The set of geometrically infinitely divisible distributions

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Abstract

Klebanov e.a. (1984) have shown that the set of geometrically infinitely divisible distributions coincides with the closure of the set of compound geometric distributions. We prove that from this it follows that (mixtures of) log-convex densities on $(0, \infty)$ are geometrically infinitely divisible. The results of Pillai and Sandhya (1990) are easy consequences of this.

1. Introduction and summary

Geometric infinite divisibility (g.i.d) was introduced by Klebanov e.a. (1984). Pillai and Sandhya (1990) consider the g.i.d. of distributions with a completely monotone density. Unfortunately, their paper contains several mistakes (one of which is attributed to me) and misprints, and somehow in the definition of g.i.d. a line seems to have disappeared. So we start by giving the definition.

Definition 1.1. A random variable (r.v.) Y (or its distribution, its density, or its characteristic function) is said to be geometrically infinitely divisible, if for every $p \in (0,1)$ i.i.d. r.v.'s $X_1(p), X_2(p), ...$ exist such that

$$Y \stackrel{d}{=} X_1(p) + ... + X_{N_p}(p) ,$$

where N_p is independent of $X_1(p), X_2(p), ...$, and

$$P(N_p = n) = (1-p)p^{n-1} \quad (n = 1, 2, ...) .$$

Writing ϕ_Z for the characteristic function (ch.f.) of a r.v. Z, and abbreviating $\phi_{X(p)}$ to ϕ_p, we see that the condition for g.i.d. is equivalent to: for every $p \in (0,1)$ there is a ch.f. ϕ_p such that

$$(1.1) \qquad \phi_Y(t) = \frac{(1-p)\phi_p(t)}{1 - p\phi_p(t)} .$$

In the subsequent sections we give simple proofs of some results of Klebanov e.a. (1984), one of which says that the set of g.i.d. distributions is equal to the closure (in the sense of wealk convergence) of the set of compound geometric distributions. This result is then used to prove that (mixtures of) log-convex densities on $(0, \infty)$ are g.i.d. The g.i.d. of completely monotone densities, considered by Pillai and Sandhya, is an easy consequence of this. Finally, it shown that a certain type of subclass of g.i.d. distributions is closed under mixing, but not the whole class.

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2. The class of g.i.d. distributions

We shall need the following result from Steutel (1968)

Lemma 2.1. The function

$$\frac{\lambda}{\lambda - h(t)}$$

is a ch.f. for every $\lambda > 1$ iff $e^{h(t)}$ is an infinitely divisible (inf.div.) ch.f.

The set of functions h such that $\exp(h)$ is an inf.div. ch.f. will be denoted by $\mathcal{H}$. From (1.1) and Definition 1.1. it follows that $\phi = \phi_Y$ is g.i.d. iff the function

$$\phi_p(t) = \frac{\phi(t)}{1 - p + p\phi(t)} = \frac{1}{p + (1-p)/\phi} \tag{2.1}$$

is a ch.f. for every $p \in (0,1)$. If we put $\lambda = (1-p)^{-1}$, from Lemma 2.1 it immediately follows that we have

Theorem 2.2. (Klebanov e.a.). A ch.f. ϕ is g.i.d. iff $\exp(1 - 1/\phi)$ is an inf.div. ch.f.

It is now quite easy to characterize the g.i.d. distributions as follows (Klebanov e.a. first prove Theorem 2.3 and then Theorem 2.4).

Theorem 2.3. (Klebanov e.a.). The set of g.i.d. distributions is equal to the closure, in the sense of weak convergence, of the set of compound geometric distributions.

Proof. From Lemma 2.1. we know that a ch.f. ϕ is g.i.d. iff $1 - 1/\phi \in \mathcal{H}$, i.e. iff

$$\phi(t) = \frac{1}{1 - h(t)}$$

with $h \in \mathcal{H}$. Now, by De Finetti's theorem (Lukacs (1970)) we have: $\exp(h)$ is inf.div. iff

$$h(t) = \lim_{m\to\infty} \alpha_m(g_m(t) - 1) \ ,$$

where the α_m are positive numbers and the g_m are ch.f.'s. It follows that ϕ is g.i.d. iff

$$\phi(t) = \lim_{m\to\infty} \frac{1}{1 - \alpha_m(g_m(t) - 1)} \ ,$$

i.e. iff ϕ is the limit of a sequence of compound geometric distributions.

Corollary 2.4. Geometrically infinitely divisible distributions are infinitely divisible.

Corollary 2.5. All compound geometric distributions are g.i.d.

Remark 1. If ϕ is compound geometric, say

$$\phi(t) = \frac{1}{1 - \alpha(g(t) - 1)} ,$$

then ϕ_p as meant in Definition 1.1 is given by

$$\phi_p(t) = \frac{1}{1 - \alpha(1-p)(g(t) - 1)} ,$$

which is again compound geometric; this immediately shows that compound geometric distributions are g.i.d.

Remark 2. It is not hard to see that the set of "Poisson infinitely divisible" ch.f.'s defined by the analogous requirement that

$$\phi(t) = \phi_\mu(t) \exp(\mu(\phi_\mu(t) - 1)) \quad (\text{all } \mu > 0) ,$$

coincides with the set of all inf.div. distributions.

3. Log-convex and completely monotone densities

A compound geometric distribution $(p_n)_0^\infty$ on $\{0, 1, 2, ...\}$ has a probability generating function (p.g.f.) P of the form

$$P(z) = \frac{1-p}{1 - pG(z)} ,$$

where G is a p.g.f. with $G(0) = 0$. Putting $1 - p = p_0$ and $pG(z) = zQ(z)$ we obtain a wellknown lemma (see Steutel (1971), p. 83/84).

Lemma 3.1. A distribution $(p_n)_0^\infty$ on $\{0, 1, 2, ...\}$, with $p_0 > 0$, is compound geometric iff the $(q_k)_0^\infty$ defined by

$$(3.1) \qquad p_{n+1} = \sum_{k=0}^{n} q_k p_{n-k} \quad (n = 0, 1, 2, ...)$$

are all nonnegative.

The following lemma is also wellknown; we present it with a simple proof.

Lemma 3.2. If $(p_n)_0^\infty$ with $p_0 > 0$ is a log-convex distribution on $\{0, 1, 2, ...\}$, i.e. if

$$(3.2) \qquad p_{n+1} p_{n-1} \geq p_n^2 \quad (n = 1, 2, ...) ,$$

then $(p_n)_0^\infty$ is compound geometric.

Proof. We show that the q_k defined by (3.1) are nonnegative. Since by (3.2) all p_n are positive we may write (3.1) as

$$(3.3) \qquad \frac{p_{n+1}}{p_n} = \sum_{k=0}^{n} q_k \frac{p_{n-k}}{p_n} \quad (n = 0, 1, 2, ...) .$$

From (3.3) together with (3.2) written as $p_{n+1}/p_n \geq p_n/p_{n-1}$ it follows that

$$(3.4) \qquad 0 \leq \frac{p_{n+1}}{p_n} - \frac{p_n}{p_{n-1}} = \sum_{k=0}^{n-1} q_k \left(\frac{p_{n-k}}{p_n} - \frac{p_{n-k-1}}{p_{n-1}} \right) + \frac{p_0}{p_n} q_n \ .$$

Since $q_0 \geq 0$ we may proceed by induction. Supposing the $q_k \geq 0$ for $k = 0, 1, ..., n-1$ from (3.4) it follows by use of (3.2) again that $q_n \geq 0$.

As a corollary to Lemma 3.2 we obtain from Corollary 2.5

Theorem 3.3. Log-convex lattice distributions are g.i.d.

By a simple approximation argument (see e.g. Steutel (1970), p. 89) together with the fact that the set of g.i.d. distributions is closed under weak convergence, we obtain

Theorem 3.4. Log-convex densities one $(0, \infty)$ are g.i.d.

Since completely monotone densities are log-convex we immdeiately have

Corollary 3.5. Completely monotone densities on $(0, \infty)$ are g.i.d.

Remark 1. The proof of Corollary 3.5. as given as Theorem 2.1 in Pillai and Sandhya (1990) makes use of the technique used in Steutel (1967), based on the location of the zeroes of $\sum_1^n p_k \lambda_k / (\lambda_k + s)$. This, elementary but rather cumbersome technique can be avoided by use of equation (3.1).

Remark 2. Since not all log-convex densities are completely monotone, not all g.i.d. densities are completely monotone (compare Theorem 2.2 in Pillai and Sandhya (1990)).

4. Mixtures

Since log-convexity is preserved under mixing, one might conjecture that the class of g.i.d. distributions is closed under mixing. The following example shows that this is not so. The p.g.f.'s $(2-z)^{-1}$ and $(2-z^2)^{-1}$ are (compound) geometric and hence g.i.d. For the mixture to be g.i.d. we need that in

$$\frac{\frac{1}{2}}{2-z} + \frac{\frac{1}{2}}{2-z^2} = \tfrac{1}{2} \left(1 - \frac{z + z^2 - z^3}{4 - z - z^2} \right)^{-1} ,$$

the function $\frac{1}{2}(z + z^2 - z^3)/(4 - z - z^2)$ is a p.g.f., which it is not as its coefficient of z^3 is negative. However we do have the following result.

Theorem 4.1. If ϕ is g.i.d., i.e. $\phi = (1-h)^{-1}$ with $h \in \mathcal{H}$, then

$$\gamma(t) = \int_0^\infty \frac{1}{1 - xh(t)} dG(x)$$

is a g.i.d. ch.f. for any distribution function G on $[0, \infty)$.

Proof. Since by Corollary 3.5.

$$g(s) = \int_0^\infty \frac{1}{1+xs} dG(x)$$

is the Laplace-Stieltjes transform (L.S.T) of a g.i.d. distribution it follows (cf. Theorem 2.2) that k defined by

$$k(s) = \exp(1 - \frac{1}{g(s)})$$

is the L.S.T. of an inf.div. distribution. As a consequence (see e.g. Feller (1971)) $k(-h(t))$ is an inf.div. ch.f. (see also Steutel (1971), p. 59), i.e. $\exp(1 - 1/\gamma(t))$ is an inf.div. ch.f., or γ is g.i.d.

Remark. The trivial identity $\phi = 1(1 - (1 - \phi^{-1}))$ implies that

$$\int_0^\infty \frac{1}{1 - x(1 - \phi^{-1})} dG(x)$$

is g.i.d. if y is g.i.d.

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