Inference rules and inferential distributions

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Abstract

We introduce the concept of inferential distributions corresponding to inference rules. Fiducial and posterior distributions are special cases. Inferential distributions are essentially unique. They correspond to or represent inference rules and are defined on the parameter space. Not all inference rules can be represented by an inferential distribution. A constructive method is given to investigate its existence for any given inference rule.

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0. Introduction

Whether or not fiducial distributions are unique has been the subject of sometimes vigorous debate, as for example the discussion following the presentation of a paper by Wilkinson (1977) shows. The proper marginalization and conditioning of observational evidence (Berkum, Linssen, Overdijk (1994)) eliminates a number of uniqueness problems. Some however seem to remain. For example, the Cornish-Fisher and Segal fiducial distributions for the multivariate normal mean yield different fiducial probabilities for a fixed inferential statement. Wilkinson argues that these distributions correspond to different inference rules. In this paper we strongly support his position.
Fiducial distributions should not be taken out of context. They determine fiducial probabilities of inferential statements. In that way they correspond to or represent inference rules. We show that every inference rule can be represented by at most one inferential distribution. More than one inference rule may be represented by the same inferential distribution. Fiducial and posterior distributions are special cases of inferential distributions.

In Section 1 and 2 we give the general form of statistical inference problems and a strict definition of inference rules. In Section 3 and 4 we discuss the uniqueness and existence of inferential distributions. Theorem 4.1 describes how to construct the inferential distribution when it exists. Section 5 contains a number of examples.

1. Inference model

Consider an experiment. The set of possible outcomes of the experiment is denoted by Ω. The σ-field of events on Ω is written as Σ. The measurable space (Ω, Σ) is the sample space of the experiment. Let $\widehat{\Sigma}$ be the set of probability measures on Σ and let $\overline{\Sigma}$ be the smallest σ-field on $\widehat{\Sigma}$ such that the map

$$p \mapsto p(A) \in [0,1], \quad p \in \widehat{\Sigma} \ ,$$

from $\widehat{\Sigma}$ into the unit interval is Borel measurable for all $A \in \Sigma$.
The probability distribution on Σ corresponding to the outcome of the experiment is not known. However, a subset $\mathcal{P} \subset \widehat{\Sigma}$ is given such that the probability distribution of the outcome of the experiment is in the set $\mathcal{P}$. The measurable space $(\mathcal{P}, \overline{\Sigma}|\mathcal{P})$, where

$$(1.1) \qquad \overline{\Sigma}|\mathcal{P} := \{A \cap \mathcal{P} | A \in \overline{\Sigma}\} \ ,$$

of probability measures on Σ is said to be the probability model of the experiment. We assume that every sufficient and every ancillary statistic is trivial. If this is not the case, then the observation should be reduced to the value of the minimal sufficient statistic and its distribution should be conditioned on the value of a maximal ancillary statistic.

Let p_0 be the probability distribution of the outcome of the experiment. It may be that one is interested only in a specific aspect of p_0, i.e. a σ-field $\mathcal{R} \subset \overline{\Sigma}|\mathcal{P}$ is specified such that every inferential statement can be written as $p_0 \in A$ with $A \in \mathcal{R}$. We refer to $\mathcal{R}$ as the σ-field of interest. The triple

sample space	(Ω, Σ) ,
probability model	$(\mathcal{P}, \overline{\Sigma}\|\mathcal{P})$,
σ-field of interest	$\mathcal{R} \subset \overline{\Sigma}\|\mathcal{P}$

is said to constitute an inference model for the experiment.

2. Inference rule

Let (Ω, Σ) be the sample space, $(\mathcal{P}, \overline{\Sigma}|\mathcal{P})$ the probability model and $\mathcal{R} \subset \overline{\Sigma}|\mathcal{P}$ the σ-field of interest of an inference model.
The product space

$$(\Omega \times \mathcal{P}, \Sigma \otimes \mathcal{R})$$

is said to be the reference space of the inference model. Let $U \in \Sigma \otimes \mathcal{R}$. For $\omega \in \Omega$ and $p \in \mathcal{P}$ we write

$$\begin{aligned} U^\omega &:= \{p \in \mathcal{P} | (\omega, p) \in U\} \in \mathcal{R} \ , \\ U_p &:= \{\omega \in \Omega | (\omega, p) \in U\} \in \Sigma \ . \end{aligned} \tag{2.1}$$

Obviously we have

$$(\omega, p) \in U \Leftrightarrow \omega \in U_p \Leftrightarrow p \in U^\omega \tag{2.2}$$

for all $U \in \Sigma \otimes \mathcal{R}$ and $(\omega, p) \in \Omega \times \mathcal{P}$.
An inference rule specifies a subset of $\mathcal{P}$ for every observation $\omega \in \Omega$. The inference states that the unknown probability distribution is an element of that subset. The subset depends not only on the observation, but also on the desired degree of certainty. We now give a strict definition of inference rule.

Definition 2.1. Let $\mathcal{U} \subset \Sigma \otimes \mathcal{R}$ be a nonempty collection of measurable subsets of the reference space, and let γ be a function from $\mathcal{U} \times \Omega$ into the unit interval. The pair $(\mathcal{U}, \gamma)$ is called an inference rule if

- $\gamma(U, \cdot) : \Omega \to [0,1]$ is Borel measurable for all $U \in \mathcal{U}$,
- γ is monotone, i.e. for all $U_1, U_2 \in \mathcal{U}$ and all $\omega \in \Omega$

$$U_1 \subset U_2 \Rightarrow \gamma(U_1, \omega) \leq \gamma(U_2, \omega) \ .$$

We refer to γ as the inferential function of the inference rule $(\mathcal{U},\gamma)$. The elements of $\mathcal{U}$ are called the tables of the inference rule $(\mathcal{U},\gamma)$. The inference corresponding to the table $U \in \mathcal{U}$ is specified as follows. If the outcome of the experiment is $\omega \in \Omega$, then we infer that the probability distribution of the outcome is in the set $U^\omega \in \mathcal{R}$; see (2.1). The number $\gamma(U,\omega) \in [0,1]$ can be interpreted as the appreciation level of the inference corresponding to the table $U \in \mathcal{U}$ and the outcome $\omega \in \Omega$. In some cases γ is the confidence level.

3. Inferential distribution and its uniqueness

Let (Ω,Σ) be the sample space, $(\mathcal{P},\overline{\Sigma}|\mathcal{P})$ the probability model and $\mathcal{R} \subset \overline{\Sigma}|\mathcal{P}$ the σ-field of interest of an inference model. Furthermore, let $(\mathcal{U},\gamma)$ be an inference rule. For $\omega \in \Omega$ let $\mathcal{R}^\omega \subset \mathcal{R}$ be the σ-field generated by the collection $\{U^\omega \in \mathcal{R}|U \in \mathcal{U}\}$. We now define inferential distributions corresponding to the inference rule $(\mathcal{U},\gamma)$.

Definition 3.1. For $\omega \in \Omega$ let P^ω be a probability measure on $(\mathcal{P},\mathcal{R}^\omega)$. The collection $\{P^\omega|\omega \in \Omega\}$ is said to be a collection of inferential distributions corresponding to the inference rule $(\mathcal{U},\gamma)$, if for all $\omega \in \Omega$ and all $U \in \mathcal{U}$ we have

$$(3.1) \qquad P^\omega(U^\omega) \geq \gamma(U,\omega) \ ,$$

and there exists $V \in \mathcal{U}$ such that

$$(3.2) \qquad U^\omega = V^\omega \ \text{ and } \ P^\omega(U^\omega) = \gamma(V,\omega) \ .$$

Two collections, say $\{P_1^\omega|\omega \in \Omega\}$ and $\{P_2^\omega|\omega \in \Omega\}$, of inferential distributions corresponding to the inference rule $(\mathcal{U},\gamma)$ are said to be identical if

$$(3.3) \qquad P_1^\omega(U^\omega) = P_2^\omega(U^\omega)$$

for all $\omega \in \Omega$ and $U \in \mathcal{U}$. The collection of inferential distributions corresponding to an inference rule is unique which is formulated in the following theorem.

Theorem 3.1. (uniqueness theorem).
The collection of inferential distributions corresponding to an inference rule is unique.

Proof. Let $\{P_1^\omega|w \in \Omega\}$ and $\{P_2^\omega|\omega \in \Omega\}$ be two collections of inferential distributions corresponding to the inference rule $(\mathcal{U},\gamma)$. Fix $\omega \in \Omega$ and $U \in \mathcal{U}$ and define

$$\beta_1 := P_1^\omega(U^\omega) \ \text{ and } \ \beta_2 := P_2^\omega(U^\omega) \ .$$

From (3.2) we conclude that there exist $V_1, V_2 \in \mathcal{U}$ such that

$$(3.4) \qquad \begin{aligned} &V_1^\omega = V_2^\omega = U^\omega \ , \\ &\beta_1 = \gamma(V_1,\omega) \ \text{ and } \ \beta_2 = \gamma(V_2,\omega) \ . \end{aligned}$$

Using (3.1) and (3.4) we obtain

$$\beta_2 = P_2^\omega(U^\omega) = P_2^\omega(V_1^\omega) \geq \gamma(V_1, \omega) = \beta_1 ,$$

and similarly $\beta_1 \geq \beta_2$. Hence, $\beta_1 = \beta_2$ which completes the proof of the uniqueness theorem; see (3.3).

4. Existence of inferential distributions

Let (Ω, Σ) be the sample space, $(\mathcal{P}, \overline{\Sigma}|\mathcal{P})$ the probability model and $\mathcal{R} \subset \overline{\Sigma}|\mathcal{P}$ the σ-field of interest of an inference model. Furthermore, let $(\mathcal{U}, \gamma)$ be an inference rule. The following theorem can be used to investigate the existence of the collection of inferential distributions corresponding to the inference rule $(\mathcal{U}, \gamma)$.

Theorem 4.1. Let $\{P^\omega | \omega \in \Omega\}$ be the collection of inferential distributions corresponding to the inference rule $(\mathcal{U}, \gamma)$. We have

$$P^\omega(U^\omega) = \max\{\gamma(V, \omega) | V \in \mathcal{U}, V^\omega = U^\omega\}$$

for all $\omega \in \Omega$ and $U \in \mathcal{U}$.

Proof. Let $\omega \in \Omega$ and $U \in \mathcal{U}$. Define

$$\eta := \sup\{\gamma(V, \omega) | V \in \mathcal{U}, V^\omega = U^\omega\} .$$

From (3.2) we conclude that there exists $V \in \mathcal{U}$ such that

$$(4.1) \qquad P^\omega(U^\omega) = \gamma(V, \omega), \quad V^\omega = U^\omega .$$

Hence,

$$(4.2) \qquad P^\omega(U^\omega) \leq \eta .$$

Conversely, let $V \in \mathcal{U}$ such that $V^\omega = U^\omega$. From (3.1) we infer

$$\gamma(V, \omega) \leq P^\omega(V^\omega) = P(U^\omega) .$$

Hence,

$$(4.3) \qquad \eta \leq P^\omega(U^\omega) .$$

The relations (4.1), (4.2) and (4.3) complete the proof of Theorem 4.1.

The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.1. If the inference rule $(\mathcal{U}, \gamma)$ satisfies the conditions

- for all $\omega \in \Omega$, $U \in \mathcal{U}$

$$\Gamma^{\omega}(U^{\omega}) := \max\{\gamma(V, \omega) | V \in \mathcal{U}, \quad V^{\omega} = U^{\omega}\}$$

  exists,

- for all $\omega \in \Omega$ there exists a probability measure P^{ω} on $(\mathcal{P}, \mathcal{R}^{\omega})$ such that

$$P^{\omega}(U^{\omega}) = \Gamma^{\omega}(U^{\omega})$$

  for all $U \in \mathcal{U}$,

then the set $\{P^{\omega} | \omega \in \Omega\}$ is the collection of inferential distributions corresponding to the inference rule $(\mathcal{U}, \gamma)$.

5. Examples

Example 5.1.
Consider the sample space (Ω, Σ), where $\Omega = I\!R^n$ and Σ the Borel σ-field $\mathcal{B}(I\!R^n)$ on $I\!R^n$. Let P be a probability measure on Σ and consider the location family

(5.1.1) $\quad \mathcal{P} := \{P_{\mu} | \mu \in I\!R^n\}$

corresponding to P. Here the probability measure P_{μ}, $\mu \in I\!R^n$, on Σ is defined by

(5.1.2) $\quad P_{\mu}(A) := P(A - \mu), \quad A \in \Sigma$.

We consider the case that we are interested in the full location parameter $\mu \in I\!R^n$, i.e. the σ-field $\mathcal{R} \subset \overline{\Sigma} | \mathcal{P}$ of interest can be written as

(5.1.3) $\quad \mathcal{R} = \overline{\Sigma} | \mathcal{P}$.

Let $\mathcal{A} \subset \Sigma$ be a nonempty collection of subsets of Ω.

For $A \in \mathcal{A}$ write

(5.1.4) $\quad U(A) := \{(\omega, P_{\mu}) \in \Omega \times \mathcal{P} | \mu \in \omega - A\}$,

and define

(5.1.5) $\mathcal{U} := \{U(A) \subset \Omega \times \mathcal{P} | A \in \mathcal{A}\}$,

(5.1.6) $\gamma(U(A),\omega) := P(A),\ A \in \mathcal{A},\ \omega \in \Omega$.

It is easily verified that $(\mathcal{U},\gamma)$ constitutes an inference rule as defined in definition 2.1. Note that for $A \in \mathcal{A},\ \omega \in \Omega$ we have

(5.1.7) $(U(A))^\omega = \{P_\mu \in \mathcal{P} | \mu \in \omega - A\}$.

We now describe the collection of inferential (in this case also fiducial) distributions corresponding to the inference rule $(\mathcal{U},\gamma)$. For $\omega \in \Omega$ the probability measure Q^ω on $\overline{\Sigma}|\mathcal{P}$ is defined by

(5.1.8) $Q^\omega(\{P_\mu \in \mathcal{P} | \mu \in C\}) := P(\omega - C),\ C \in \mathcal{B}(\mathbb{R}^n)$.

For $\omega \in \Omega$ we defined in section 3 the σ-field $\mathcal{R}^\omega$ on $\mathcal{P}$ as the σ-field generated by the collection

$$\{(U(A))^\omega \in \overline{\Sigma}|\mathcal{P}\ | A \in \mathcal{A}\} .$$

The restriction of the probability measure Q^ω on $\overline{\Sigma}|\mathcal{P}$ to the σ-field $\mathcal{R}^\omega \subset \overline{\Sigma}|\mathcal{P}$ is denoted by P^ω. The collection $\{P^\omega | \omega \in \Omega\}$ is the collection of inferential distributions corresponding to the inference rule $(\mathcal{U},\gamma)$. This is easily verified as follows. For $A \in \mathcal{A}, \omega \in \Omega$ we have

$$P^\omega((U(A))^\omega) = P^\omega(\{P_\mu \in \mathcal{P} | \mu \in \omega - A\}) = P(A) = \gamma(U(A),\omega) ;$$

see definition 3.1.

Example 5.2.
We illustrate example 5.1 in a special case. Let $(\Omega,\Sigma) = (\mathbb{R},\mathcal{B}(\mathbb{R}))$ and P on Σ is the standard normal distribution. Furthermore, take

(5.2.1) $\mathcal{A} := \{(a,b) \subset \Omega | a < b\}$.

Evidently $\mathcal{R}^\omega = \overline{\Sigma}|\mathcal{P}$ for all $\omega \in \Omega$. For the inferential distribution $P^\omega, \omega \in \Omega$, on $\mathcal{R}^\omega = \overline{\Sigma}|\mathcal{P}$ we obtain

(5.2.2) $P^\omega(\{P_\mu \in \mathcal{P} | a < \mu < b\}) = \Phi(b - \omega) - \Phi(a - \omega),\ a < b$,

where Φ is the distribution function of P.

Example 5.3.
Consider the sample space $(\Omega,\Sigma) = (\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$. Let P be a spherically symmetric probability measure on Σ, i.e. for every orthogonal operator H on $\Omega = \mathbb{R}^n$ and every $A \in \Sigma = \mathcal{B}(\mathbb{R}^n)$ we have

$$P(H(A)) = P(A) .$$

For sake of simplicity we suppose that P is absolutely continuous with respect to the Lebesgue measure on Σ. Let $\mathcal{P}$ be the location family corresponding to P as described in (5.1.1,2). Now we consider the case that we are interested in the distance $|\mu|$ of the location parameter $\mu \in \mathbb{R}^n$ from the origin. So the σ-field $\mathcal{R} \subset \overline{\Sigma}|\mathcal{P}$ of interest is generated by the collection

$$\{W_\nu \subset \mathcal{P} | \nu > 0\} ,$$

where for $\nu > 0$

$$(5.3.1) \quad W_\nu := \{P_\mu \in \mathcal{P} | \, |\mu| < \nu\} .$$

According to section 6 in Berkum, Linssen, Overdijk (1994) the statistic

$$(5.3.2) \quad I(\omega) := |\omega|, \quad \omega \in \Omega ,$$

is the unique minimal invariant statistic on Ω, and therefore the sample space (Ω, Σ) has to be reduced to the new sample space

$$(5.3.3) \quad (\Omega, \sigma(I)) .$$

Here $\sigma(I)$ is the σ-field on Ω generated by the statistic I, i.e. the σ-field of the spherically symmetric sets in Σ. For $\mu_1, \mu_2 \in \mathbb{R}^n$ with $|\mu_1| = |\mu_2|$ and $A \in \sigma(I)$ we have

$$P_{\mu_1}(A) = P(A - \mu_1) = P(A - \mu_2) = P_{\mu_2}(A) .$$

So for $\nu \geq 0$ the marginal distribution Q_ν on $\sigma(I)$ corresponding to P_μ with $|\mu| = \nu$ on Σ is well defined. The new probability model can be written as

$$(5.3.4) \quad (\mathcal{P}_1, \overline{\sigma(I)}|\mathcal{P}_1) ,$$

where

$$\mathcal{P}_1 := \{Q_\nu \in \widehat{\sigma(I)} | \nu \geq 0\} .$$

Here $(\widehat{\sigma(I)}, \overline{\sigma(I)})$ is the space of probability measures on $\sigma(I) \subset \Sigma$.
The new σ-field $\mathcal{R}_1 \subset \overline{\sigma(I)}|\mathcal{P}_1$ of interest is written as

$$(5.3.5) \quad \mathcal{R}_1 = \overline{\sigma(I)}|\mathcal{P}_1 .$$

For details with respect to the invariant reduction of inference models the reader is referred to Berkum, Linssen, Overdijk (1994). We now construct an inference rule and its collection of inferential (in this case also fiducial) distributions for the inference model:

sample space $(\Omega, \sigma(I))$,

probability model $(\mathcal{P}_1, \overline{\sigma(I)}|\mathcal{P}_1)$,

σ-field of interest $\mathcal{R}_1 = \overline{\sigma(I)}|\mathcal{P}_1$.

Introduce the notation

$$B_r := \{\omega \in \Omega |\ |\omega| \leq r\} \in \sigma(I),\ r \geq 0 \ .$$

For $0 \leq \alpha \leq 1$ write

(5.3.6) $U(\alpha) := \{(\omega, Q_\nu) \in \Omega \times \mathcal{P}_1 | \nu > 0 \Rightarrow Q_\nu(B_{|\omega|}) > 1 - \alpha\}$.

Note that for $\omega \in \Omega,\ 0 \leq \alpha \leq 1$ we have

(5.3.7) $(\omega, Q_0) \in U(\alpha)$.

Furthermore, define

(5.3.8) $\mathcal{U} := \{U(\alpha) \subset \Omega \times \mathcal{P}_1 | 0 \leq \alpha \leq 1\}$,

and for $\omega \in \Omega,\ 0 \leq \alpha \leq 1$

(5.3.9) $\gamma(U(\alpha), \omega) := \alpha$.

It is easily verified that $(\mathcal{U}, \gamma)$ constitutes an inference rule as defined in definition 2.1.
We now describe the collection of inferential distributions corresponding to the inference rule $(\mathcal{U}, \gamma)$. First introduce the following functions for later use

$$\varphi(\omega, \alpha) := \sup\{\nu \geq 0 | (\omega, Q_\nu) \in U(\alpha)\},\ \omega \in \Omega,\ 0 \leq \alpha \leq 1 \ ,$$

$$\psi(\alpha) := \inf\{|\omega| \geq 0 | \omega \in \Omega,\ \varphi(\omega, \alpha) \neq 0\},\ \ 0 \leq \alpha \leq 1 \ .$$

For $\omega \in \Omega,\ 0 \leq \alpha \leq 1$ we have

(5.3.10) $\varphi(\omega, \alpha) \neq 0 \Rightarrow Q_{\varphi(\omega,\alpha)}(B_{|\omega|}) = 1 - \alpha$,

(5.3.11) $\varphi(\omega, \alpha) = 0 \Rightarrow |\omega| \leq \psi(\alpha)$,

(5.3.12) $Q_0(B_{\psi(\alpha)}) = 1 - \alpha$.

For sake of simplicity we suppose that for $\omega \in \Omega,\ 0 \leq \alpha \leq 1$ we have

(5.3.13) $(U(\alpha))^\omega = \{Q_\nu \in \mathcal{P}_1 | \nu > 0 \Rightarrow \nu < \varphi(\omega, \alpha)\}$.

For $\omega \in \Omega$ we defined in section 3 the σ-field $\mathcal{R}^\omega$ on $\mathcal{P}_1$ as the σ-field generated by the collection

$$\{(U(\alpha))^\omega \subset \mathcal{P}_1 |\, 0 \le \alpha \le 1\} .$$

Use (5.3.13) to verify

$$\text{(5.3.14)} \qquad \begin{aligned} &|\omega| \neq 0 \Rightarrow \mathcal{R}^\omega = \mathcal{R}_1 = \overline{\sigma(I)}|\mathcal{P}_1 , \\ &|\omega| = 0 \Rightarrow \mathcal{R}^\omega = \{\emptyset, \{Q_0\}, \mathcal{P}_1 \backslash \{Q_0\}, \mathcal{P}_1\} . \end{aligned}$$

The probability distribution P^ω, $\omega \in \Omega$, on the σ-field $\mathcal{R}^\omega$ on $\mathcal{P}_1$ is defined by

$$\text{(5.3.15)} \quad |\omega| \neq 0 \Rightarrow P^\omega(\{Q_\nu \in \mathcal{P}_1 | \nu < a\}) := 1 - Q_a(B_{|\omega|}),\ a > 0 ,$$

$$\text{(5.3.16)} \quad |\omega| = 0 \Rightarrow P^\omega(\{Q_0\}) := 1 .$$

Note that for $\omega \in \Omega$ we have

$$\text{(5.3.17)} \quad P^\omega(\{Q_0\}) = 1 - Q_0(B_{|\omega|}) = 1 - P(B_{|\omega|}) .$$

We now prove that $\{P^\omega | \omega \in \Omega\}$ is the collection of inferential distributions corresponding to $(\mathcal{U}, \gamma)$. Let $\omega \in \Omega$, $0 \le \alpha \le 1$. If $\varphi(\omega, \alpha) \neq 0$, then use (5.3.13,15,10,9) to obtain

$$P^\omega((U(\alpha))^\omega) = P^\omega(\{Q_\nu \in \mathcal{P}_1 | 0 \le \nu < \varphi(\omega, \alpha)\}) = 1 - Q_{\varphi(\omega,\alpha)}(B_{|\omega|}) = \alpha = \gamma(U(\alpha), \omega) ;$$

see definition 3.1. If $\varphi(\omega, \alpha) = 0$, then use (5.3.13,17,11,12,9) to obtain

$$P^\omega((U(\alpha))^\omega) = P^\omega(\{Q_0\}) = 1 - Q_0(B_{|\omega|}) \ge 1 - Q_0(B_{\psi(\alpha)}) = \alpha = \gamma(U(\alpha), \omega) .$$

If $P^\omega((U(\alpha))^\omega) > \gamma(U(\alpha), \omega)$, then determine $\overline{\alpha} > \alpha$ such that $|\omega| = \psi(\overline{\alpha})$. We now have

$$(U(\alpha))^\omega = (U(\overline{\alpha}))^\omega = \{Q_0\} ,$$

and

$$P^\omega((U(\overline{\alpha}))^\omega) = 1 - Q_0(B_{\psi(\overline{\alpha})}) = \overline{\alpha} = \gamma(U(\overline{\alpha}), \omega) ;$$

see definition 3.1.

Example 5.4.
We illustrate example 5.3 in a special case. Let $(\Omega, \Sigma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and P on Σ is the standard normal distribution. Since $|\mu|$ is the interesting aspect of the location parameter μ, the unique minimal invariant statistic on Ω can be written as

$$I(\omega) = |\omega|,\ \omega \in \Omega .$$

After invariant reduction of (Ω, Σ) the new sample space is

$$(\Omega, \sigma(I)) ,$$

where

$$\sigma(I) = \{A \in \Sigma | A = -A\} .$$

The marginal probability distribution Q_ν, $\nu \geq 0$, on $\sigma(I)$ is specified by

$$(5.4.1) \quad Q_\nu((-a, a)) = \Phi(a - \nu) - \Phi(-a - \nu), \; a \geq 0 ,$$

where Φ is the distribution function of P.
Performing the calculations as described in example 5.3, we obtain the following collection $\{P^\omega | \omega \in \Omega\}$ of inferential distributions corresponding to the inference rule given in example 5.3. For $\omega \in \Omega$ and $\omega \neq 0$ we get

$$(5.4.2) \quad P^\omega(\{Q_\nu | \nu < a\}) = \Phi(a - |\omega|) + \Phi(-a - |\omega|), \quad a > 0 ,$$

and

$$(5.4.3) \quad P^0(\{Q_0\}) = 1 .$$

As we stated in the introduction fiducial distributions should not be taken out of the inferential context. In example 5.2 we are interested in the location parameter μ, while now the interesting aspect is $|\mu|$. However, in both inference rules symmetric inferential statements with respect to the location parameter occur. It should not be astonishing that the inferential probabilities differ. This can be seen as follows. Let $\omega_0 \in \Omega = I\!R$ be an observation and consider the inferential statement S_a, $a > 0$, that the probability distribution of the experiment is an element of the set

$$\{P_\mu | \; |\mu| < a\} .$$

If we are interested in μ and use the inference rule described in example 5.2, then the inferential probability of the inferential statement S_a is equal to

$$(5.4.4) \quad \Phi(a - \omega_0) - \Phi(-a - \omega_0) ;$$

see (5.2.2).
If we are interested in $|\mu|$ and use the inference rule described in this example, then the inferential probability of the inferential statement S_a is equal to

$$(5.4.5) \quad \Phi(a - |\omega_0|) + \Phi(-a - |\omega_0|) ;$$

see (5.4.2). This inferential probability is always larger than that given by (5.4.4). Compare the discussion in Wilkinson (1977) on pp. 138, 163.

Example 5.5.
Consider the sample space (Ω, Σ) with $\Omega = \{1,2\}$ and $\Sigma = \{\emptyset, \{1\}, \{2\}, \Omega\}$. Let p_1 be defined by $p_1(\{1\}) = \frac{1}{3}$, $p_1(\{2\}) = \frac{2}{3}$ and p_2 by $p_2(\{1\}) = \frac{3}{4}$, $p_2(\{2\}) = \frac{1}{4}$, and let $\mathcal{P} = \{p_1, p_2\}$. First we define the tables V, W, Y and Z by

$$V^1 = \{p \in \mathcal{P} | (1,p) \in V\} := \{p_2\},\ V^2 := \{p_1\}\ ,$$

$$W^1 := \mathcal{P},\ W^2 := \{p_1\}\ ,$$

$$Y^1 = Y^2 := \mathcal{P}\ ,$$

$$Z^1 = \{p_1\},\ Z^2 := \{p_2\}\ .$$

Let $\mathcal{U}$ be defined by $\mathcal{U} := \{V, W, Y\}$ and $\gamma : \mathcal{U} \times \Omega \to [0,1]$ by

$$\gamma(V,\omega) = \gamma_1,\ \gamma(W,\omega) = \gamma_2,\ \gamma(Y,\omega) = 1$$

for all $\omega \in \Omega$, with $\gamma_1 \leq \gamma_2 < 1$. If $\gamma_1 = \frac{2}{3}$ and $\gamma_2 = \frac{3}{4}$, then the interpretation of this γ is

$$\gamma(U,\omega) = \inf\{p(U_p) | p \in \mathcal{P}\}$$

for all $\omega \in \Omega$ and $U \in \mathcal{U}$.
The pair $(\mathcal{U}, \gamma)$ is an inference rule. Evidently we have

$$\mathcal{R}^\omega = \{\emptyset, \mathcal{P}, \{p_1\}, \{p_2\}\},\ \ \omega \in \Omega\ .$$

We investigate the existence of the collection of inferential distributions corresponding to $(\mathcal{U}, \gamma)$. According to corollary 4.1 we have

$$\Gamma^1(V^1) = \gamma_1,\ \Gamma^1(W^1) = \Gamma^1(Y^1) = 1\ ,$$

$$\Gamma^2(V^2) = \Gamma^2(W^2) = \gamma_2,\ \Gamma^2(Y^2) = 1\ .$$

There exist probability measures P^ω $(\omega = 1,2)$ on $(\mathcal{P}, \mathcal{R}^\omega)$ such that $P^\omega(U^\omega) = \Gamma^\omega(U^\omega)$ for all $U \in \mathcal{U}$, defined by

$$P^1(\{p_2\}) = \gamma_1,\ P^1(\{p_1\}) = 1 - \gamma_1\ ,$$

$$P^2(\{p_2\}) = 1 - \gamma_2,\ P^2(\{p_1\}) = \gamma_2\ .$$

Remark 1.
If we define $\mathcal{U}_2 := \{V, W\}$ and $\overline{\gamma}$ as the restriction of γ to $\mathcal{U}_2$, then there does not exist a collection of inferential distributions corresponding to $(\mathcal{U}_2, \overline{\gamma})$, because

$$\Gamma^1(W^1) = \Gamma^1(\mathcal{P}) = \gamma_2 < 1\ .$$

Remark 2.
If we define $\mathcal{U}_3 := \{V, Y\}$ and $\hat{\gamma}$ accordingly, then there does exist a collection of inferential distributions corresponding to $(\mathcal{U}_3, \hat{\gamma})$, but in general the collection is different from the one above, because now we have $P^2(\{p_1\}) = \gamma_1$.

Remark 3.
If we define $\mathcal{U}_4 := \{V, Z\}$ and the function $\delta : \mathcal{U}_4 \times \Omega \to [0, 1]$ by

$$\delta(V, \omega) = \gamma_1,\ \delta(Z, \omega) = \gamma_3$$

for all $\omega \in \Omega$, then there does not exist a collection of inferential distributions corresponding to $(\mathcal{U}_4, \delta)$ in general, because $\Gamma^1(V^1) = \Gamma^1(\{p_2\}) = \gamma_1$, $\Gamma^1(Z^1) = \Gamma^1(\{p_1\}) = \gamma_3$ and $\gamma_1 + \gamma_3$ unequal one in general.

References

Berkum, E.E.M. van, Linssen, H.N. and Overdijk, D.A. (1994). The reduction and ancillary classification of observations. (Submitted for publication).

Wilkinson, G.N. (1977). On Resolving the Controversy in Statistical Inference, J.R. Statist. Soc. B, 39, 119-171.