Solution to Problem 92-11* : On alternating multiple sums

theorem" by the authors and yet, if 92-1* were solved, I doubt that it could be made use of in order to prove the validity of (5). The underlying reason is, in my opinion, the pronounced asymmetry manifesting itself in the above problem when $b \neq c$, in contrast to $G(t)$ being of even parity in the theorem of [1, §3]. Hence, I wonder whether the latter can really be called a general sampling theorem. I think that proving (5) on the basis of a theorem as in [1, §3], requires two generalizations of the latter.

(1) The entire function $G(t)$ should be deprived of the requirement of having parity or being deduced from an entire function possessing parity by way of a linear transformation, as it is for instance the case with (4) when $b = c$:

$$(G(\lambda))_{b=c} = \frac{1}{\Gamma\left(\frac{b-\lambda}{a}\right)\Gamma\left(1-\frac{b-\lambda}{a}\right)} = \frac{1}{\pi}\sin\pi\frac{b-\lambda}{a};$$

(2) The theory should be extended to other types of boundary-value problems, not involving a differential equation but for instance a difference equation. This can be illustrated by means of the problem proposed in 92-1*. $\Gamma(x)$ is known not to satisfy a differential equation of finite order, but rather

$$\Gamma(x+1) = x\Gamma(x).$$

Hence, $1/\Gamma(x)$ is an entire function of

$$\Delta y(x) = \frac{1-x}{x}y(x), \quad \left(\frac{y(x)}{x}\right)_{x=0} = 1, \quad y(x) \neq 0 \quad \forall x \in \left[-\frac{1}{2}, \frac{1}{2}\right[\setminus\{0\},$$

where $\Delta y(x) := y(x+1) - y(x)$. The zeros of $y(x)$ are given by $\mathbb{Z}\setminus\mathbb{N}$. Similarly, (4) is not a solution of a differential equation, but satisfies

$$G(\lambda + a) = \frac{b-a-\lambda}{\lambda+a-c}G(\lambda).$$

In contrast, (4) with $b = c$ is a solution of the regular boundary-value problem

$$(G''(\lambda))_{b=c} = -\frac{\pi^2}{a^2}(G(\lambda))_{b=c}, \qquad G(b) = 0.$$

In my opinion, no solution of this kind to 92-1* can be found.

REFERENCE

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On Alternating Multiple Sums

Problem 92-11*, *by* MALTE HENKEL (Université de Genève, Switzerland) *and* R. A. WESTON (University of Durham, Durham, UK).

Consider the functions

(1) $$S(v) = \sum_{q_x=1}^{\infty}\sum_{q_y=0}^{\infty}(-1)^{q_x+q_y}q^{-1}\sin(2vq),$$

(2) $$C(v) = \sum_{q_x=1}^{\infty}\sum_{q_y=0}^{\infty}(-1)^{q_x+q_y}q^{-1}\cos(2vq),$$

where $q = \sqrt{q_x^2 + q_y^2}$. Numerical studies have led to the conjectures

$$(3) \qquad S(v) = -v/2 \quad \text{if } -\frac{\pi}{\sqrt{2}} < v < \frac{\pi}{\sqrt{2}}; \qquad C(v) = 0 \quad \text{if } v = \pm 5/4.$$

Prove or disprove (3). Find the general expressions for $C(v)$ and $S(v)$ for v arbitrary.

These sums arose in finite-size scaling studies of the three-dimensional spherical model.

Solution by J. BOERSMA *and* P. J. DE DOELDER (Eindhoven University of Technology, Eindhoven, the Netherlands).

Since $S(v)$ and $C(v)$ are odd and even functions of v, respectively, we may restrict the analysis to the case $v \geq 0$. Consider the function $f(v; x, y)$, with $v \geq 0$, defined by

$$f(v; x, y) = \begin{cases} \dfrac{\exp\left[2iv\sqrt{x^2+y^2}\right] - 1}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 2iv, \qquad (x, y) = (0, 0). \end{cases}$$

Its two-dimensional Fourier transform is found to be

$$\begin{aligned} \mathcal{F}\{f(v; x, y)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[2iv\sqrt{x^2+y^2}\right] - 1}{\sqrt{x^2+y^2}} e^{i\epsilon x + i\eta y} dx\, dy \\ &= 2\pi \int_0^{\infty} (e^{2iv\rho} - 1) J_0\left(\rho\sqrt{\xi^2+\eta^2}\right) d\rho \\ &= \begin{cases} 2\pi[i(4v^2 - \xi^2 - \eta^2)^{-1/2} - (\xi^2 + \eta^2)^{-1/2}], & \xi^2 + \eta^2 < 4v^2, \\ 2\pi[(\xi^2 + \eta^2 - 4v^2)^{-1/2} - (\xi^2 + \eta^2)^{-1/2}], & \xi^2 + \eta^2 > 4v^2, \end{cases} \end{aligned}$$

by means of some Hankel transforms from [1, form. 8.2(1), (32), (42)]. Next, by inversion of $\mathcal{F}\{f\}$ we obtain

$$\begin{aligned} f(v; x, y) &= \frac{1}{2\pi} \iint_{\xi^2+\eta^2<4v^2} [i(4v^2 - \xi^2 - \eta^2)^{-1/2} - (\xi^2 + \eta^2)^{-1/2}] e^{-ix\xi - iy\eta} d\xi d\eta \\ &+ \frac{1}{2\pi} \iint_{\xi^2+\eta^2>4v^2} [(\xi^2 + \eta^2 - 4v^2)^{-1/2} - (\xi^2 + \eta^2)^{-1/2}] e^{-ix\xi - iy\eta} d\xi d\eta. \end{aligned}$$

This representation is used in the double series

$$T(v) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (-1)^{m+n} f(v; m, n) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (-1)^{m+n} \frac{\exp\left[2iv\sqrt{m^2+n^2}\right] - 1}{\sqrt{m^2+n^2}},$$

yielding

$$\begin{aligned} T(v) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} (-1)^{m+n} \Bigg\{ &\iint_{\xi^2+\eta^2<4v^2} [i(4v^2 - \xi^2 - \eta^2)^{-1/2} \\ &\qquad - (\xi^2 + \eta^2)^{-1/2}] e^{-im\xi - in\eta} d\xi d\eta \\ &+ \iint_{\xi^2+\eta^2>4v^2} [(\xi^2 + \eta^2 - 4v^2)^{-1/2} \\ &\qquad - (\xi^2 + \eta^2)^{-1/2}] e^{-im\xi - in\eta} d\xi d\eta \Bigg\}. \end{aligned}$$

After a formal interchange of the summation and integrations we employ Poisson's summation formula to rewrite

$$\sum_{m=-\infty}^{\infty}(-1)^m e^{-im\xi} = 2\pi\sum_{k\text{ odd}}\delta(\xi-k\pi),$$
$$\sum_{n=-\infty}^{\infty}(-1)^n e^{-in\eta} = 2\pi\sum_{l\text{ odd}}\delta(\eta-\ell\pi).$$

As a result we find

$$T(v) = 2\pi\sum_{k,\ell\text{ odd}}\left\{\iint_{\xi^2+\eta^2<4v^2}[i(4v^2-\xi^2-\eta^2)^{-1/2}-(\xi^2+\eta^2)^{-1/2}]\delta(\xi-k\pi)\delta(\eta-\ell\pi)d\xi d\eta\right.$$
$$\left.+\iint_{\xi^2+\eta^2>4v^2}[(\xi^2+\eta^2-4v^2)^{-1/2}-(\xi^2+\eta^2)^{-1/2}]\delta(\xi-k\pi)\delta(\eta-\ell\pi)d\xi d\eta\right\}$$
$$=8\sum_{\substack{k,\ell\text{ odd }>0\\ h^2+\ell^2<4v^2/\pi^2}}[i(4v^2/\pi^2-k^2-\ell^2)^{-1/2}-(k^2+\ell^2)^{-1/2}]$$
$$+8\sum_{\substack{k,\ell\text{ odd }>0\\ k^2+\ell^2>4v^2/\pi^2}}[(k^2+\ell^2-4v^2/\pi^2)^{-1/2}-(k^2+\ell^2)^{-1/2}].$$

On the other hand, the double series $T(v)$ can be expressed as

$$T(v) = f(v;0,0)+4\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{\cos(2v\sqrt{m^2+n^2})-1+i\sin(2v\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}$$
$$=2iv-4\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{\sqrt{m^2+n^2}}+4C(v)+4iS(v).$$

The latter double series is evaluated by means of a result of Glasser and Zucker [2], viz.

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{\sqrt{m^2+n^2}} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{\sqrt{m^2+n^2}}$$
$$=-\log 2+\lim_{s\to\frac{1}{2}}[(1-2^{1-2s})\zeta(2s)-(1-2^{1-s})\beta(s)\zeta(s)]$$
$$=-\log 2+\log 2+(-1+2^{1/2})\beta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2}\right)=Q.$$

Finally, by identifying the real parts and the imaginary parts of the two results for $T(v)$, we find the general expressions

$$S(v) = -\frac{1}{2}v+2\sum_{\substack{k,\ell\text{ odd }<0\\ k^2+\ell^2>4v^2/\pi^2}}(4v^2/\pi^2-k^2-\ell^2)^{-1/2},$$
$$C(v) = -2\sum_{\substack{k,\ell\text{ odd }>0\\ k^2+\ell^2<4v^2/\pi^2}}(k^2+\ell^2)^{-1/2}+2\sum_{\substack{k,\ell\text{ odd }>0\\ k^2+\ell^2>4v^2/\pi^2}}[(k^2+\ell^2-4v^2/\pi^2)^{-1/2}$$
$$-(k^2+\ell^2)^{-1/2}]+Q,$$

valid for $v \geq 0$.

Let $0 \le v < \pi/\sqrt{2}$, so that $4v^2/\pi^2 < 2$; then

$$S(v) = -v/2, \qquad C(v) = 2 \sum_{k,\ell \text{odd}>0} [(k^2 + \ell^2 - 4v^2/\pi^2)^{-1/2} - (k^2 + \ell^2)^{-1/2}] + Q.$$

Thus the conjecture for $S(v)$ is correct. By expansion of $(k^2 + \ell^2 - 4v^2/\pi^2)^{-1/2}$ and by use of another result of Glasser [3], $C(v)$ can be expressed in terms of a single series, viz.

$$C(v) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(1-2^{-n-1/2})\beta\left(n+\frac{1}{2}\right)\zeta\left(n+\frac{1}{2}\right)\left(\frac{2v^2}{\pi^2}\right)^n, \quad v^2 < \pi^2/2,$$

where $\zeta(s)$ is the well-known Riemann zeta function and

$$\beta(s) = \sum_{\ell-0}^{\infty}(-1)^{\ell}(2\ell+1)^{-s}, \qquad s > 0.$$

We disprove the second conjecture by computing, from this expansion that the smallest zero of $C(v)$ is approximately 1.252129830.

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Also solved by SCOTT ALLEN *and* RAJ PATHRIA (University of Waterloo), DAVID BORWEIN (University of Western Ontario) *and* JONATHAN BORWEIN (University of Waterloo), C. C. GROSJEAN (University of Ghent, Belgium), NORBERT ORTNER *and* PETER WAGNER (University of Innsbruck).

Editorial Note. Since the solutions for this problem were of uniformly high quality, although lengthy, the editors regret that space limitations preclude publishing all of them. The Borweins decided to submit their solution in the form of a paper, *On some trigonometric and exponential lattice sums*, to the SIAM Journal on Mathematical Analysis. Grosjean presented accurate, detailed, numerical procedures, which included two finite, high precision approximations to compute $C(v)$ in $\frac{\pi}{\sqrt{2}} < v < \frac{\pi}{\sqrt{2}}$. Ortner and Wagner gave a careful discussion of the distributional setting and convergence questions. Generalizations to Bessel functions were mentioned by Grosjean, as well as Allen and Pathria who also considered sums in arbitrary dimensions and refer to their paper, *Analytical evaluation of a class of phase-modulated lattice sums*, J. Math. Phys. (in press). Grosjean proved the surprising result, for all real x:

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} J_0\left(x\sqrt{m^2+n^2}\right) = 0.$$

Circular Binary Sequences

Problem 92-12, *by* MARK STAMP (Worcester Polytechnic Insitute).

The binomial coefficient $\binom{n}{k}$ can be interpreted as the number of distinct binary sequences of length n with exactly k ones. Let $\left\langle {n \atop k} \right\rangle$ be the number of binary sequences of length n with k