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L.C.G.J.M. Habets

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The Netherlands

Eindhoven University of Technology
Department of Mathematics and Computing Science
Probability theory, statistics, operations research and systems theory
P.O. Box 513
5600 MB Eindhoven - The Netherlands

Secretariate: Dommelbuilding 0.03
Telephone: 040-47 3130

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L.C.G.J.M. Habets
Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
NL-5600 MB Eindhoven
The Netherlands

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Abstract

The investigation of the stability of an exponential polynomial is a well-known problem in the literature. Although exact analytic conditions to check stability are known, they are often too difficult to check, certainly in practical problems. Therefore mostly a graphical test is used, based on the well-known circle-criterion. In this paper an algorithmization of such a test is presented. Although the method can be applied to low order exponential polynomials, it is especially suitable to test the stability of high order exponential polynomials, often needed for the stabilization of time-delay systems. The method proposed in this paper is designed to carry out this test in a reliable and efficient way.

Key Words: Exponential polynomials, Stability, Circle-criterion, Variable step-length, Curvature.

1 Introduction

Exponential polynomials are analytic functions of the form

$$f(z) = z^n + \sum_{i=0}^{n-1} p_i(e^{-\tau_1 z}, \dots, e^{-\tau_k z}) \cdot z^i, \qquad z \in \mathbf{C}, \tag{1}$$

where for all $i \in \{0, 1, \dots, n-1\}$, the function $p_i(e^{-\tau_1 z}, \dots, e^{-\tau_k z})$ is a polynomial in the variables $e^{-\tau_1 z}, \dots, e^{-\tau_k z}$ with real coefficients, and all $\tau_j > 0$ $(j = 1, \dots, k)$. Note that we assume that an exponential polynomial is monic, i.e. the coefficient of the highest power of z is 1. After substitution of $\sigma_j := e^{-\tau_j z}$, $(j = 1, \dots, k)$, f becomes a real polynomial in the indeterminates $z, \sigma_1, \dots, \sigma_k$. This explains the name *exponential polynomial*: there is a strong relationship with both exponential functions and polynomials in more than one variable. Nevertheless, exponential polynomials are often called *quasi-polynomials* (see for example [4]).

Exponential polynomials are useful for the characterization of the concept of stability for systems with time-delays. Let $\tau_1, \dots, \tau_k$ be an k-tuple of incommensurate time-delays and introduce the delay operators σ_j $(j = 1, \dots, k)$, which act on the state x of the system: $\sigma_j x(t) = x(t - \tau_j)$. Let $A(\sigma_1, \dots, \sigma_k)$ be an $n \times n$ matrix with entries in the polynomial ring $\mathbf{R}[\sigma_1, \dots, \sigma_k]$. Interpreting the indeterminates $\sigma_1, \dots, \sigma_k$ as delay-operators as defined above, the equation

$$\dot{x}(t) = A(\sigma_1, \dots, \sigma_k) x(t) \tag{2}$$

describes an autonomous dynamical system with time-delays. The stability of this system is determined by its characteristic function $\chi_A(z) = \det(zI - A(\sigma_1, \dots, \sigma_k))$. After substitution of $\sigma_j := e^{-\tau_j z}$ $(j = 1, \dots, k)$ we obtain an exponential polynomial

$$\tilde{\chi}_A(z) = \det(zI - A(e^{-\tau_1 z}, \dots, e^{-\tau_k})). \tag{3}$$

Now the system (2) is stable if and only if the exponential polynomial $\tilde{\chi}_A(z)$ has no zeros in the closed right half-plane (RHP) (see [5, p. 182]). Therefore an exponential polynomial f itself is called stable if it satisfies the condition

$$\forall z \in \mathbf{C} : f(z) = 0 \Longrightarrow z \in \mathbf{C}^-.$$

Exact analytic conditions to test the stability of an exponential polynomial are known in the literature (see [1, ch. 13] or the original paper [7] of Pontrjagin for the commensurate delay case. The non-commensurate delay case is described in [3]). Unfortunately, in somewhat more complicated examples these conditions are impossible to check. They are simply too difficult. Therefore in practice a graphical test, based on the well-known circle-criterion, is mostly used. However, an algorithm to carry out this test can only check the stability of an exponential polynomial based on a finite number of points on the curve. From our own experience we know that this does not always yield reliable results. In the present paper we propose an algorithmization of this graphical method which overcomes this problem.

The paper is organized as follows. In Section 2 we give a short introduction on the well-known circle-criterion with special emphasis on exponential polynomials. With this result it is possible to determine the exact number of RHP-zeros of an exponential polynomial with a graphical test. The next two sections are devoted to the practical implementation. It is shown that only a part of the image of the imaginary axis has to be considered to determine the number of RHP-zeros. Moreover, a reliable method is given to search along this part of the imaginary axis for the crucial information. Variable step-length is the key word here. Then an example is given to illustrate the method derived in this paper. Although it is not really faster, the method is far more reliable than the algorithm that was commonly used for this problem up to now. Finally we draw some conclusions.

2 The circle-criterion for exponential polynomials

The circle-criterion is a well-known and often used result from complex analysis to determine the number of zeros of an analytic function in an area enclosed by a Jordan-curve. In this section this criterion is specialized to the case of exponential polynomials. Given an exponential polynomial f it is shown how the number of RHP-zeros of f is determined with help of the image of the imaginary axis under the function f.

Lemma 2.1 *An exponential polynomial has only a finite number of zeros in the closed RHP.*

Proof Consider an arbitrary exponential polynomial f given by

$$f(z) = z^n + \sum_{i=0}^{n-1} p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z}) \cdot z^i.$$

Because for all $i \in \{0, 1, \ldots, n-1\}$, $p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z})$ is a polynomial in $e^{-\tau_1 z}, \ldots, e^{-\tau_k z}$, we know that $|p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z})|$ is bounded in the closed RHP. So, when $\text{Re}(z) \geq 0$ and $|z|$ becomes large, the term z^n is the dominant term in f. Therefore there exists an $R \in \mathbf{R}$ such that

$$\forall z \in \mathbf{C}, |z| > R, \text{Re}(z) \geq 0 : |f(z)| \geq |z|^n - \sum_{i=0}^{n-1} |p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z})| \cdot |z|^i > 0.$$

Hence, all the RHP-zeros of f lie in the half disk $D = \{z \in \mathbf{C} \mid \text{Re}(z) \geq 0, |z| \leq R\}$. Since f is an analytic function and D is a compact set, f has only a finite number of zeros inside D. This proves the claim. ∎

The exact number of RHP-zeros of an exponential polynomial f can be determined with the following well-known result from complex analysis (see for example [9, pp. 115,116]).

Proposition 2.2 *Let Γ be a Jordan-curve in the complex plane. Consider a function f which is analytic inside and on the curve Γ. Assume that f has no zeros on Γ. Then the number of zeros of f inside Γ is given by*

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz. \tag{4}$$

∎

We will apply Proposition 2.2 to exponential polynomials, with Jordan-curves defined in the following way.

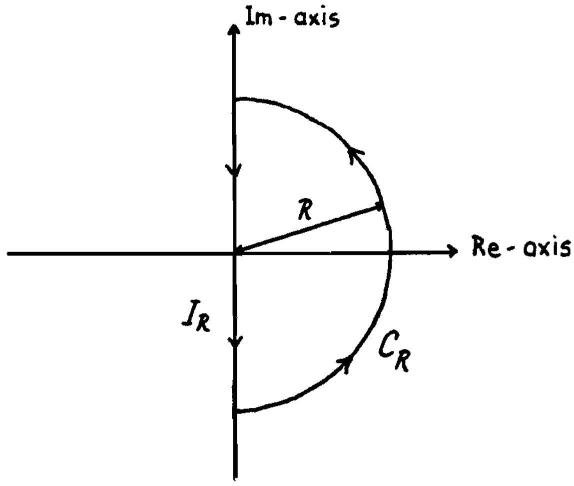

Figure 1: Jordan-curve $\mathcal{J}_R$

Definition 2.3 Let $R \in \mathbf{R}^+$. Then the half circle C_R is defined as

$$C_R := \{z \in \mathbf{C} \mid |z| = R \wedge \mathrm{Re}(z) \geq 0\},$$

the part I_R of the imaginary axis as

$$I_R := \{z \in \mathbf{C} \mid \mathrm{Re}(z) = 0 \wedge |z| < R\},$$

and the Jordan-curve $\mathcal{J}_R$ as

$$\mathcal{J}_R := C_R \cup I_R.$$

This Jordan-curve is traversed in counter clockwise direction as depicted in Figure 1.

Now suppose that the exponential polynomial f has no zeros on the imaginary axis. Let N_f denote the number of zeros of f in the open RHP. When R becomes large enough, $\mathcal{J}_R$ will enclose all the N_f zeros of f. Hence

$$N_f = \lim_{R \to \infty} \frac{1}{2\pi\imath} \oint_{\mathcal{J}_R} \frac{f'(z)}{f(z)} dz. \tag{5}$$

The computation of the integral (5) can be split into two parts: the integral over the half circle C_R, and over the imaginary axis I_R. The first term can be determined analytically, but before we can state this result we need some preliminary lemmas.

Lemma 2.4 *Let $f(z)$ be an exponential polynomial of the form*

$$f(z) = z^n + \sum_{i=0}^{n-1} p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z}) z^i. \tag{6}$$

Define

$$A(z) := \frac{d}{dz}(p_{n-1}(e^{-\tau_1 z}, \ldots, e^{-\tau_k z})). \tag{7}$$

Then for large values of $|z|$, such that $\mathrm{Re}(z) \geq 0$ *we have*

$$\frac{f'(z)}{f(z)} = \frac{n}{z} + \frac{A(z)}{z} + O(\frac{1}{z^2}). \tag{8}$$

Proof Because all the functions $p_i(e^{-\tau_1 z},\dots,e^{-\tau_k z})$ are polynomial in $e^{-\tau_1 z},\dots,e^{-\tau_k z}$, they are bounded in the closed RHP, and the same holds true for their derivatives ($A(z)$ is also bounded in the RHP). For large values of $|z|$ in the closed RHP this implies that

$$f'(z) = nz^{n-1} + A(z)\cdot z^{n-1} + O(z^{n-2}).$$

Hence

$$\frac{f'(z)}{f(z)} - \frac{n}{z} - \frac{A(z)}{z} = \frac{zf'(z) - nf(z) - A(z)f(z)}{zf(z)} =$$

$$= \frac{nz^n + A(z)z^n - nz^n - A(z)z^n + O(z^{n-1})}{z^{n+1} + O(z^n)} = O(\frac{1}{z^2}).$$

■

Lemma 2.5 *Let $\alpha \in \mathbf{R}$, $\alpha > 0$. Then*

$$\lim_{R\to\infty} \frac{1}{2\pi\imath}\int_{C_R} \frac{e^{-\alpha z}}{z} dz = 0. \tag{9}$$

Proof

$$\left|\frac{1}{2\pi\imath}\int_{C_R} \frac{e^{-\alpha z}}{z} dz\right| = \left|\frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-\alpha R(\cos(\omega)+\imath\sin(\omega))} d\omega\right| \le \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} e^{-\alpha R\cos(\omega)} d\omega.$$

Since $\alpha > 0$, the inequality $-\alpha R\cos(\omega) \le \alpha R(\frac{2}{\pi}\omega - 1)$ holds on the whole interval $[0, \frac{\pi}{2}]$. Therefore:

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} e^{-\alpha R\cos(\omega)} d\omega \le \frac{1}{\pi}\int_0^{\frac{1}{2}\pi} e^{\alpha R(\frac{2}{\pi}\omega - 1)} d\omega = \frac{1}{2\alpha R}(1 - e^{-\alpha R}) \longrightarrow 0$$

when R tends to infinity. ■

Proposition 2.6 *Let f be an exponential polynomial as defined in (6). Then*

$$\lim_{R\to\infty} \frac{1}{2\pi\imath}\int_{C_R} \frac{f'(z)}{f(z)} dz = \frac{n}{2}. \tag{10}$$

Proof Because we are interested in the asymptotic behavior of (10) for $R \to \infty$, while z remains in the RHP, we can use (8) to prove (10). First of all

$$\frac{1}{2\pi\imath}\int_{C_R} \frac{n}{z} dz = \frac{n}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\omega = \frac{n}{2}. \tag{11}$$

$A(z)$ was defined in (8) as the derivative of $p_{n-1}(e^{-\tau_1 z},\dots,e^{-\tau_k z})$, where p_{n-1} is considered as a polynomial in $e^{-\tau_1 z},\dots,e^{-\tau_k z}$. Therefore $A(z)$ is a linear combination of functions of the form $e^{-\alpha z}$, with $\alpha > 0$. Applying Lemma 2.5, we obtain

$$\lim_{R\to\infty} \frac{1}{2\pi\imath}\int_{C_R} \frac{A(z)}{z} dz = 0. \tag{12}$$

Finally,

$$\lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{C_R}\frac{1}{|z|^2}dz=\lim_{R\to\infty}\frac{1}{2\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}e^{\imath\omega}d\omega=\lim_{R\to\infty}\frac{1}{\pi R}=0. \tag{13}$$

Combination of (8), (11), (12) and (13) yields the required result. ∎

With help of the last proposition it is possible to rewrite equality (5). In this way the following expression for the number of zeros in the RHP of an exponential polynomial f is obtained:

$$N_f=\frac{n}{2}+\lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{I_R}\frac{f'(z)}{f(z)}dz=\frac{n}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f'(\imath\omega)}{f(\imath\omega)}d\omega. \tag{14}$$

To compute N_f we need a method to determine the second term in (14). The next result shows how this can be done by inspection of the image under f of the imaginary axis.

Theorem 2.7 *Let f be an exponential polynomial as defined in (6). Assume that $f(0)>0$ and that f has no zeros on the imaginary axis. Let N_f denote the number of zeros of f in the closed RHP. Then*

$$N_f=\frac{n}{2}-\frac{1}{\pi}\cdot\mathrm{totarg}(f(\imath\infty)), \tag{15}$$

where $\mathrm{totarg}(f(\imath\infty))$ *is the net increase of the argument of $f(z)$ when z traverses along the imaginary axis from $z=0$ to $z=\imath\infty$.*

Proof Because the function f is a real function on the real axis, the second term of (14) can be written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f'(\imath\omega)}{f(\imath\omega)}d\omega=\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{f'(\imath\omega)}{f(\imath\omega)}\right)d\omega.$$

Define $u(\omega):=\mathrm{Re}\,(f(\imath\omega))$ and $v(\omega):=\mathrm{Im}\,(f(\imath\omega))$. Then

$$\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{f'(\imath\omega)}{f(\imath\omega)}\right)d\omega=\frac{1}{\pi}\int_0^{\infty}\frac{v'(\omega)u(\omega)-u'(\omega)v(\omega)}{u^2(\omega)+v^2(\omega)}d\omega=$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{d}{d\omega}\arctan\left(\frac{v(\omega)}{u(\omega)}\right)=\frac{1}{\pi}\int_0^{\infty}\frac{d}{d\omega}\arctan(\tan(\arg(f(\imath\omega))))=$$

$$=\frac{1}{\pi}\cdot(\mathrm{totarg}(f(\imath\infty))-\arg(f(0)))=\frac{1}{\pi}\mathrm{totarg}(f(\imath\infty)).$$

Substitution of this last formula in (14) completes the proof. ∎

Theorem 2.7 gives rise to a graphical method to compute the number of RHP-zeros N_f of an exponential polynomial f. The image under the function f of the positive imaginary axis is needed to determine the final result. The rest of this paper is devoted to the question how this search along the positive imaginary axis and its image can be carried out in a reliable and efficient way.

3 Bounds on the search along the imaginary axis

As already mentioned in the proof of Lemma 2.1, the term z^n of the exponential polynomial f in (6) is the dominant term of this function for large values of $|z|$ in the closed RHP, including the imaginary axis. This implies that $\arg(f(\imath\omega))$ converges for large values of ω because with growing ω the term $(\imath\omega)^n$ becomes more and more dominant. It is not difficult to see that

$$\lim_{\omega\to\infty} \arg(f(\imath\omega)) = \frac{n \bmod 4}{2} \cdot \pi. \tag{16}$$

(Note however that $f(\imath\omega)$ itself does not converge.)

From the asymptotic behavior of the argument of $f(\imath\omega)$, it follows that the function $f(\imath\omega)$ itself has a half-plane of convergence. This means that there exists an $K \in \mathbf{R}$ such that for all $\omega > K$, the value of $f(\imath\omega)$ remains in the half-plane determined by the degree of f in z as described in (16). Choose this K as small as possible:

$$K := \min\{\beta \in \mathbf{R} \mid \forall\omega > \beta : |\mathrm{totarg}(f(\imath\omega)) - \mathrm{totarg}(f(\imath\infty))| < \frac{\pi}{2}\}. \tag{17}$$

Then it is clear that at $\omega = K$, i.e. at the moment that $f(\imath\omega)$ enters the half-plane of convergence for the last time to remain there forever, the value of $\mathrm{totarg}(f(\imath\infty))$ is completely known: the difference between $\mathrm{totarg}(f(\imath K))$ and its final value is at most $\frac{\pi}{2}$.

The observations above lead to the following conclusion. For the computation of the value of $\mathrm{totarg}(f(\imath\infty))$, only the behavior of $f(\imath\omega)$ for $\omega \in [0, K]$ is important. Thus for an efficient test, we should look for a sharp upper bound $K_{\max}$ for K. In this way we obtain a search interval $[0, K_{\max}]$ which is as small as possible, but still contains all the information required for the determination of the number of RHP-zeros of f.

Let f be an exponential polynomial as defined in (6) and let K_1 be a positive real number such that

$$\forall\omega > K_1 : |\imath\omega|^n > \left|\sum_{i=0}^{n-1} p_i(e^{-\imath\tau_1\omega}, \ldots, e^{-\imath\tau_k\omega})(\imath\omega)^i\right|. \tag{18}$$

Such an K_1 exists because on the imaginary axis z^n is the dominant term of f. But then it is clear that we can choose $K_{\max} = K_1$, because for $\omega > K_1$, $|\imath\omega|^n$ is so large that $(\imath\omega)^n$ determines the half-plane in which $f(\imath\omega)$ will be for all $\omega > K_1$. This must be the half-plane of convergence. Therefore formula (18) can be used to obtain an upper bound $K_{\max}$ for K.

Proposition 3.8 *Let f be an exponential polynomial as defined in (6) and suppose that $n \geq 2$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbf{R}$ be such that*

$$\forall i \in \{0, 1, \ldots, n-1\}\ \forall\omega \in \mathbf{R} : |p_i(e^{-\imath\tau_1\omega}, \ldots, e^{-\imath\tau_k\omega})| \leq \alpha_i \tag{19}$$

(Because all p_i are polynomials in $e^{-\tau_1 z}, \ldots, e^{-\tau_k z}$, and for $z = \imath\omega$ we have $|e^{-\imath\tau_j\omega}| = 1$, an upper bound for α_i can be obtained by summation of all the absolute values of the coefficients of p_i). Define

$$\alpha_{\max} := \max\{\alpha_i \mid i = 0, \ldots, n-2\}, \qquad K_{\max} := \sqrt{\alpha_{\max}} + \max(1, \alpha_{n-1}).$$

Then $K_{\max}$ is an upper bound for K.

Proof It suffices to show that $K_{\max}$ satisfies (18). Let $\omega > K_{\max}$ and $z = \imath\omega$. Then $|z| - 1 > \sqrt{\alpha_{\max}}$. So

$$\left|\sum_{i=0}^{n-1} p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z}) z^i\right| \le \sum_{i=0}^{n-1} |p_i(e^{-\tau_1 z}, \ldots, e^{-\tau_k z})|\,|z|^i \le \alpha_{n-1}|z|^{n-1} + \sum_{i=0}^{n-2} \alpha_i |z|^i \le$$

$$\le \alpha_{n-1}|z|^{n-1} + \alpha_{\max} \sum_{i=0}^{n-2} |z|^i = \alpha_{n-1}|z|^{n-1} + \alpha_{\max} \frac{|z|^{n-1} - 1}{|z| - 1} \le$$

$$\le \alpha_{n-1}|z|^{n-1} + \sqrt{\alpha_{\max}}(|z|^{n-1} - 1) \le (\alpha_{n-1} + \sqrt{\alpha_{\max}})|z|^{n-1} \le K_{\max}|z|^{n-1} < |z|^n.$$

■

For higher order exponential polynomials with large coefficients it is possible to derive sharper bounds for K. The next proposition states such an alternative result. The proof is omitted; except for some technical details it is analogous to the previous proof.

Proposition 3.9 *Let f be an exponential polynomial as defined in (6), and suppose that $n \ge 3$. Choose $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbf{R}$ such that they satisfy*

$$\forall i \in \{0, 1, \ldots, n-1\}\ \forall \omega \in \mathbf{R} : |p_i(e^{-\imath\tau_1\omega}, \ldots, e^{-\imath\tau_k\omega})| \le \alpha_i.$$

Define $\alpha_{\max} := \max\{\alpha_i \mid i = 0, \ldots, n-3\}$. Then $K_{\max} := \sqrt[3]{\alpha_{\max}} + \max(1, \alpha_{n-1} + \sqrt{\alpha_{n-2}})$ is an upper bound for K. ■

In the same way we can continue. Based on formula (18), smaller upper bounds for K can be derived by taking more terms α_i separately into account. It depends on the exponential polynomial f under consideration which estimation method is the best. Clearly, the definitive value of $K_{\max}$ is chosen to be the minimum of all the computed upper bounds.

4 Determination of the number of RHP-zeros

In the last section it was shown that the number of RHP-zeros of an exponential polynomial f is determined by the behavior of the function $f(\imath\omega)$ on the finite interval $[0, K]$. Moreover, an upper bound $K_{\max}$ for K was derived. This section is devoted to the question how the value of $\operatorname{totarg}(f(\imath\infty))$ can be computed with help of the image $f(\imath\omega)$ of f on the interval $\omega \in [0, K_{\max}]$. Again we assume that f has no zeros on the imaginary axis, and that $f(0) > 0$.

From formula (17) it is obvious that $\operatorname{totarg}(f(\imath\infty))$ is completely determined by the value of $\operatorname{totarg}(f(\imath K_{\max}))$. This value is the sum of $\arg(f(\imath K_{\max}))$ and 2π times the number of complete encirclements of the curve $\Gamma = \{f(\imath\omega) \mid \omega \in [0, K_{\max}]\}$ of the origin of the complex plane. In order to count these encirclements, the curve Γ has to be followed accurately from $\omega = 0$ to $\omega = K_{\max}$. However, for the determination of the number of encirclements of the origin, it is only interesting where the curve Γ enters and leaves the half-plane of convergence and in what direction. Because the boundary of this half-plane is the real axis (when the degree n of the exponential polynomial is odd) or the imaginary axis (when n is even), only intersections with these axes have to be considered. Then it is easily seen that an intersection with one of the axes contributes to the number of encirclements as depicted in Figure 2.

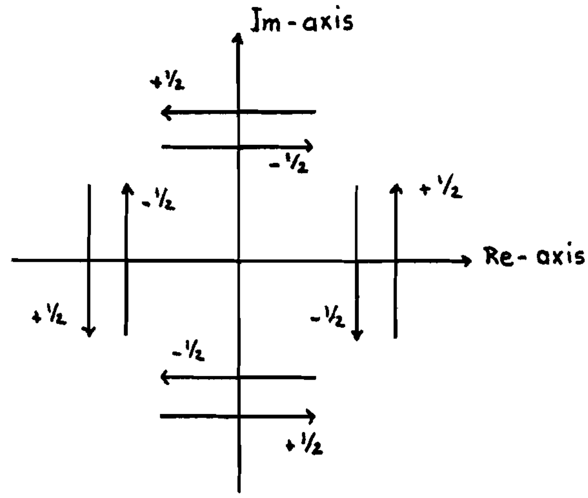

Figure 2: Counting the number of encirclements

The only problem left is to track the curve Γ of $f(\omega)$ for $\omega \in [0, K_{\max}]$ in an accurate way that ascertains that all the intersections with the real axis (when n is odd) or the imaginary axis (when n is even) are detected. However, in an algorithm only a finite number of points on the curve Γ can be calculated. Therefore the tracking problem can be reformulated as the question to find a method for the selection of a finite number of points on the curve Γ in such a way that all the intersections of Γ with the real and imaginary axis can be detected from this finite set of points. For this purpose, linear search in the parameter space with constant step length ℓ is until now most commonly used. Unfortunately, this method is not always reliable as will be illustrated by an example in the next section. Therefore we propose another, more reliable method to overcome this problem. The main idea is to make the step length variable depending on the curvature of the curve Γ.

For a curve $\Gamma = \{(u(\omega), v(\omega)) \mid \omega \in [0, K]\}$ in a two-dimensional plane, parametrized by the variable ω, the *curvature* in a point $\bar{x}(\omega_0) = (u(\omega_0), v(\omega_0))$ at $\omega = \omega_0$, is given by

$$\frac{\dot{u}(\omega_0)\ddot{v}(\omega_0) - \dot{v}(\omega_0)\ddot{u}(\omega_0)}{(\dot{u}^2(\omega_0) + \dot{v}^2(\omega_0))^{3/2}}. \tag{20}$$

(See for example [8, pp. 13-15]; the formula can be found literally in [2, p. 590]). The curvature in a point $\bar{x}(\omega_0)$ on Γ is a measure for the rate of change of the tangent in $\bar{x}(\omega_0)$, when proceeding along the curve. For example, a circle with radius R has in every point a curvature of $\frac{1}{R}$. It is easily seen that in a small neighborhood of the point $\bar{x}$ on the curve Γ, with curvature $k_{\bar{x}}$, the curve Γ itself behaves like a circle with radius $|\frac{1}{k_{\bar{x}}}|$. This implies that to track the curve accurately we have to take small steps along the curve when the curvature is large in absolute value, and we can take somewhat larger steps when the absolute value of the curvature is small. In this way we obtain the following rule:

$$|\text{curvature}| \times \text{step length along the curve} = \text{constant} \tag{21}$$

Note that the curve $\{(u(\omega), v(\omega)) \mid \omega \in [0, K]\}$ is parametrized by ω and not by its length along the curve. The length of the curve between ω_0 and $\omega_0 + \Delta\omega$ is given by

$$\int_{\omega_0}^{\omega_0+\Delta\omega} \sqrt{\dot{u}^2(\omega) + \dot{v}^2(\omega)}d\omega. \tag{22}$$

For small values of $\Delta\omega$, this integral is approximately $\sqrt{\dot{u}^2(\omega_0) + \dot{v}^2(\omega_0)}\Delta\omega$. Substitution of this formula and (20) in (21) yields

$$\frac{|\dot{u}(\omega_0)\ddot{v}(\omega_0) - \dot{v}(\omega_0)\ddot{u}(\omega_0)|}{\dot{u}^2(\omega_0) + \dot{v}^2(\omega_0)} \cdot \Delta\omega = C.$$

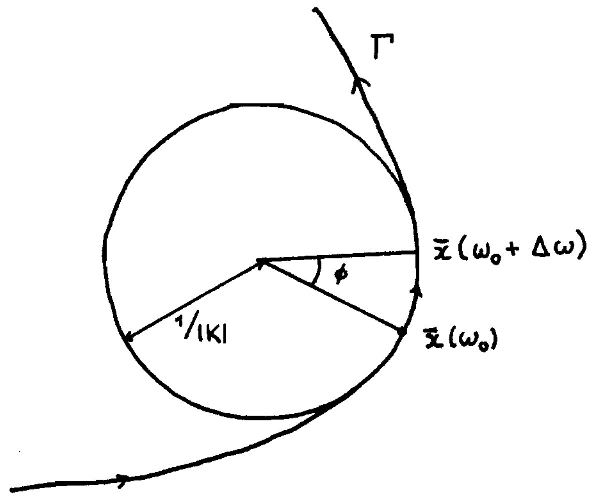

Figure 3: The relation between C and ϕ

In this way we obtain the following formula for the step length at $\omega = \omega_0$ in terms of the parameter ω:

$$\Delta\omega = C \cdot \frac{\dot{u}^2(\omega_0) + \dot{v}^2(\omega_0)}{|\dot{u}(\omega_0)\ddot{v}(\omega_0) - \dot{v}(\omega_0)\ddot{u}(\omega_0)|}. \tag{23}$$

However, in this formula there is still one unknown: the parameter C.

Consider in Figure 3 the point $\bar{x}$ on the curve Γ. Assume that the curvature of Γ in $\bar{x}$ is K. Then the curve Γ behaves in a neighborhood of $\bar{x}$ like a circle with radius $\frac{1}{|K|}$, as depicted in the figure. For small values of the angle ϕ, the step-length along the curve Γ and along the circle are almost the same. So the step-length along the curve Γ is approximately $\phi/|K|$. Therefore $C = |K| \cdot \frac{1}{|K|} \cdot \phi = \phi$. Hence, C can be seen as the angle of rotation along the curve between two successively calculated points. A smaller choice of C leads to a more accurate tracking of the curve, but also to more computations. In practice one has to find a trade-off between accuracy and computational expenses. In our case, the accuracy has to be large enough to detect all intersections of the curve $\{f(\imath\omega) \mid \omega \in [0, K_{\max}]\}$ with the real and imaginary axis. The choice of the actual value of C must be based on this condition, and the interpretation of this parameter as given above.

Remark 4.10 It is possible that in some points on the curve Γ the absolute value of the curvature is very small. In this case, the step length obtained with formula (23) is probably too large. This undesirable situation can be solved beforehand by defining an upper bound for the maximal step length. Moreover, in a practical implementation it is recommendable to restrict the growth of the step length by an (exponential) growth bound.

5 An example

To illustrate the advantage of the variable step length method proposed in the last section in comparison with the linear search method, we have implemented both algorithms in MATLAB. This can be done rather straightforwardly: points on the curve $\Gamma = \{f(\imath\omega) \mid \omega \in [0, K_{\max}]\}$ are easily calculated; for the variable step length, formula (23) is used. Intersections with the real and imaginary axis are detected by a change of the signs of the imaginary and real parts of $f(\imath\omega)$ in two successive points. The rest of this section describes an experiment to compare the performances of both methods.

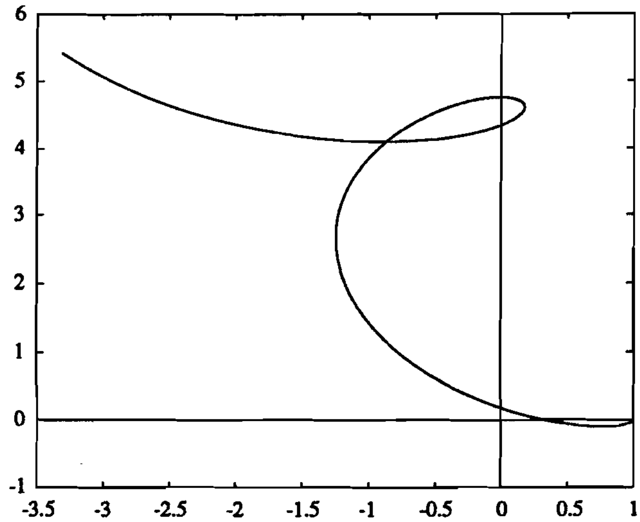

Figure 4: The curve Γ

Consider the following exponential polynomial:

$$f(z) = z^2 + (2 - e^{-z} - e^{-2z} - e^{-3z} - e^{-4z}) \cdot z + (2 - e^{-z}). \tag{24}$$

This polynomial has degree $n = 2$, so the LHP is the half-plane of convergence. First we apply Proposition 3.8 to find an upper bound $K_{\max}$ for the search interval $[0, K]$. In the notation of Proposition 3.8, we have $\alpha_1 = 6$, $\alpha_{\max} = \alpha_0 = 3$. So $K_{\max} = 6 + \sqrt{3}$. The interesting part of the curve $\Gamma = \{f(\imath\omega) \mid \omega \in [0, 6 + \sqrt{3}]\}$, for $\omega \in [0, 2]$, is depicted in Figure 4. The rest of the curve Γ, for $\omega \in [2, 6 + \sqrt{3}]$ is not very important; on this interval Γ remains in the LHP. This indicates that the upper bound $K_{\max}$ for K is not very sharp. From Figure 4 it is immediately clear that $\text{totarg}(f(\imath\infty)) = \pi$ because the curve Γ crosses the *positive* imaginary axis on its way to the half-plane of convergence. So, according to Theorem 2.7, the number N_f of RHP-zeros of f is given by:

$$N_f = \frac{n}{2} - \frac{1}{\pi} \cdot \text{totarg}(f(\imath\infty)) = \frac{2}{2} - \frac{1}{\pi} \cdot \pi = 0.$$

Hence f is a stable exponential polynomial.

The same conclusion on the number of RHP-zeros can be obtained by a linear search with step length 0.1 along the interval $[0, 6+\sqrt{3}]$. An intersection of the curve Γ with the imaginary axis is detected by a change of the signs of the real parts of two successive points. In this way only a half encirclement around the origin is counted, and in this way we obtain exactly the same result. Also the search method using variable step length, based on the curvature can be applied. In this simple case, this advanced method is of course not necessary, but it yields the same result.

The advantage of the variable step length method becomes clear when we consider the exponential polynomial

$$g(z) = (f(z))^6, \tag{25}$$

where f is defined by (24). This polynomial has degree $n = 12$, so the RHP is the half-plane of convergence. Moreover, because f is a stable polynomial, g is stable too. However, in this case the application of Theorem 2.7 is not so easy because it is very complicated to obtain the curve $\Gamma = \{g(\imath\omega) \mid \omega \in [0, K_{\max}]\}$ explicitly.

First $K_{\max}$ is computed with help of Proposition 3.9. In this way $K_{\max} = 88.4$ is obtained. This upper bound for K is not very sharp. Using an anologous estimation method, and taking one more term (α_{n-3}) separately into account, it is possible to reduce $K_{\max}$ in this case to $K_{\max} = 45.3$.

To carry out a linear search along the interval $[0, 45.3]$, first the step length ℓ has to be determined. This is a rather difficult problem because a step length which is too large can lead to wrong conclusions on the number of RHP-zeros. It is possible that some crossings of the real or imaginary axis are not detected. This is dangerous because it is impossible to see this from the data afterwards. The choice of an appropriate step length is dependent on the problem under consideration and almost impossible to predict beforehand. In the case of the exponential polynomial g, it turns out (using trial and error) that the step length must be smaller or equal to $\ell = 0.008$. This means that at least $\frac{45.3}{0.008} = 5663$ points on the curve Γ have to be computed to derive a correct conclusion on the number of RHP-zeros of g.

Application of the method with variable step length is much less tricky. Only the value of C has to be chosen. Fortunately, this parameter C has a clear interpretation as explained in Section 4. Therefore the choice it not so problem dependent. From our practical experience, it turned out that a choice of $C = 0.25 \approx \frac{\pi}{12}$ is appropriate in most cases. However, also Remark 4.10 has to be taken into account. In our case we defined an upper bound of 0.25 for the step length, and imposed an exponential growth bound of 2. This means that the step length in a new step is at most twice as large as in the last step. This method was started with a beginning step length of 0.001, because especially at low frequencies the step length has to be very small. In this way, the number of RHP-zeros of g was correctly determined, computing only 631 points on the graph of $g(\imath\omega)$. However, note that the computational expenses for each step are much higher than in the linear search method. Therefore the variable step length method is not really faster. Its main advantage is its improved reliability: it takes small steps where this is required and somewhat larger steps when it is possible.

6 Conclusions

In this paper we proposed an algorithm for the determination of the number of RHP-zeros of an exponential polynomial. We started with a specialization of the well-known circle-criterion to the case of exponential polynomials. In this way it was shown how the number of RHP-zeros of an exponential polynomial f can be determined from the curve $\Gamma = \{f(\imath\omega) \mid \omega \in [0, K]\}$ in the complex plane. First upper bounds $K_{\max}$ for K were derived. Then two possible algorithmizations to replace an explicit graphical test were introduced. The first, based on linear search along the interval $[0, K_{\max}]$ is only well suited for low order exponential polynomials. In the second method, one passes through the interval $[0, K_{\max}]$ with a variable step length, based on the curvature of Γ in the successively calculated points. With help of an example it was shown that the second method is especially useful in rather complicated cases. For low order exponential polynomials this method is unnecessarily advanced; in this case the linear search method is probably good enough. However, high order exponential polynomials do often occur, for example in the constructive solution of the stabilization problem for time-delay systems (see [6]). To solve this problem, the stability of high order exponential polynomials has to be investigated. For this purpose the algorithm with variable step length is an appropriate method: it is far more reliable than linear search.

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