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An arbitrary high-order discontinuous Galerkin method with local time-stepping for linear acoustic wave propagation

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ABSTRACT:
This paper presents a numerical scheme of arbitrary order of accuracy in both space and time, based on the arbitrary high-order derivatives methodology, for transient acoustic simulations. The scheme combines the nodal discontinuous Galerkin method for the spatial discretization and the Taylor series integrator (TSI) for the time integration. The main idea of the TSI is a temporal Taylor series expansion of all unknown acoustic variables in which the time derivatives are replaced by spatial derivatives via the Cauchy-Kovalewski procedure. The computational cost for the time integration is linearly proportional to the order of accuracy. To increase the computational efficiency for simulations involving strongly varying mesh sizes or material properties, a local time-stepping (LTS) algorithm accompanying the arbitrary high-order derivatives discontinuous Galerkin (ADER-DG) scheme, which ensures correct communications between domains with different time step sizes, is proposed. A numerical stability analysis in terms of the maximum allowable time step sizes is performed. Based on numerical convergence analysis, we demonstrate that for nonuniform meshes, a consistent high-order accuracy in space and time is achieved using ADER-DG with LTS. An application to the sound propagation across a transmissive noise barrier validates the potential of the proposed method for practical problems demanding high accuracy. © 2021 Acoustical Society of America.

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I. INTRODUCTION
Recent years have witnessed rapid developments and applications of the time-domain discontinuous Galerkin (TD-DG) method for modeling wave propagation phenomena,1–8 which possesses many favorable properties, such as high-order accuracy, geometric flexibility, and its capability of handling inhomogeneous media and physically absorbing media.9–11 It first discretizes the volumetric space into nonoverlapping mesh elements that can be locally refined to meet the geometrical constraints. Then, solutions of governing acoustic equations are approximated spatially with a polynomial basis. Because the locally defined basis functions can be discontinuous across element interfaces, the resulting mass matrix is block diagonal, which favors a fully explicit time-marching scheme with intrinsic parallel computations.12,13 Last, following the numerical methodology of the method of lines, the resulting semi-discrete formulations in the form of coupled ordinary differential equations (ODEs) are integrated in time by an ODE solver. Explicit Runge-Kutta (RK) methods, which involve only a linear combination of the right-hand-side evaluations of the semi-discrete formulation, are usually used because they are easy to implement, and each unknown solution only needs one extra memory unit to store its intermediate stage values with low-storage RK methods.14–20

However, despite the abovementioned advantages, industrial applications of the TD-DG method for computational acoustics remain hindered by the high computational cost for longtime simulations, and advancements to improve the computational efficiency are needed. One major criticism arises from the severely restrictive maximum allowable time step imposed by the well-known Courant-Friedrichs-Lewy (CFL) conditional stability condition accompanying explicit time integration schemes. As far as practical acoustic simulations are concerned, local mesh refinement is usually needed to capture complex geometry features, such as abruptly varying surfaces, resulting in stiff terms in ODEs that necessitate a much smaller time step. When global time-stepping (GTS) is used, as is often the practice with the method of lines, the uniformly defined time step is constrained by the smallest element in the mesh, excessively slowing down the time-marching over the whole domain. Apart from the mesh-induced stiffness, the modeling of physically absorbing media9,10 is a potential source of stiffness as well. A natural solution to the restrictive time steps required for stability of explicit GTS methods is to switch to unconditionally stable implicit methods. However, for three-dimensional problems with a large number of unknowns, the computational efficiency of such an approach is debatable because solving a large linear system at every time step is time consuming.21 An alternative is to use implicit-explicit (IMEX) schemes,22–24 which applies an explicit integrator to the non-stiff/coarse part and an implicit

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integrator to the stiff/refined part such that the same time step is used everywhere. Additionally, exponential-based time integrators alleviate the stability restrictions by removing the stiffness on the explicit time integration through transforming the differential equations. However, pitfalls, such as lower-order coupling errors, coupling stability problems, ill-conditioning issues for highly disparate meshes, and cumbersome implementations, may occur. An alternative approach to mitigate the time step restriction is to use explicit local time-stepping (LTS) methods, also called multiple or multi-rate time-stepping (MTS) in the literature, which use smaller time steps where necessary while remaining fully explicit in the entire mesh. This increases the efficiency drastically as larger elements can be advanced with large local time steps. Over the past decades, various LTS methods have been developed in the scientific computing community targeting either general ODEs or specific spatial discretization methods. As far as the DG method is concerned, LTS schemes based on the second-order leap-frog (LF2) method, pth-order Runge-Kutta (RKp, p ≤ 4) methods, pth-order Adams-Bashforth multistep schemes (ABp, p ≤ 4), and the causal-path concept, combining LF2 and RK4 (Ref. [41]), have been proposed for wave propagation problems. Recently, the high-order p-th order derivatives discontinuous Galerkin (ADER-DG) method with the LTS scheme was used to model transient electromagnetic, elastic, and acoustic wave propagation. The arbitrary high-order derivatives (ADER) time integration approach follows the Taylor series method and expands the unknown solutions directly into a temporal Taylor series in which all of the time derivatives are converted into spatial derivatives by repeatedly using the governing partial differential equations (the so-called Cauchy-Kovalewski procedure). The high-order spatial derivatives are approximated with the DG method. The ADER approach is a one-step explicit scheme that does not require intermediate stages while maintaining the same approximation order in space and time. Despite such attractive properties, it has a drawback that the high-order spatial derivatives of the polynomial basis are rather tedious to derive and implement, especially when non-affine (curved) elements with varying Jacobians are involved.

This work aims to further develop the TD-DG method to be more robust and efficient in simulating transient acoustic wave propagation from the following two aspects. First, inspired by a new class of high-order scheme recently introduced for seismic and electromagnetic wave modeling, we present a variant of the ADER-DG approach to solve the time-dependent linear acoustic equations with an arbitrary order of accuracy in time and space and incorporate necessary formulations of time-domain impedance and transmission boundary conditions for indoor-outdoor acoustic simulation purposes. The governing acoustic equations are first spatially discretized with the DG method by projection onto the space-dependent polynomial basis functions to obtain the semi-discrete formulation in terms of time-dependent expansion coefficients. Then, these coefficients are time integrated by the Taylor series integrator (TSI). The needed time derivatives are replaced with the numerically approximated spatial derivatives by recursively using the semi-discrete formulation. These constitute a discrete version of the Cauchy-Kovalewski procedure. Second, we propose a new LTS algorithm to accompany the ADER-DG scheme to overcome the inefficiency and stability issues occurring in stiff systems. Without any overlapping subdomains or extrapolation involved, the proposed coupling procedure between domains with different time steps maintains the same high-order accuracy as that of the GTS scheme by using the underlying TSI in a straightforward and efficient manner and is valid for arbitrary ratios of time steps. Furthermore, the computational cost grows linearly with respect to temporal orders of accuracy. The CFL stability conditions are investigated numerically with the aim of providing guidance on time step sizes in practical simulations. An application to the sound propagation across a transmissive noise barrier exhibits the computational benefits of the proposed LTS schemes.

The paper is organized as follows. In Sec. II, spatial discretization of linear acoustic equations with impedance and transmission boundary conditions by the DG method is briefly reviewed. A detailed description of the proposed time-stepping scheme follows in Sec. III. Numerical stability analysis and convergence rate tests are presented in Sec. IV. Application to the sound propagation across a transmissive noise barrier is shown in Sec. V. Finally, concluding remarks can be found in Sec. VI.

II. SPATIAL DISCRETIZATION OF LINEAR ACOUSTIC EQUATIONS WITH THE NODAL DG METHOD

To avoid redundant repetitions with previous works while introducing necessary notations for the convenience of discussion, first, we briefly review the main ingredients of the nodal DG method, which is used for the spatial discretization. Acoustic wave propagation can be described by the following coupled system of linear acoustic equations:

\[
\frac{\partial v}{\partial t} + \frac{1}{\rho} \nabla p = 0, \\
\frac{\partial p}{\partial t} + \rho c^2 \nabla \cdot v = 0, \quad \text{in} \Omega \times [0, t],
\]

(1)

where \(v(x, t) = [u, v, w]^T\) is the particle velocity vector, \(p(x, t)\) is the sound pressure, \(x\) is the position in the spatial domain of interest \(\Omega\), \(\rho\) is the constant density of air, and \(c\) is the constant speed of sound \((c = 343 \text{ m/s and } \rho = 1.2 \text{ kg/m}^3 \text{ in this work})\). Equivalently, Eq. (1) reads

\[
\frac{\partial q}{\partial t} + \nabla \cdot F(q) = \frac{\partial q}{\partial t} + A \nabla \frac{\partial q}{\partial x_j} = 0,
\]

(2)
where \( q(x, t) = [u, v, w, p]^T \) is the unknown acoustic variable vector and \( A_j \) is the constant flux Jacobian matrix with spatial coordinate index \( j \in \{1, 2, 3\} \) and \( x_1 = x, x_2 = y, x_3 = z \). The computational domain \( \Omega \) is divided into non-overlapping simplex elements \( D^k \), i.e., \( \Omega = \bigcup_{k=1}^K D^k \). In this work, the quadrature-free approach\(^1\) is adopted and the nodal discontinuous Galerkin algorithm as presented in Ref. 4 is followed. The local solution \( q_h^i(x, t) \) in element \( D^k \), where subscript \( h \) denotes the numerically approximated variable, is given by

\[
q_h^i(x, t) = \sum_{i=1}^{N_p} q_h^i(x^i, t) \ell_i^h(x),
\]

where \( q_h^i(x^i, t) \) are the unknown nodal values, \( \ell_i^h(x^i) \) is the multidimensional Lagrange polynomial basis of order \( N_p \), which satisfies \( \ell_i^h(x^i) = \delta_{ij} \), and indices \( i, j \) denote the ordering of the nodes. \( N_p \) is the number of local basis functions (or nodes) inside a single element and is equal to \((N + d)!/(Nd!)\) for simplex elements, where \( d \) is the dimensionality. The basis (shape) function \( \ell_i^h(x) \) is determined by the nodal distribution \( x^i \), and in this study, the Legendre-Gauss-Lobatto (LGL) quadrature points are used for one-dimensional (1D) problems and the \( x \)-optimized nodal distribution\(^51\) is used for multidimensional elements due to its low Lebesque constants. Also, the test function space is spanned by the same basis polynomials \( \ell_i^h(x) \). After the Galerkin projection of Eq. (2) and twice integrations by parts, the strong formulation reads

\[
\int_{D^k} \left( \frac{\partial q_h^i}{\partial t} + \nabla \cdot F_h^i(q_h^i) \right) \ell_i^h \, dx = \int_{\partial D^k} \mathbf{n} \cdot \left( F_h^i(q_h^i) - F^i(q_h^i, q_h^i) \right) \ell_i^h \, dx,
\]

where \( \mathbf{n} = [n_x, n_y, n_z] \) is the outward normal vector of the element surface \( \partial D^k \). \( F^i(q_h^i, q_h^i) \), the so-called numerical flux across element intersection \( \partial D^k \), is a function of both the solution value from the interior side of the intersection, i.e., \( q_h^i \) and the neighboring exterior value \( q_h^i \). In this study, the upwind numerical flux is used throughout the whole domain because of its low dispersion and dissipation errors\(^52,53\). It is defined by considering the direction of propagating characteristic waves, i.e.,

\[
\mathbf{n} \cdot F^i(q_h^i, q_h^i) = L(\Lambda^+ L^{-1} q_h^- + \Lambda^- L^{-1} q_h^+),
\]

where \( \Lambda \) is a diagonal matrix with diagonal entries \( [0, 0, c, -c] \), corresponding to the speed of each characteristic wave. \( \Lambda^+ \) and \( \Lambda^- \) contain the positive and negative entries of \( \Lambda \) respectively. \( L \) is the eigenmatrix of the normally projected flux Jacobian, i.e.,

\[
A_n = (n_x A_x + n_y A_y + n_z A_z) = L \Lambda L^{-1},
\]

Physically, \( \Lambda^+ (\Lambda^- \), respectively) corresponds to the characteristic waves propagating along (opposite to) the outward normal direction \( \mathbf{n} \), which are referred to as outgoing waves out of \( D_k \) (incoming waves into \( D_k \)). Therefore, the outgoing waves are associated with the interior solution \( q_h^- \), whereas the incoming waves are dependent on the exterior (neighboring) solution \( q_h^+ \). Finally, the semi-discrete formulation is obtained by substituting the nodal basis expansion [Eq. (3)] and the upwind flux [Eq. (5)] into the strong formulation [Eq. (4)]. The resulting vector-matrix form of the formulation and additional descriptions of implementations can be found in Ref. 50.

Besides the spatial discretization inside the computational domain, proper formulations of the boundary conditions are of critical importance because boundaries of various shapes introduce a considerable amount of complexities into the wave patterns over a long time, such as absorption, transmission, and coupling with structural vibrations\(^9,11,54,55\). For the DG method, boundary conditions are enforced weakly through the numerical flux. In this work, the formulations of broadband time-domain impedance and transmission boundary conditions for locally reacting surfaces are considered\(^10,11\). The essential idea is to reformulate the numerical flux based on the characteristic waves of the linear acoustic equations, together with the plane wave reflection coefficient \( R \) and transmission coefficient \( T \) at normal incidence. The use of coefficients at normal incidence is consistent with the fact that only the normal component of the numerical flux to the boundary surface contributes to the surface integral in Eq. (4). As shown in Refs. 10 and 11, the inputs required for the boundary formulation are the complex-valued plane wave reflection coefficient \( R(\omega) \) and transmission coefficient \( T(\omega) \) as a function of the angular frequency \( \omega \) in the frequency domain, which may be obtained from (semi-)analytical impedance models or measured discrete impedance values. Then, both coefficients are approximated within the frequency range of interest using multipole models\(^56,58\). For example, \( R(\omega) \) is approximated as
reacting boundaries are discussed in Ref. 11. By applying the multipole model and parameter fitting procedures can be obtained as [Eq. (9)] into \[ \int_{-\infty}^{\infty} \sigma_{n}^{\text{out}}(\tau)R(t-\tau)d\tau \] yields
\[ \sigma_{n}^{\text{out}}(t) = R_{\infty} \sigma_{n}^{\text{out}}(t) + \sum_{k=1}^{S} A_{k} \phi_{k}(t) \]
\[ + \sum_{l=1}^{T} \left[ B_{l} \psi_{l}^{(1)}(t) + C_{l} \psi_{l}^{(2)}(t) \right], \]
where \( \phi_{k}(t), \psi_{l}^{(1)}(t), \psi_{l}^{(2)}(t) \) are the so-called accumulators or auxiliary variables and are given, respectively, by the following ODEs:
\[ \frac{d\phi_{k}}{dt} + \zeta_{k}\phi_{k}(t) = \sigma_{n}^{\text{out}}(t), \]
\[ \frac{d\psi_{l}^{(1)}}{dt} + \beta_{l}\psi_{l}^{(1)}(t) + \beta_{l}\psi_{l}^{(2)}(t) = \sigma_{n}^{\text{out}}(t), \]
\[ \frac{d\psi_{l}^{(2)}}{dt} + \beta_{l}\psi_{l}^{(2)}(t) - \beta_{l}\psi_{l}^{(1)}(t) = 0. \]
The above system is numerically integrated from zero initial values using the same time integration scheme as for the semi-discrete DG formulation in Eq. (4), which will be shown in the following.

III. THE ADER-DG SCHEME AND LTS

A. The ADER Taylor Series time integrator

After the spatial discretization by the nodal DG method, the total semi-discrete system can be expressed in a general form of ODEs as
\[ \frac{\partial q_{h}}{\partial t} = \mathcal{L}(q_{h}(t)), \]
where \( \tilde{q}_{h} \) denotes the union of all unknown solutions, including acoustic variables \( q_{h} \) as in Eq. (4) and auxiliary variables from the boundary conditions. Here, \( \mathcal{L} \) considers both the spatial discretization operator of DG in Eq. (4) and the ADEs of Eqs. (13a)–(13c). To introduce the Taylor Series time integrator scheme, we expand \( \tilde{q}_{h} \) into a truncated Taylor series with respect to time and omit the time derivative terms of order higher than \( N_{t} \), obtaining
\[ \tilde{q}_{h}(t + \Delta t) = \tilde{q}_{h}(t) + \sum_{i=1}^{N_{t}} \frac{\Delta t^{i}}{i!} \frac{\partial^{i} \tilde{q}_{h}(t)}{\partial \Delta t^{i}}. \]

Then, the discrete Cauchy-Kovalewski procedure is used to replace the time derivatives in the above Taylor series by spatial derivatives through the repeated use of Eq. (14), i.e.,
\[ \frac{\partial \tilde{q}_{h}}{\partial \tau} = \frac{\partial \tilde{q}_{h}}{\partial \tau - 1} = \mathcal{L} \left( \frac{\partial^{i} \tilde{q}_{h}}{\partial \tau^{i}} \right) = \mathcal{L}^{i} \tilde{q}_{h}. \]
Then, the stability condition becomes

\[ \Delta t = C_{CFL} \frac{1}{N_t^2 \min(\Delta e)} \frac{1}{c}, \tag{22} \]

where \( \Delta e \) is a measure of the element size and \( C_{CFL} \) is a constant of order \( O(1) \). Recall that \( N \) is the spatial approximation order. Here, we split the conventionally defined CFL number, which is, in practice, set to \( 1/(2N + 1) \) for a discretization combination of the \( N \)-th order DG method and an \( (N + 1) \)-th order explicit RK method,\(^6\) as a product of \( C_{CFL} \) and \( 1/N^2 \). The factor \( 1/N^2 \) is attributed to the fact\(^4,64\) that the eigenvalue spectrum of the spatial operator of DG grows at a rate slightly slower than does \( O(N^2) \), whereas \( C_{CFL} \) accounts for the effect of the size of the absolute stability region of time integration schemes of various orders. Because both the eigenvalue spectrum of the DG operator and the absolute stability region of the TSI schemes are defined implicitly, the exact values of \( C_{CFL} \) will be determined numerically in Sec. IV A.

### B. LTS strategy

Practical simulations typically use meshes of different sizes for regions of various dimensions to capture the necessary geometrical details while maintaining the computational efficiency. In addition, spatial variations in the sound speed \( c \) across elements influence the stiffness or the eigenvalue spectrum of the semi-discrete system (14). Furthermore, the fitting parameters of the multipole model for the impedance boundary condition as in Eq. (8) may introduce extra stiffness. To satisfy the CFL stability condition (21), the global time step is dictated by the most demanding factor, e.g., the smallest element. To overcome this inefficiency, an explicit LTS strategy accompanying the TSI scheme is introduced.

For a convenient demonstration but without loss of generality, consider a 1D computational domain with two sub-domains that are composed of coarse and fine meshes and denoted as \( \Omega_c \) and \( \Omega_f \), respectively, with their local time steps \( \Delta t_c \) and \( \Delta t_f \), as illustrated in Fig. 1. Suppose the local time steps follow the relation \( \Delta t_f = \sigma \Delta t_c \), where the ratio of time step sizes \( \sigma \) is determined based on mesh partitioning and efficiency considerations. As we will see, the proposed LTS strategy can be extended to a ratio of an arbitrary value. However, for ease of explanation, the case with an integer \( \sigma \)

\[ S = \{ z : |R(z)| \leq 1 \}. \tag{20} \]

To evaluate the fully discrete stability of the ADER-DG scheme with a given time step \( \Delta t \), the eigenvalues \( \lambda_N(L) \) of the semi-discrete operator \( L \) are substituted into Eq. (18). Then, the stability condition becomes

\[ \lambda \Delta t \subset S, \quad \forall \lambda \in \lambda_N(L). \tag{21} \]

Equation (21) is a necessary condition for the absolute stability in a general sense.\(^62,63\) It can also serve as an excellent guideline for the time step choice.\(^4\) In this work, we follow this guideline and determine the time step in the following way:\(^4\)

![FIG. 1. Illustration of the LTS algorithm in the 1D space.](https://example.com/fig1.png)
is presented in detail as follows. First, starting from a certain synchronized time \( t^r = t_0 + n\Delta t_c \) (as it happens at the initial time \( t = t_0 \)), the global solutions step forward simultaneously using their local time steps. To be specific, the solutions inside the coarse region \( \hat{q}^r_c \) is advanced to the next synchronous time level \( t^{r+1} = t^r + \Delta t_c \) by the TSI formula

\[
\hat{q}^{r+1}_c = \hat{q}^r_c + \sum_{i=1}^{N_i} \frac{\gamma_i}{\Delta t_c} \frac{\partial \hat{q}^n_c}{\partial t^r} ,
\]

whereas the solutions inside the fine region \( \hat{q}^r_f \) are advanced to its next intermediate time level \( t^{r+1/\sigma} = t^r + \Delta t_f \) by the TSI formula

\[
\hat{q}^{r+1/\sigma}_f = \hat{q}^r_f + \sum_{i=1}^{N_i} \frac{\gamma_i}{\Delta t_f} \frac{\partial \hat{q}^n_f}{\partial t^f} .
\]

The time derivatives needed for the local TSI advancements in Eqs. (23) and (24) can be calculated using the known solutions \( \hat{q}^r \) at the synchronous time level \( t^r \) in the same iterative way as for a GTS scheme as shown in Eq. (16). The next step is to advance the newly obtained solutions \( \hat{q}^{r+1/\sigma}_f \) inside the fine region to its next intermediate time level \( t^{r+2/\sigma} = t^r + 2\Delta t_f \) in a manner similar to that of Eq. (24). However, in this case, the time derivatives \( \frac{\partial \hat{q}^{r+1/\sigma}_f}{\partial t^r} \) inside the fine region \( \Omega_f \) cannot be calculated directly using Eq. (16). Recall that the spatial discretization operator \( L \) for the discrete Cauchy-Kovalewski procedure involves the numerical flux approximation, which depends on the solution values along the interface belonging to neighboring elements.

Therefore, to calculate the spatial discretization operator of fine elements that lie next to the interface between the coarse and fine meshes, the solution values along the interface on the coarse mesh are required. Specifically, along the dividing interface, solution values \( \hat{q}^{r+1/\sigma}_c \) are needed to get \( \frac{\partial \hat{q}^{r+1/\sigma}_f}{\partial t^r} \) and \( \frac{\partial \hat{q}^{r+1/\sigma}_c}{\partial t^r} \) are needed to get \( \frac{\partial^2 \hat{q}^{r+1/\sigma}_f}{\partial t^r^2} \), and so on. These desired solution/derivative values on the coarse mesh at different intermediate time levels \( (t^{r+j/\sigma} = t^r + j\Delta t_f, j \in \{1, \ldots, \sigma\}) \), are obtained by using their own Taylor series expansions around the synchronous time level, which is supposed to be valid throughout \( [t^r, t^{r+1}] \), i.e.,

\[
\frac{\partial q^{r+j/\sigma}_c}{\partial t^r} = \frac{\partial q^n_c}{\partial t^r} + \sum_{i=1}^{N_i} \gamma_i (j\Delta t_f)^{j-k} \frac{\partial^k q^n_c}{\partial t^r^k},
\]

where the derivative terms \( \frac{\partial q^n_c}{\partial t^r} \) are pre-saved while advancing the coarse elements using Eq. (23). The presented coupling procedure provides an accurate evolution of solutions by exploiting the nature of the Taylor series expansion and ensures the correct communication between the coarse and fine regions. Because only the solution/derivative values at the nodes along the interface are needed for flux computations during the coupling procedure, the incurred memory and computation costs are negligible for the entire system. As will be seen later, the same order of accuracy as that of the GTS scheme is maintained, and the stability is ensured locally without a huge compromise on time steps.

To theoretically appreciate the potential speedup that can be achieved by such a LTS scheme, we assume that the fine domain \( \Omega_f \) takes up \( \chi \) fraction of the whole domain, whereas the proportion of the remaining coarse domain \( \Omega_c \) is \( 1 - \chi \). If GTS is used, all elements are advanced with the small time step \( \Delta t_f \), and the computational cost (or, equivalently, the total number of element-wise right-hand-side evaluations) is proportional to

\[
\frac{N^r}{(1 - \chi)N^c}.
\]

\[
\text{TABLE I. Stability limits } C_{\text{CFLmax}} \text{ for ADER-DG with upwind fluxes and a uniform mesh.}
\]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N = 3 )</th>
<th>( N = 4 )</th>
<th>( N = 5 )</th>
<th>( N = 6 )</th>
<th>( N = 7 )</th>
<th>( N = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_i = 3 )</td>
<td>1.171</td>
<td>1.434</td>
<td>1.653</td>
<td>1.835</td>
<td>1.994</td>
<td>2.134</td>
</tr>
<tr>
<td>( N_i = 4 )</td>
<td>1.308</td>
<td>1.600</td>
<td>1.841</td>
<td>2.043</td>
<td>2.217</td>
<td>2.372</td>
</tr>
<tr>
<td>( N_i = 5 )</td>
<td>1.511</td>
<td>1.848</td>
<td>2.126</td>
<td>2.359</td>
<td>2.562</td>
<td>2.739</td>
</tr>
<tr>
<td>( N_i = 6 )</td>
<td>1.669</td>
<td>2.042</td>
<td>2.348</td>
<td>2.605</td>
<td>2.832</td>
<td>3.028</td>
</tr>
<tr>
<td>( N_i = 7 )</td>
<td>1.858</td>
<td>2.272</td>
<td>2.612</td>
<td>2.901</td>
<td>3.150</td>
<td>3.368</td>
</tr>
<tr>
<td>( N_i = 8 )</td>
<td>2.026</td>
<td>2.478</td>
<td>2.849</td>
<td>3.164</td>
<td>3.437</td>
<td>3.677</td>
</tr>
<tr>
<td>LS – RK5</td>
<td>1.980</td>
<td>2.437</td>
<td>2.815</td>
<td>3.135</td>
<td>3.412</td>
<td>3.656</td>
</tr>
</tbody>
</table>

FIG. 2. (Color online) Distribution of eigenvalues of the amplification matrix \( R \) for a uniform time step with \( N = N_i = 6 \). \( C_{\text{CFL}} = C_{\text{CFLmax}} \).

(a) \( \sigma = 1 \) and (b) \( \sigma = 5 \).
In contrast, if LTS is used, the computational cost scales as

\[ N_{LTS} = \frac{Z}{\Delta t_f} + \frac{1 - Z}{\Delta t_x}. \]

Therefore, the theoretical speedup \( \nu \) is obtained as the ratio

\[ \nu = \frac{N_{GTS}}{N_{LTS}} = \frac{\sigma}{1 - \chi + \sigma \chi} \geq 1. \]  

(26)

Two extreme cases are noteworthy to understand Eq. (26). One case is that the two time steps are extremely disparate, i.e., \( \sigma \gg 1 \), then we obtain

\[ \lim_{\sigma \to \infty} \nu = \frac{1}{\chi}, \]

indicating that a dramatic speedup is achieved if a small fraction of elements need a very small time step. The other case is when there are a small number of fine elements, i.e., \( \chi \to 0 \), and then the speedup is determined by the ratio of time steps as

\[ \lim_{\chi \to 0} \nu = \sigma. \]

The above two-level formulation with integer-valued ratios can be extended to general real-valued time step ratios of arbitrarily many levels of local time steps to accommodate the needs arising from practical simulations, e.g., mesh partitioning. It should be mentioned that the cost of the coupling procedure [Eqs. (25a) and (25b)], which can be performed efficiently with matrix-vector multiplication, together with accompanying storage needs, is negligible compared to the overall computational work. A common approach to save the memory cost is to group elements with similar maximum allowable time steps together and, then, to adopt one uniform time step for each group.

IV. NUMERICAL STUDIES OF THE ADER-DG SCHEME WITH LTS

In this section, numerical experiments are performed to investigate the numerical properties of the ADER-DG scheme with both GTS and LTS.

A. Numerical stability analysis

The stability properties are studied by applying the ADER-DG scheme to the 1D linear acoustic equation (1) with periodic boundary conditions. Consider a computational domain of size \( x \in [0, 4] \text{ m} \), where \( [0, 2] \text{ m} \) is discretized with an equidistant mesh of size \( \Delta x \), whereas \( [2, 4] \text{ m} \) is discretized with an equidistant mesh of size \( \Delta x' = \Delta x / \sigma \). First, the stability with a uniform time step is evaluated. To that end, we assemble the DG spatial discretization operator
and insert it into an $N_t$th-order TSI time integration [Eq. (17)]. Then, it is straightforward to obtain the amplification matrix $R$ that relates the solutions between two time levels as $q_{n+1} = Rq_n$, where

$$R = I + \sum_{i=1}^{N_t} \gamma_i \Delta t \mathcal{L}^i,$$

(27)

and $I$ is the identity matrix. The fully discrete formulation is stable if all eigenvalues of matrix $R$, denoted as $\lambda_N(R)$, have a modulus less or equal to unity, i.e.,

$$\max(|\lambda_N(R)|) \leq 1.$$

(28)

For the uniform time step case, eigenvalues of matrix $R$ are related to the product of eigenvalues of the spatial operator $\mathcal{L}$ and the time step as $\lambda_N(R) = R(\lambda_N(\mathcal{L})\Delta t)$, and the condition of Eq. (28) is equivalent to the stability condition of Eq. (21). The eigenvalues based on a uniform mesh with $\Delta x = \Delta y = 0.1$ m are calculated, and a simple bisection method is used to find the maximum allowable time step by iteratively reducing the time step subject to the stability condition of Eq. (28). Table I presents the maximum allowable $C_{CFL}$ values as in Eq. (22) for different combinations of time and space discretization orders, which match exactly the corresponding CFL number limit for the standard RK method as reported in Ref. 2. For each fixed spatial order $N$, $C_{CFL_{max}}$ increases with the time integration order $N_t$ as the absolute stability region grows. Furthermore, since the eigenvalue spectrum of the DG discretization operator grows at a rate slightly slower than $O(N^2)$, a slight increase in $C_{CFL_{max}}$ with respect to $N$ is observed as well. As a comparison, the low-storage five-stage fourth-order RK (LS-RK5) scheme, which is devised and optimized to enhance stability, has a much larger stability region than the fifth-order TSI scheme at the cost of a compromising accuracy.

The stability limit also applies to the nonuniform mesh case, where $C_{CFL}$ is determined by the finer mesh. Figure 2 displays the distribution of eigenvalues of the amplification matrix $R$ for a uniform mesh with $\Delta x = \Delta y = 0.1$ m as well as a nonuniform mesh with $\Delta x = 5\Delta y = 0.1$ m.

Next, the stability of the proposed LTS scheme is evaluated, and we now consider the domain with a nonuniform mesh of ratio $\sigma = 5$. To obtain the amplification matrix $R$, the TSI for both the coarse and fine regions [Eqs. (23) and (24), respectively], together with the coupling procedure [Eqs. (25a) and (25b)], are applied to the semi-discrete formulation [Eq. (14)]. Clearly, the eigenvalues of $R(\Delta t_c, \Delta t_f)$, arising from the LTS scheme, are different from those for a GTS scheme. Ideally, the LTS scheme would introduce no further constraint on the time step size and, therefore, the same CFL condition as in the GTS case would be obtained. However, it is found that the maximum allowable CFL-like constants $C_{CFL_{max}}$ for the LTS scheme are slightly smaller than the maximum allowable CFL-like constants for the GTS case, which is typical for a spatially nonoverlapping coupling procedure. Their ratios are shown in Table II.

Figure 3 shows two examples of distributions of the eigenvalues. Similar numerical experiments with different values of $\sigma$ have been performed to test the CFL stability conditions for the LTS scheme. It is found that a ratio value of 0.95 would yield a stable scheme in all of the tests, indicating that the developed coupling procedure does not have a significant negative impact on the stability limit of the TSI, in general.
B. Convergence rate of ADER-DG

1. Periodic boundary with GTS

To investigate the convergence rate of the proposed ADER-DG method for free field propagation, the 1D linear acoustic equation [Eq. (1)] on the interval \( x \in [0, 4] \) m with uniform meshes are solved. Periodic boundary conditions are prescribed on both ends, and the initial condition is a sine wave of wavelength \( \lambda_w = 4 \) m,

\[
p(x, t = 0) = \sin(-0.5\pi x), \\
u(x, t = 0) = \frac{1}{\rho c} \sin(-0.5\pi x).
\]

First, numerical tests are conducted for various orders of accuracy in space and time with a uniform maximum allowable CFL number on a sequence of mesh sizes \( \Delta x \). The \( L^2 \) space-time error defined as \( \epsilon_{L^2} = \|p_{ana}(T_f) - p_{num}(T_f)\|_{L^2} \) is used to evaluate the accuracy, where \( p_{ana}(T_f) \) and \( p_{num}(T_f) \) denote the analytical solution and the numerical solution, respectively, at the final time \( T_f = (16 \text{ m})/c \) across the whole domain. \( \| \cdot \|_{L^2} \) denotes the \( L^2 \) norm, which is carried out numerically and accurately up to the order of the polynomial approximation. As shown in Fig. 4, the observed convergence rates \( k_c \), which are calculated by linear fitting with the least square method, are as expected.

Next, to investigate the effects of temporal error on the spatial convergence rate, we repeat previous tests with a fixed time integration order of \( N_t = 5 \) and different time step sizes. Figure 5(a) displays the results obtained using the maximum allowable time step for ADER-DG as prescribed from Table I. It can be seen that above a certain spatial resolution, the time integration error becomes dominant over the spatial error, and the spatial convergence rate is deteriorated. As shown in Fig. 5(b), a smaller time step reduces the temporal error and the expected spatial convergence rate of ADER-DG scheme is recovered.

2. Impedance and transmission boundary with GTS

Now, convergence rates of the ADER-DG scheme for wave propagation involving time-domain impedance and transmission boundary conditions are verified, where the ADEs [Eqs. (13a)–(13c)] are integrated using the same time integration scheme. As illustrated in Fig. 6, the transmission and impedance boundary is located at \( x = 0 \) m, and the source at \( x_s = -2.5 \) m initiates the simulation with the Gaussian-shaped pressure conditions

\[
p(x, t = 0) = e^{(-\ln 2/b^2)(x-x_s)^2}, \\
u(x, t = 0) = 0,
\]

where \( b = 0.15 \) m is the half-bandwidth of the Gaussian pulse. Convergence tests similar to those in the periodic boundary case are performed. The numerical solutions at the final time \( T_f = 4/c \) s, when both the reflected and transmitted waves are present in the domain, are compared against the analytical solutions. Without loss of generality, one single real pole \( [A, \zeta]_R = [6.4 \times 10^3, 8 \times 10^3] \) is used for fitting the reflection coefficient \( R \) as in Eq. (8) while the transmission coefficient \( T \) is defined by \( [A, \zeta]_R = [5 \times 10^3, 9 \times 10^3] \). The \( L^2 \) error with conforming spatial and temporal approximation order is plotted in Fig. 7. The expected convergence rates are yielded.

C. Convergence of ADER-DG with LTS and comparison against GTS

To verify the convergence rate of ADER-DG with the proposed LTS scheme, we repeat the above numerical tests using nonuniform meshes. For the periodic boundary case, the original computational domain \( x \in [0, 4] \) m is split into two parts with different mesh sizes. The coarse part \( x \in [0, 2] \) m is discretized with an equidistant mesh of size \( \Delta x_c \), whereas the fine part \( x \in [2, 4] \) m has a mesh of size \( \Delta x_f = \Delta x_c / \sigma \). For the impedance and transmission boundary case, the interface between the coarse region \( x \in [-4, 0] \) m and the fine region \( x \in [0, 2] \) m coincides with the reflective and transmissive boundary. With a uniform CFL value, the time step sizes are different between the two regions. The \( \zeta_{L^2} \) error for different values of mesh size ratio \( \sigma \) and a fixed order of accuracy is presented in Fig. 8 for both boundary cases. As a comparison, the results obtained from the GTS scheme are shown as well. It can be seen that for all of the tested time step ratios \( \sigma \), the expected convergence rates \( k_c \) are observed for LTS, and the scheme introduces no extra error compared to the GTS scheme.
V. APPLICATION

To demonstrate the functionality and speedup potential of the proposed ADER-DG scheme with LTS for practical applications, wave propagation across a finite-height noise barrier made of porous glass-wool material is simulated, which is a configuration pertaining to outdoor sound propagation. The schematic diagram is shown in Fig. 9(a) with the outer boundaries far away from the barrier to prevent spurious reflected waves from influencing the pressure response. The source is located at \( (x_s, y_s) = (-2, 0) \) and four receivers are located at \( (-1, 0), (-1.3, 0), (1,0) \), and \( (1,3) \) are distributed symmetrically with respect to the barrier, which has a height of \( H = 4 \text{ m} \) and a width of \( W = 0.04 \text{ m} \). To mimic a real noise barrier mounted on a hard ground surface, the rigid boundary condition is imposed on the top edge of the barrier, whereas both side edges are modeled with the reflective and transmissive boundary conditions. The characteristic impedance of the locally reacting glass-wool layer is assumed to follow the phenomenological Johnson-Champoux-Allard-Lafarge (JCAL) model with its physical parameters measured experimentally. To model the reflection and transmission properties of the layer, four and five real poles are used to fit the reflection and transmission coefficients from the time-domain boundary conditions, respectively, as shown in Ref. 11. The other boundaries are assumed to be rigid.

The two-dimensional domain is discretized with 13,427 unstructured triangles. As illustrated in Fig. 9(b), the mesh is locally refined near the top edge of the sound barrier in order to capture the geometry features and diffraction effects. A relatively simple mesh size measure, which is the shortest edge length \( h \), is chosen for determining the time step size based on the CFL condition. In-depth discussions on appropriate triangular mesh measures can be found in Refs. 68 and 69. Mesh elements far away from the sound barrier have an edge length of approximately 0.5 m. Here, suppose the shortest edge length is denoted by \( h_{\text{min}} \), we treat all of the elements with the shortest length that satisfies \( h \leq 6h_{\text{min}} \) as the fine mesh elements, i.e., the green triangles in Fig. 9(c). There are, in total, 2729 fine triangles scattered around the barrier, which suggests a speedup ratio of \( \nu \approx 3 \) with the LTS scheme according to Eq. (26).

To evaluate the accuracy of the LTS scheme, we use the numerical solutions obtained by the RK-DG method with the GTS scheme and the same mesh, which have been verified against the finite element solutions in the frequency domain as the reference solutions. Following the same numerical setup of the reference solution, a uniform CFL-like constant \( C_{\text{CFL}} = 1 \) is used, as well as a spatial polynomial basis of order \( N = 8 \), together with a fifth-order TSI, which shares the same computational cost as the LS-RK5 scheme, are used. The time steps in two regions follow the relation \( \Delta t_c = 6\Delta t_f \). The Gaussian distribution with \( b = 0.25 \text{ m} \) as in Eqs. (30a) and (30b) acts as the source and initiates the simulation. The comparison of pressure signals at two receiver locations obtained from both the GTS scheme and LTS scheme are shown in Fig. 10, where a high level of agreement is observed. For receiver 2, the main components are the direct sound and the reflected sound from the barrier, whereas receiver 4 experiences the pressure signal diffracted over the hard barrier top as well as the faint pressure signal transmitted through the barrier. Besides the time-domain comparisons, the relative sound pressure level, which is obtained by normalizing the total spectrum against the spectrum of the free field solution, and its phase angle at different receiver locations are shown in Figs. 11(a) and 11(b), respectively. Again, a perfect match of the frequency-domain results between the GTS scheme and the LTS scheme is found.

VI. CONCLUSION

In this work, we present an ADER-DG approach with the LTS scheme to solve the time-dependent linear acoustic equations with an arbitrary order of accuracy in time and space. The scattered fine mesh elements near the top edge of the barrier are shown in Fig. 9(c). The mesh is locally refined to capture the geometry features and diffraction effects. The source is located at \( (x_s, y_s) = (-2, 0) \) and four receivers are distributed symmetrically with respect to the barrier. The characteristics of the locally reacting glass-wool layer are modeled using the JCAL model with parameters measured experimentally. The wave propagation is simulated using the LTS scheme, and the results are compared with those obtained using the GTS scheme. The accuracy and speedup of the LTS scheme are demonstrated through time-domain and frequency-domain comparisons.

FIG. 9. (Color online) Geometry and mesh of the noise barrier. (a) Schematic diagram, (b) overall mesh, and (c) zoomed-in view of the fine mesh near the top edge.

FIG. 10. (Color online) Time history of the pressure signal at receivers \( r_2 \) and \( r_4 \).
space. Built upon the well-known nodal DG method for the spatial discretization, it performs the time integration using TSI in an efficient and accurate manner. Necessary formulations of time-domain acoustic boundary conditions are included for indoor-outdoor acoustic simulation purposes. Numerical experiments and applications are performed to validate the stability and convergence properties. It is shown that the proposed LTS scheme enables a significant reduction in computational cost while maintaining the desired high-order accuracy. Applications to more practical sound propagation problems involving complex geometries and material properties will be addressed in future work.