , then the set  $S$  is partitioned into  $k^*$  blocks  $S_k$  of size  $s_k \geq 1, k = 1, \dots, k^*$ . Denote the elements of  $S_k$  by  $\pi_k(1), \dots, \pi_k(s_k)$ , and assume that

$$t_{\pi_k(j)} \leq t_{\pi_k(j+1)}, \quad 1 \leq j \leq s_k - 1, \quad 1 \leq k \leq k^*, \quad \text{and} \quad t_{\pi_k(s_k)} \leq t_{\pi_{k+1}(1)}, \quad 1 \leq k \leq k^* - 1.$$

Since  $(x, t)$  satisfies (3), we have

$$t_{\pi_k(i),\pi_k(i+1)} = t_{\pi_k(i-1),\pi_k(i)} + p_{\pi_k(i-1),\pi_k(i)}, \quad 2 \leq i \leq s_k - 1, \quad 1 \leq k \leq k^*.$$

Furthermore, if node  $\pi_1(1)$  is visited first in the tour, then  $x_{0,\pi_1(1)} = 1$  and  $t_{\pi_1(1)} = p_{0,\pi_1(1)}$ . Otherwise,  $\sum_{i \in S} p_{0i} x_{0i} = 0$ . Also notice that if  $i$  is the last node of a block, then the terms  $t_{ij}$  and  $x_{ij}$  in (9) are all equal to zero.

Combining the above observations yields

$$t_{\pi_k(i),\pi_k(i+1)} \geq p_{0,\pi_1(1)} x_{0,\pi_1(1)} + \sum_{l=1}^{k-1} \sum_{j=1}^{s_l-1} p_{\pi_l(j),\pi_l(j+1)} + \sum_{j=1}^{i-1} p_{\pi_k(j),\pi_k(j+1)} \quad (10)$$

for  $1 \leq i \leq s_k - 1, 1 \leq k \leq k^*$ , and

$$\begin{aligned} \sum_{i,j \in S} p_{ij} t_{ij} &= \sum_{k=1}^{k^*} \sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)} t_{\pi_k(i),\pi_k(i+1)} \\ &\stackrel{(10)}{\geq} \sum_{k=1}^{k^*} \sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)} \cdot \left( p_{0,\pi_1(1)} x_{0,\pi_1(1)} + \sum_{l=1}^{k-1} \sum_{j=1}^{s_l-1} p_{\pi_l(j),\pi_l(j+1)} + \sum_{j=1}^{i-1} p_{\pi_k(j),\pi_k(j+1)} \right) \\ &= \frac{1}{2} \left( \sum_{k=1}^{k^*} \sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)} \right)^2 - \frac{1}{2} \sum_{k=1}^{k^*} \sum_{i=1}^{s_k-1} p_{\pi_k(i),\pi_k(i+1)}^2 \\ &= \frac{1}{2} \left( \sum_{i \in S_0, j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i \in S_0, j \in S} p_{ij}^2 x_{ij}. \end{aligned} \quad (11)$$

This concludes the proof of the validity of inequality (9) for all tours.  $\square$

If all travel times are strictly positive, then (9) is satisfied at equality by all tours  $(x, t)$  for which  $x(0, S) = 1, x(S) = |S| - k^*$  for some  $k^* \geq 1$ , and  $|S_k| = 1$  for  $1 < k \leq k^*$ . This can be seen as follows. If  $p_{ij} > 0$  for all  $i, j$ , then  $t_{\pi_k(1)} > t_{\pi_{k-1}(s_{k-1})}$  for  $1 < k \leq k^*$  and  $t_{\pi_1(1)} \geq \sum_i p_{i0}x_{i0}$ . Hence, equality holds in (10), and, consequently, in (11), if and only if  $x(0, S) = 1, k = 1$ , and  $1 \leq i \leq s_1 - 1$ . These restrictions are equivalent to the ones mentioned before.

Observe that when we set  $p_{ij} = p_i$  in the above inequality, we do not get inequality (8), unless  $S = V$ . In order to obtain (8), we would need for every  $i \in S$  the terms  $p_{ij}t_{ij}$  and  $p_{ij}x_{ij}$  for every  $j \in V$ . However, these terms appear in (9) only for  $j \in S$ .

As mentioned before, we can only deal with linear inequalities. In the sequel, we discuss a linearization of the righthand side of (9) that yields valid linear inequalities.

Let  $S \subseteq V$  and define  $\Gamma(S)$  to be the set of all values that  $\sum_{i \in S_0, j \in S} p_{ij}x_{ij}$  can attain, thus,  $\Gamma(S) = \{\sum_{i \in S_0, j \in S} p_{ij}x_{ij} \mid (x, t) \text{ is a tour}\}$ . Note that the assumption that the travel times are integral implies that all elements of  $\Gamma(S)$  are integral. Let  $\gamma_1$  and  $\gamma_2$ ,  $\gamma_1 < \gamma_2$ , be two consecutive elements of  $\Gamma(S)$ , i.e., every  $\gamma \in \Gamma(S)$  satisfies  $\gamma \leq \gamma_1$  or  $\gamma \geq \gamma_2$ . Then

$$\left( \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_1 \right) \left( \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_2 \right) \geq 0$$

and, hence,

$$\begin{aligned} \left( \sum_{i \in S_0, j \in S} p_{ij}x_{ij} \right)^2 &= \left( \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_1 \right) \left( \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_2 \right) + (\gamma_1 + \gamma_2) \cdot \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_1 \gamma_2 \\ &\geq (\gamma_1 + \gamma_2) \cdot \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_1 \gamma_2 \end{aligned}$$

for every tour  $(x, t)$ . This is represented in Figure 1, in which the term  $\sum_{i \in S_0, j \in S} p_{ij}x_{ij}$  is considered as one variable  $y$ . Hence, replacing the quadratic term in the righthand side of (9)

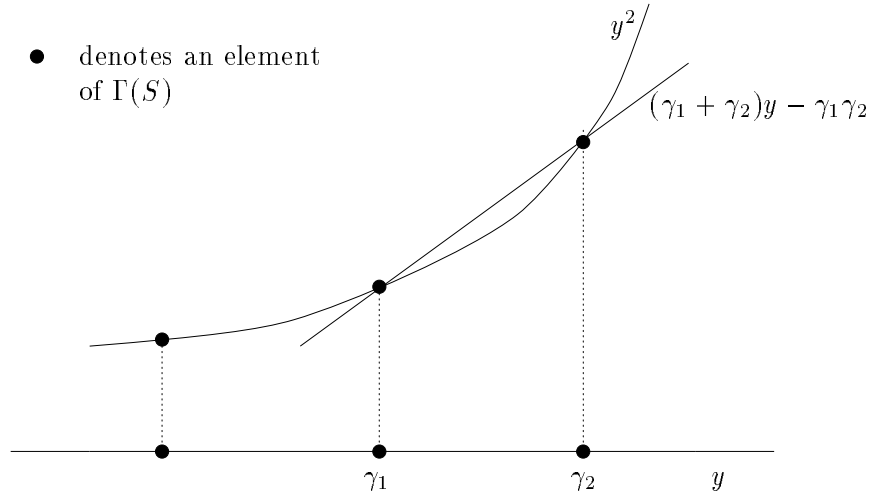


Figure 1: Linearization of  $y^2$

by  $(\gamma_1 + \gamma_2) \cdot \sum_{i \in S_0, j \in S} p_{ij}x_{ij} - \gamma_1 \gamma_2$  yields a valid linear inequality. This proves the following statement.

**Lemma 2** *Let  $S \subseteq V, S \neq \emptyset$ . Then for every pair  $(\gamma_1, \gamma_2)$  of consecutive elements of  $\Gamma(S)$*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \sum_{i \in S_0, j \in S} p_{ij} (\gamma_1 + \gamma_2 - p_{ij}) x_{ij} - \frac{1}{2} \gamma_1 \gamma_2 \quad (12)$$

*is a valid inequality.*

In this way we obtain for every  $S$  a set of valid linear inequalities. However, in a separation algorithm for (12) we only have to check violation for at most one inequality for every subset  $S$ . Notice that these inequalities differ in the coefficients of the  $x$ -variables and the constant term, but *not* in the coefficients of the  $t$ -variables. For every triple  $(\gamma_1, \gamma_2, \gamma_3)$  of consecutive elements of  $\Gamma(S)$ , where  $\gamma_1 < \gamma_2 < \gamma_3$ , and  $y \in \mathbb{R}$ , we have

$$(\gamma_2 + \gamma_3)y - \gamma_2 \gamma_3 - [(\gamma_1 + \gamma_2)y - \gamma_1 \gamma_2] = (\gamma_3 - \gamma_1)(y - \gamma_2) > 0 \text{ if and only if } y > \gamma_2.$$

Let  $(\hat{x}, \hat{t})$  be a solution to the LP-relaxation of (M1) and let  $S \subseteq V$ . Define  $\hat{y} = \sum_{i \in S_0, j \in S} p_{ij} \hat{x}_{ij}$ . Then it is easily seen from the above observation that  $(\gamma_1 + \gamma_2)\hat{y} - \gamma_1 \gamma_2$  is maximal for  $\gamma_1 = \max\{\gamma \in \Gamma(S) \mid \gamma \leq \hat{y}\}$  and  $\gamma_2 = \min\{\gamma \in \Gamma(S) \mid \gamma > \hat{y}\}$ . From this it follows that when for a given subset  $S$  some inequalities of the form (12) are violated by  $(\hat{x}, \hat{t})$ , the inequality with  $\gamma_1$  and  $\gamma_2$  as defined above is the most violated one. Thus, in a separation routine it suffices to consider at most one inequality for every subset  $S$ .

However, we expect the inequalities (12) to be rather weak in general. Let us therefore consider the tours that satisfy (12) at equality. Clearly, such a tour also satisfies (9) at equality. If we restrict ourselves to the case that all travel times are strictly positive, then it was observed before that for every tour  $(x, t)$  satisfying (9) at equality there is a subset  $S' \subseteq S$  such that  $x(S') = x(S) = |S'| - 1$  and  $x(0, S') = 1$ . In general, we cannot say much about the number of such tours for which  $\sum_{i \in S_0, j \in S} p_{ij} x_{ij}$  has a particular value, but we expect it to be small. Notice that even if there would exist a linear inequality that is satisfied at equality by all tours that satisfy (9) at equality, then this would not define a facet. This follows from the observation that every tour  $(x, t)$  satisfying (9) at equality also satisfies  $x(0, S) = 1$ , hence  $x_{0i} = 0$  for every  $i \notin S$ .

Furthermore, there will not be an efficient way in general to determine  $\Gamma(S)$ . In a separation routine that uses the ideas described previously, it will usually be too time-consuming to determine  $\gamma_1 = \max\{\gamma \in \Gamma(S) \mid \gamma \leq \hat{y}\}$  and  $\gamma_2 = \min\{\gamma \in \Gamma(S) \mid \gamma > \hat{y}\}$ . Hence, checking violation will have to be restricted to the inequality with  $\gamma_1 = \lfloor \hat{y} \rfloor$  and  $\gamma_2 = \lfloor \hat{y} \rfloor + 1$ . From the assumption that all travel times are integral, it is easily seen that these inequalities are always valid. However, if neither  $\gamma$  nor  $\gamma + 1$  is an element of  $\Gamma(S)$ , then there is no feasible solution that satisfies such an inequality at equality.

A second class of valid linear inequalities can be derived in a similar way. In this case we start from the following class of quadratic inequalities. For  $S \subseteq V$ , define

$$T_S = \{(i, j) \in V \times V \mid \text{if } i \in \bar{S}, \text{ then } j \notin \bar{S}\},$$

where  $\bar{S} = V \setminus S$ . Thus, the set  $T_S$  consists of all pairs  $(i, j)$  for which at most one of  $i$  and  $j$  is in  $\bar{S}$ .

**Proposition 3** *Let  $S \subseteq V, S \neq \emptyset$ , and let  $T_S$  be as defined above. Then*

$$\sum_{(i,j) \in T_S} p_{ij} t_{ij} \geq \frac{1}{2} \left( \sum_{j \in V} p_{0j} x_{0j} + \sum_{(i,j) \in T_S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{j \in V} p_{0j}^2 x_{0j} - \frac{1}{2} \sum_{(i,j) \in T_S} p_{ij}^2 x_{ij} \quad (13)$$



is a valid (quadratic) inequality.

**Proof.** We first restrict ourselves to tours for which the following holds:

$$\text{if } j \in \bar{S} \text{ and } x(S_0, j) = 1, \text{ then } x(j, S) = 1 \text{ or } t_i \leq t_j \text{ for all } i \in S. \quad (14)$$

Thus, no two nodes in  $\bar{S}$  are visited successively before the last node of  $S$  in the tour. Now consider a tour  $(x, t)$  that satisfies the above restriction. Define  $\bar{S}' = \{j \in \bar{S} \mid x(S \cup \{0\}, j) = 1\}$ . Furthermore, let  $S' = S \cup \bar{S}' = \{\pi(1), \dots, \pi(s')\}$  and suppose  $x_{\pi(i), \pi(i+1)} = 1, 1 \leq j \leq s' - 1$ . Note that  $\pi(s') \in \bar{S}$  when  $S' \neq V$ . By definition of  $\bar{S}'$ , we have

$$\sum_{i,j \in S'} p_{ij} t_{ij} = \sum_{i \in S, j \in \bar{S} \setminus S'} p_{ij} t_{ij} = \sum_{i \in \bar{S} \setminus S', j \in S} p_{ij} t_{ij} = 0. \quad (15)$$

Hence,

$$\begin{aligned} \sum_{i,j \in S'} p_{ij} t_{ij} &= \sum_{i,j \in S} p_{ij} t_{ij} + \sum_{i,j \in \bar{S}'} p_{ij} t_{ij} + \sum_{i \in S, j \in \bar{S}'} p_{ij} t_{ij} + \sum_{i \in \bar{S}', j \in S} p_{ij} t_{ij} \\ &= \sum_{i,j \in S} p_{ij} t_{ij} + \sum_{i \in S, j \in \bar{S}} p_{ij} t_{ij} + \sum_{i \in \bar{S}, j \in S} p_{ij} t_{ij} = \sum_{(i,j) \in T_S} p_{ij} t_{ij}. \end{aligned}$$

Clearly, (15) and the above equality also hold when the  $t$ -variables are replaced by  $x$ -variables. Furthermore,  $\sum_{j \in S'} p_{0j} x_{0j} = \sum_{j \in V} p_{0j} x_{0j}$ . Combining these results with the observation that (9) is satisfied at equality for the subset  $S'$ , we get

$$\begin{aligned} \sum_{(i,j) \in T_S} p_{ij} t_{ij} &= \sum_{i,j \in S'} p_{ij} t_{ij} = \frac{1}{2} \left( \sum_{i \in S'_0, j \in S'} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i \in S'_0, j \in S'} p_{ij}^2 x_{ij} \\ &= \frac{1}{2} \left( \sum_{j \in S'} p_{0j} x_{0j} + \sum_{i,j \in S'} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{j \in S'} p_{0j}^2 x_{0j} - \frac{1}{2} \sum_{i,j \in S'} p_{ij}^2 x_{ij} \\ &= \frac{1}{2} \left( \sum_{j \in V} p_{0j} x_{0j} + \sum_{i,j \in T_S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{j \in V} p_{0j}^2 x_{0j} - \frac{1}{2} \sum_{i,j \in T_S} p_{ij}^2 x_{ij}. \end{aligned}$$

From the proof it immediately follows that (13) is satisfied at equality by all tours for which restriction (14) holds. Hence, unlike in the case of (9), there exist tours  $(x, t)$  such that  $x(0, \bar{S}) = 1$  which satisfy (13) at equality.

The proof that (13) is also valid for all other tours is analogous to the corresponding proof in Proposition 1 and is therefore omitted.  $\square$

To these quadratic inequalities we can apply a linearization that is similar to the one described for (9). Therefore, we will not discuss it in detail. Define  $\Gamma(T_S) = \{ \sum_{j \in V} p_{0j} x_{0j} + \sum_{(i,j) \in T_S} p_{ij} x_{ij} \mid (x, t) \text{ is a tour} \}$ .

**Lemma 4** *Let  $S \subseteq V, S \neq \emptyset$ . Then for every pair  $(\gamma_1, \gamma_2)$  of consecutive elements of  $\Gamma(T_S)$*

$$\sum_{(i,j) \in T_S} p_{ij} t_{ij} \geq \frac{1}{2} \sum_{j \in V} p_{0j} (\gamma_1 + \gamma_2 - p_{0j}) x_{0j} + \frac{1}{2} \sum_{(i,j) \in T_S} p_{ij} (\gamma_1 + \gamma_2 - p_{ij}) x_{ij} - \frac{1}{2} \gamma_1 \gamma_2 \quad (16)$$

is a valid inequality.

### 3.2 Inequalities for the delivery man problem with time windows

Let us now consider tours  $(x, t)$  that satisfy constraints (1), (2), (5) - (7). Constraint (6) yields that the departure time at node  $i$  may be strictly larger than the arrival time. In this case we say that waiting time occurs at node  $i$ . Note that waiting affects the lefthand side of the inequalities derived previously, but not the righthand side. Hence, inequalities (12) and (16) are also valid for the problem with time windows.

In the remainder of this section we show how time windows can be incorporated in the inequalities derived previously. We first present two classes of valid quadratic inequalities that involve earliest and latest departure times, respectively. Detailed proofs of their validity are omitted, since these are similar to the proof of Proposition 1. They will be linearized in the same way as the quadratic inequalities derived for the problem without time windows.

The first class of inequalities can be considered as a generalization of (9). Let  $S$  be a set of nodes and let  $e_S = \min_{i \in S} e_i$ . Introduce an extra node  $0'$  that has to be left at time  $e_S$  and for which  $p_{0'i} = p_{i0'} = 0$  for every  $i \in V$ . This extra node can be considered as a depot for the set  $S$ . Let  $(x, t)$  be a tour satisfying  $x(S) = s - 1$ . Subtracting  $e_S$  from all departure times yields a vector for which (9) holds, if  $0$  is replaced by  $0'$ . Recalling that  $p_{0'i} = 0$  for all  $i \in V$ , this shows the validity of the following inequality for tours satisfying  $x(S) = s - 1$ .

**Proposition 5** *Let  $S \subseteq V$ ,  $S \neq \emptyset$ , and define  $e_S = \min_{i \in S} e_i$ . Then*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \left( \sum_{i,j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i,j \in S} p_{ij}^2 x_{ij} + e_S \sum_{i,j \in S} p_{ij} x_{ij} \quad (17)$$

*is a valid (quadratic) inequality.*

The validity of the above inequality for all other tours can be easily proven, using similar arguments as in Proposition 1. From the above interpretation it immediately follows that (17) is satisfied at equality by all tours for which the departure time at the first node of  $S$  equals  $e_S$ ,  $x(S) = s - 1$ , and no waiting time occurs at any node of  $S$ , except possibly at the one visited first. Thus, in this case,  $t_{\pi(i), \pi(i+1)} = e_S + \sum_{j=1}^{i-1} p_{\pi(j), \pi(j+1)}$  holds for  $i = 1, \dots, s - 1$ , where it is assumed that  $S = \{\pi(1), \dots, \pi(s)\}$  and the nodes are visited in this order. Note that such a tour might not exist.

Until now, we discussed inequalities that involved a certain moment *from* which the tours were considered. For the second class of quadratic inequalities, tours will be considered *until* a moment  $l_S$ , defined as the maximum latest departure time of the nodes in a set  $S$ .

**Proposition 6** *Let  $S \subseteq V$ ,  $S \neq \emptyset$ , and define  $l_S = \max_{i \in S} l_i$ . Then*

$$\sum_{i,j \in S} p_{ij} t_{ij} \leq l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \left( \sum_{i,j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i,j \in S} p_{ij}^2 x_{ij} \quad (18)$$

*is a valid (quadratic) inequality.*

We show that equality holds for all tours for which the departure time at the last node of  $S$  equals  $l_S$ ,  $x(S) = s - 1$ , and no waiting time occurs at any node in  $S$ , except possibly in the one visited first. As in the previous case, such a tour might not exist. Let

$S = \{\pi(1), \pi(2), \dots, \pi(s)\} \subseteq V$  and suppose  $(x, t)$  satisfies  $x_{\pi(i), \pi(i+1)} = 1, 1 \leq j \leq s - 1$ . Furthermore, let  $t_{\pi(s)} = l_S$  and

$$t_{\pi(i-1), \pi(i)} = t_{\pi(i), \pi(i+1)} - p_{\pi(i-1), \pi(i)} = l_S - \sum_{j=i}^{s-1} p_{\pi(j), \pi(j+1)} \quad (19)$$

for  $i = 1, \dots, s - 1$ . Then

$$\begin{aligned} \sum_{i,j \in S} p_{ij} t_{ij} &= \sum_{i=1}^{s-1} p_{\pi(i), \pi(i+1)} t_{\pi(i), \pi(i+1)} = \sum_{i=1}^{s-1} p_{\pi(i), \pi(i+1)} \left( l_S - \sum_{j=i}^{s-1} p_{\pi(j), \pi(j+1)} \right) \\ &= l_S \sum_{i=1}^{s-1} p_{\pi(i), \pi(i+1)} - \frac{1}{2} \left( \sum_{i=1}^{s-1} p_{\pi(i), \pi(i+1)} \right)^2 - \frac{1}{2} \sum_{i=1}^{s-1} p_{\pi(i), \pi(i+1)}^2 \\ &= l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \left( \sum_{i,j \in S} p_{ij} x_{ij} \right)^2 - \frac{1}{2} \sum_{i,j \in S} p_{ij}^2 x_{ij}. \end{aligned}$$

Notice that for a tour  $(x, t)$  such that  $x_{\pi(i), \pi(i+1)} = 1, 1 \leq i < s$ , (19) is the latest possible departure time for every  $i \in S$ . Hence, if  $t_{\pi(i-1), \pi(i)} + x_{\pi(i-1), \pi(i)} < t_{\pi(i), \pi(i+1)}$  for some  $i, 2 < i < s$ , then the lefthand side of (18) is *less* than when equality holds. As the righthand side is the same in both cases, this shows the validity of (18) for all tours  $(x, t)$  with  $x(S) = s - 1$ . For tours  $(x, t)$  satisfying  $x(S) < s - 1$  validity can be proven in a similar way as in Proposition 1.

To (17) and (18) almost the same linearization as the one described for (9) can be applied. Define  $\Gamma(S) = \{\sum_{i,j \in S} p_{ij} x_{ij} \mid (x, t) \text{ is a tour}\}$ .

**Lemma 7** *Let  $S \subseteq V, S \neq \emptyset$ , and let  $e_S$  and  $l_S$  be as defined before. Then for every pair  $(\gamma_1, \gamma_2)$  of consecutive elements of  $\Gamma(S)$*

$$\sum_{i,j \in S} p_{ij} t_{ij} \geq \frac{1}{2} \sum_{i,j \in S} p_{ij} (\gamma_1 + \gamma_2 - p_{ij}) x_{ij} - e_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \gamma_1 \gamma_2 \quad (20)$$

and

$$\sum_{i,j \in S} p_{ij} t_{ij} \leq l_S \sum_{i,j \in S} p_{ij} x_{ij} - \frac{1}{2} \sum_{i,j \in S} p_{ij} (\gamma_1 + \gamma_2 + p_{ij}) x_{ij} + \frac{1}{2} \gamma_1 \gamma_2 \quad (21)$$

are valid inequalities.

It was already noticed that when time windows are involved, the number of tours satisfying the quadratic inequalities (17) and (18) at equality is expected to be small in general, especially if there is only one element of  $S$  with earliest departure time  $e_S$  or latest departure time  $l_S$ . We expect inequalities (20) and (21) to be of more help in obtaining better lower bounds for the DMPTW when subsets of nodes have the same time window. In our computational experiments we will therefore consider, among other things, instances for which the nodes are partitioned into clusters such that nodes in the same cluster have the same time window.

## 4 A cutting plane procedure

The main purpose of our computational experiments is to study the quality of the lower bounds for the DMP and DMPTW given by the LP-relaxation and the improvement of these lower bounds by adding cuts from some specific classes. We compare these bounds for the three models discussed in Section 2, i.e., the model introduced in this paper with variables  $x_{ij}$  and  $t_{ij}$  (model 1), the model proposed in [8] with variables  $x_{ij}, t_{ij}, y_{ij}$ , and  $u_{ij}$  (model 2), and the model with variables  $x_{ij}$  and  $t_i$  (model 3). In this section we briefly describe the steps of our cutting plane procedure. All experiments have been performed using MINTO (version 1.6), which is a linear programming based branch-and-bound software system that can be provided with problem specific functions such as cut generation ([9]).

Before solving the initial LP one usually tries to reduce the size of the problem or improve the formulation by preprocessing techniques such as fixing variables and improving bounds. We restrict ourselves to fixing variables  $x_{ij}$  in the presence of time windows. Since for all our instances the travel times  $p_{ij}$  will satisfy the triangle inequality, variable fixing can be done in the following way: if  $e_i + p_{ij} > l_j$  ( $e_i > \min_j l_j$ ,  $l_i < \max_j e_j$ ), then  $x_{ij}$  ( $x_{0i}, x_{i0}$ ) is set to zero. After a variable has been fixed, it is eliminated from the formulation. Observe that  $x_{ij} = 0$  implies  $t_{ij} = 0$  (model 1, 2) and  $y_{ij} = 0, u_{ij} = 0$  (model 2). Therefore, preprocessing may considerably reduce the large number of variables in our models (e.g.,  $2n^2 + n$  in model 1), especially when the time windows are tight.

It is a trivial observation that all inequalities derived for the TSP, such as subtour elimination constraints (SEC's), 2-matching constraints, comb inequalities, etc. (cf. [6]), are also valid for the three models considered here and, hence, can be used to strengthen these formulations. We will restrict ourselves to the addition of SEC's, which have the following form:

$$x(S) \leq |S| - 1, \quad S \subset \{0, \dots, n\}, \quad 2 \leq |S| \leq n.$$

The exact separation algorithm described in [10] is applied to check whether an LP-solution satisfies all SEC's. Violated SEC's found in this way are added to the formulation and the LP is solved again. This step is repeated until the LP-solution satisfies all SEC's.

Next, we check whether violated inequalities of the form (12) or (16) can be identified. Notice that this step can only be applied to the formulations with variables  $t_{ij}$ , i.e., to model 1 and 2. Our (not very sophisticated) separation algorithms are inspired by the  $O(n \log n)$  exact algorithm for inequalities (8), i.e.,  $\sum_{i \in S} p_i t_i \geq \sum_{i, j \in S, i < j} p_i p_j$  for  $S \subseteq V$ . Given an LP-solution  $\hat{t}$ , the separation problem for these inequalities amounts to checking whether (8) is violated for  $S = \{\pi(1), \dots, \pi(i)\}, 1 \leq i \leq n$ , where the permutation  $\pi : V \rightarrow V$  satisfies  $\hat{t}_{\pi(i)} \leq \hat{t}_{\pi(i+1)}$  (cf. [11]).

Let us now give an outline of the separation algorithm for inequalities (12). Let  $(\hat{x}, \hat{t})$  be the LP-solution and let the permutation  $\pi : V \rightarrow V$  be such that  $\hat{t}_{\pi(i)} \leq \hat{t}_{\pi(i+1)}$ . We check whether  $(\hat{x}, \hat{t})$  violates (12) for  $S = \{\pi(1), \dots, \pi(i)\}, 1 \leq i \leq n$ ,  $\gamma_1 = \lfloor \hat{y} \rfloor$  and  $\gamma_2 = \lfloor \hat{y} + 1 \rfloor$ , where  $\hat{y} = \sum_{i \in S_0, j \in S} p_{ij} \hat{x}_{ij}$  (cf. Section 3.2). This procedure is repeated a fixed number of times with a permutation  $\pi$  that is obtained from the permutation  $\bar{\pi}$  in the previous iteration by putting  $\bar{\pi}(i) = \pi(i+1)$  and  $\bar{\pi}(i+1) = \pi(i)$  for  $i \in V'$ , where  $V'$  is a randomly chosen subset of  $V$  of size at most  $|V|/2$ .

A similar separation heuristic is used to identify violated inequalities (16). The heuristics for (20) and (21) differ slightly from the one described above, but we will not discuss them

since we never found violated inequalities of these types. This is possibly due to the fact that these separation algorithms were only called when neither violated SEC's nor violated inequalities of type (12) or (16) were identified.

## 5 Computational results

We report results for twelve sets of five randomly generated instances with  $n = 15$ . These sets were constructed from two sets of five matrices  $(p_{ij})$ , which are denoted by *grid* and *sched*, respectively. For *grid* the 15 nodes and the depot are randomly generated lattice points of a  $20 \times 20$  grid. The travel time  $p_{ij}$  equals the Manhattan distance between  $i$  and  $j$ , i.e.,  $p_{ij} = |a_i - a_j| + |b_i - b_j|$ , where  $(a_i, b_i)$  denotes the pair of coordinates of node  $i$ . The second set of matrices, *sched*, results from the interpretation of the DMP as a machine scheduling problem with sequence-dependent processing times. Here  $p_{ij}$  can be considered as the sum of a fixed processing time  $p_i$  for job  $i$  and a changeover time  $s_{ij}$ . The (integer) processing times  $p_i, i \neq 0$ , and changeover times  $s_{ij}, j \neq 0$ , are randomly chosen from the interval  $[5, 15]$  and  $[0, 4]$ , respectively. Furthermore,  $p_0 = 0$  and  $s_{i0} = 0$ . Observe that only in *grid* the matrices are symmetric. However, in both sets the travel times  $p_{ij}$  satisfy the triangle inequality. Furthermore, for fixed  $i$  the value of  $p_{ij}$  in the first set is in the interval  $[1, 38]$ , whereas in the second set this value only ranges from  $p_i$  to  $p_i + 4$ .

Each of these two sets of matrices gave rise to six sets of five instances by the addition of time windows. These sets are denoted by *grid* $_k$  and *sched* $_k$ , where  $k \in \{0, \dots, 5\}$ .

- For  $k = 0$  the instances are instances for the ordinary DMP, i.e., no time windows are involved.
- For  $k = 1$  the nodes are partitioned into three clusters of size five. Every node in cluster  $c, 1 \leq c \leq 3$ , has a time window  $[(c - 1)W_{inst}, cW_{inst}]$ , where  $W_{grid} = 80$  and  $W_{sched} = 100$ .
- For  $k = 2$  the nodes are partitioned into five clusters of size three. Every node in cluster  $c, 1 \leq c \leq 5$ , has a time window  $[(c - 1)W_{inst}, cW_{inst}]$ , where  $W_{grid} = 50$  and  $W_{sched} = 70$ .
- The instances in *grid* $_3$  and *sched* $_3$  have the following form. First, a random solution to the DMP is generated. The departure time at node  $i$  in this solution is denoted by  $t_i^*$ . Then for each node an earliest departure time  $e_i$  is randomly generated such that  $t_i^*$  is in the interval  $[e_i, e_i + W]$ , where  $W = 60$ . For  $k = 4$  and  $k = 5$  the instances are generated in the same way as the instances for  $k = 3$  with  $W = 40$  and  $W = 20$ , respectively.

All instances were tested with respect to the objective function  $\sum_{i=1}^n t_i$ .

Table 1 shows integrality gaps with respect to a lower bound  $Z_{LB}$  and the best known upper bound  $Z_{UB}^*$  to the value of the optimal solution to the DMPTW, where the integrality gap is defined as

$$\frac{Z_{UB}^* - Z_{LB}}{Z_{UB}^*} \times 100\%.$$

The value  $Z_{UB}^*$  was found by a branch-and-cut algorithm in which in every node a feasible solution to the DMPTW was constructed from the LP-solution (max. 2000 nodes). The last

column of Table 1 lists the number of problems (out of five) for which the upper bound  $Z_{UB}^*$  was proven to be optimal.

For all three models column  $G_0$  shows the average gap over five instances, where  $Z_{LB}$  is the value of the LP-relaxation of the model. The average gap after SEC's have been added to the formulation is reported in column  $G_1$ . Finally, for model 1 and 2 column  $G_2$  shows the average gap after both SEC's and inequalities (12) and (16) have been added.

inst	model 1			model 2			model 3		opt
	$G_0$	$G_1$	$G_2$	$G_0$	$G_1$	$G_2$	$G_0$	$G_1$	
<i>grid_0</i>	56.5	53.5	33.2	56.5	53.5	30.4	64.9	64.9	0
<i>grid_1</i>	25.6	23.3	19.6	23.1	21.0	17.5	35.1	35.1	1
<i>grid_2</i>	17.2	15.0	14.2	12.9	11.2	10.1	24.3	24.3	1
<i>grid_3</i>	11.1	9.5	7.2	8.7	7.4	4.5	16.9	16.9	4
<i>grid_4</i>	6.1	5.4	5.3	3.3	2.8	2.7	8.4	8.4	4
<i>grid_5</i>	6.3	2.8	2.3	4.8	2.5	1.9	10.2	9.9	5
<i>sched_0</i>	81.0	81.0	5.9	81.0	81.0	5.0	97.1	97.1	0
<i>sched_1</i>	27.3	26.3	7.0	26.1	25.0	6.1	39.2	39.2	0
<i>sched_2</i>	11.9	11.0	5.8	10.4	9.6	3.6	21.5	21.5	2
<i>sched_3</i>	16.6	16.0	4.8	16.3	15.6	3.5	32.0	32.0	2
<i>sched_4</i>	4.4	4.2	3.4	3.2	3.0	1.8	11.0	11.0	5
<i>sched_5</i>	1.7	1.5	1.2	1.2	1.2	0.7	12.8	12.7	5

Table 1: Quality of the lower bounds

Since every solution to the LP-relaxation of model 2 yields a feasible solution to the LP-relaxation of model 1, the lower bounds obtained from the latter cannot exceed the lower bounds obtained from the first. Table 1 shows that the bounds obtained from model 2 can be considerably better than the bounds obtained from model 1. The LP-relaxation of the third formulation yields bounds that are inferior to the corresponding bounds obtained from model 1 and which are rarely improved by the addition of SEC's. Also for model 1 and 2 we conclude that the reduction of the gap by the addition of SEC's is rather limited. This especially holds for the second set of problem instances. However, the addition of inequalities (12) and (16) substantially improves the lower bounds for these instances. For the instances of the sets *grid\_k*, the gaps are also reduced by the addition of inequalities (12) and (16), but clearly not as much as for the instances of the sets *sched\_k*.

For model 1 the instances of *grid\_k* and *sched\_k*,  $k \in \{1, 2, 4\}$ , were also tested with respect to the ordinary TSP objective, i.e., minimizing  $\sum_{i,j} p_{ij}x_{ij}$ . Table 2 shows the results for both objective functions. As in the previous table,  $G_0$  shows the average gap obtained from the LP-relaxation of model 1,  $G_1$  gives the average gap after SEC's have been added, and  $G_2$  shows the average gap after the addition of SEC's and inequalities (12) and (16). As far as the last column is concerned, we mention that all instances of the TSPTW were solved to optimality by N. Ascheuer.

We also compare the effect of preprocessing as described in the previous section for the two problems. Column  $\underline{G}_0$  reports the average gap for the value of the LP-relaxation of model 1 without preprocessing. We observe that variable fixing hardly improves the value of the

obj	inst	$\underline{G}_0$	$G_0$	$G_1$	$G_2$	opt
$\sum_i t_i$	<i>grid_1</i>	25.6	25.6	23.3	19.6	1
	<i>grid_2</i>	17.5	17.2	15.0	14.2	1
	<i>grid_4</i>	6.4	6.1	5.4	5.3	4
$\sum_{i,j} p_{ij} x_{ij}$	<i>grid_1</i>	32.6	27.3	12.9	12.9	5
	<i>grid_2</i>	36.3	27.8	18.1	18.1	5
	<i>grid_4</i>	41.2	22.8	11.6	11.6	5
$\sum_i t_i$	<i>sched_1</i>	27.3	27.3	26.3	7.0	0
	<i>sched_2</i>	11.9	11.9	11.0	5.8	2
	<i>sched_4</i>	4.5	4.4	4.2	3.4	5
$\sum_{i,j} p_{ij} x_{ij}$	<i>sched_1</i>	2.1	1.6	1.4	1.4	5
	<i>sched_2</i>	3.2	2.4	2.3	2.3	5
	<i>sched_4</i>	4.7	3.0	2.8	2.8	5

Table 2: Comparison of DMPTW and TSPTW

LP-relaxation with respect to the DMP objective. Only for the instances with the smallest time windows ( $k = 5$ , not reported in Table 2) we found that preprocessing increased the value of the LP-relaxation by more than 0.5%. With respect to the TSP objective, however, preprocessing actually leads to an improved formulation for all instances.

Furthermore, we conclude from Table 2 that the addition of SEC's may be much more effective for the TSPTW than for the DMPTW. However, this does not hold for the inequalities derived in Section 3. Although violated inequalities of this kind were identified for the instances of the TSPTW, the value of the objective function was not improved by adding these inequalities. Apparently, the addition of violated inequalities of the form (12) and (16) hardly influences the value of the  $x$ -variables, but only changes the value of the  $t$ -variables.

## 6 Concluding remarks

In this paper we presented a mixed integer programming formulation for the DMP and DMPTW and derived additional classes of valid inequalities. The quality of the lower bounds obtained from the LP-relaxation and the effectiveness of the new inequalities in reducing the gap were studied computationally. The results show that the addition of the inequalities derived in this paper can substantially improve the lower bound. Although our study is too limited to reach solid conclusions about the kind of matrices  $(p_{ij})$  for which this holds, the results point in the direction of instances for which  $p_{ij}$  does not vary much for fixed  $i$ . In general, however, we have to conclude from our computational experiments that further study is necessary in order to obtain strong lower bounds from the polyhedral approach to the DMP and DMPTW.

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