

A theorem on choice functions

Citation for published version (APA):

Bruijn, de, N. G. (1957). A theorem on choice functions. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen: Series A: Mathematical Sciences*, 60(4), 409-411.

Document status and date:

Published: 01/01/1957

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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MATHEMATICS

A THEOREM ON CHOICE FUNCTIONS

BY

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(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of March 30, 1957)

Let M be an infinite set, and let its cardinal number be denoted by $|M| = \mathfrak{m}$. Let, for each element $\mu \in M$, a non-empty set A_μ be given. For our purpose it is immaterial whether A_μ 's with different μ 's have or do not have intersections.

A function f that attaches to every $\mu \in M$ an element of A_μ (so $f(\mu) \in A_\mu$ for all $\mu \in M$) will be called a *choice function*. Two choice functions f_1, f_2 are different if there is a $\mu \in M$ such that $f_1(\mu) \neq f_2(\mu)$. If there are three different choice functions, however, there need not be a μ such that the three values at μ are mutually different. It certainly will not be the case if all A_μ 's have just two elements.

Denote the collection of all choice functions by F . For several purposes it is of some importance to have a large subcollection of choice functions which are *independent*, in the following sense: if we take any finite number of elements f_1, \dots, f_k from the collection, then there is a $\mu \in M$ such that $f_1(\mu), \dots, f_k(\mu)$ are mutually different. From what we just remarked it is clear that it will be necessary that the order $|A_\mu|$ of A_μ is unbounded if μ runs through M .

We shall impose a slightly stronger condition: we shall assume that the $|A_\mu|$'s are not essentially bounded, i.e. we assume that for every positive integer n there are still \mathfrak{m} μ 's such that $|A_\mu| > n$. Under this condition we can show that there is a collection of $2^{\mathfrak{m}}$ independent choice functions. In many cases this number is equal to the number of all choice functions: if $|A_\mu| \leq 2^{\mathfrak{m}}$ for all $\mu \in M$, then the total number of choice functions is $(2^{\mathfrak{m}})^{\mathfrak{m}} = 2^{\mathfrak{m}}$ (assuming that $|A_\mu| \geq 2$ for \mathfrak{m} values of μ).

Theorem. If $|A_\mu|$ is not essentially bounded, then there is an $F^* \subset F$, with $|F^*| = 2^{\mathfrak{m}}$, such that the elements of F^* are independent.

Proof. We first show that M can be divided into a countable number of disjoint subsets M_0, M_1, M_2, \dots , such that

$$\begin{aligned} |M_j| &= \mathfrak{m} & (j = 0, 1, 2, \dots), \\ |A_\mu| &\geq 2^j & (\mu \in M_j, j = 0, 1, 2, \dots). \end{aligned}$$

This can be done as follows. Let M^i ($i = 0, 1, 2, \dots$) denote the set of all μ 's with $|A_\mu| \geq 2^i$, and put $M^\infty = \bigcap_{i=0}^{\infty} M^i$. We distinguish two cases. (i) If $|M^\infty| = \mathfrak{m}$, then we divide M^∞ into a countable number of disjoint subsets $M_0^\infty, M_1^\infty, \dots$, each having power \mathfrak{m} , and we take $M_j = M_j^\infty$ ($j = 1, 2, \dots$), $M_0 = M_0^\infty \cup (M \setminus M^\infty)$. (ii) If $|M^\infty| < \mathfrak{m}$, then we take $M_0 = M^\infty \cup (M^0 \setminus M^1)$,

where i_1 is an integer > 0 with the property that $M^0 \setminus M^{i_1}$ has power \mathbf{m} . Next we take $M_1 = M^{i_1} \setminus M^{i_2}$, where i_2 is an integer $> i_1$ such that $M^{i_1} \setminus M^{i_2}$ has power \mathbf{m} . We take $M_2 = M^{i_2} \setminus M^{i_3}$, where $i_3 > i_2$, such that $|M_3| = \mathbf{m}$, etc. In both cases it is obvious that the M_j 's have the required properties.

Let X be any set with cardinal number \mathbf{m} . It is well known that the cartesian products

$$X^2 = X \times X, X^3 = X \times X \times X, X^4, X^5, \dots$$

all have cardinal number \mathbf{m} . So for $j=1, 2, 3, \dots$, M_j can be mapped one-to-one onto X^j , and there is no objection against assuming that $M_j = X^j$ ($j=1, 2, \dots$).

Let T be a set of two elements, and put $T \times T = T^2$, $T \times T \times T = T^3$, etc. If $j=1, 2, \dots$, $\mu \in M_j$, then T^j can be embedded into A_μ . Our construction of F will only use choice functions whose value at μ is taken from T^j .

The set M_0 is unimportant. For each $\mu \in M_0$ we choose an element $a_\mu \in A_\mu$, and we take $f^*(\mu) = a_\mu$ ($\mu \in M_0$) for all $f^* \in F^*$. The independence will come from the M_j 's with $j > 0$. There is no harm in omitting M_0 entirely.

The problem has thus been reduced to the following case:

$$(1) \quad M = X \cup X^2 \cup X^3 \cup \dots,$$

and $A_\mu = T^j$ if $\mu \in X^j$ ($j=1, 2, \dots$). The elements of X^j can be described as j -tuples (x_1, \dots, x_j) ($x_1 \in X, \dots, x_j \in X$); the elements of T^j are j -tuples (t_1, \dots, t_j) ($t_1 \in T, \dots, t_j \in T$).

Let F_0 denote the set of all mappings f of X into T . There are $2^{\mathbf{m}}$ such f 's. For every $f \in F_0$ we shall define its continuation f^* as a choice function on M . If $f \in F_0$, $j=1, 2, \dots$, we simply define f^* on X^j by

$$(2) \quad f^*((x_1, \dots, x_j)) = (f(x_1), \dots, f(x_j));$$

the right-hand-side is an element of T^j . Obviously $f^*(\mu) = f(\mu)$ if $\mu \in X$. By this extension procedure we have mapped F_0 one-to-one into F , for if f_1 and f_2 are different on X , then their extensions to M are different, because $X \subset M$. Our collection F^* will now be defined as the set of all f^* , if f runs through F_0 .

It is not difficult to show that the elements of F^* are independent. Let k be any positive integer, and take k distinct elements of F^* . These can be represented as f_1^*, \dots, f_k^* , where f_1, \dots, f_k are k distinct elements of F .

If $1 \leq p < q \leq k$, then f_p and f_q are different. That is to say, there is an element $x_{pq} \in X$ such that $f_p(x_{pq}) \neq f_q(x_{pq})$. Now take $j = \frac{1}{2}k(k-1)$, and in X^j consider the point

$$(3) \quad \mu = (x_{12}, x_{13}, \dots, x_{1k}, x_{23}, \dots, x_{2k}, \dots, x_{k-1,k}).$$

At this point the functions f_1^*, \dots, f_k^* all have different values. In order to show that $f_p^*(\mu) \neq f_q^*(\mu)$, we just look at the place corresponding to the

entry x_{pq} in (3). At that place, $f_p^*(\mu)$ has $f_p(x_{pq})$ (according to (2)), whereas $f_q^*(\mu)$ has $f_q(x_{pq})$. So the points $f_p^*(\mu)$ and $f_q^*(\mu)$ differ in one coordinate, whence they are different points of T^j . This completes the proof.

Remarks. 1. It is not difficult to show that the collection F^* , constructed above, even has the property that its elements are independent in a somewhat stronger sense. For any finite number of distinct elements f_1^*, \dots, f_k^* of F^* we can state that there are \mathbf{m} μ 's such that (for each of these μ 's) the k elements $f_1^*(\mu), \dots, f_k^*(\mu)$ are mutually different. This can be proved as follows: instead of (3) we consider the point

$$(x_0, x_{12}, \dots, x_{k-1, k}) \in T^{j+1},$$

where x_0 is an arbitrary element of X . For every choice of x_0 this is a point where the f_1^*, \dots, f_k^* have different values. And there are \mathbf{m} possibilities for x_0 .

2. Prof. J. DE GROOT mentioned to me that the following special case was known to him: If S is an infinite set, with $|S| = \mathbf{m}$, then there is a collection of $2^{\mathbf{m}}$ subsets S_α (α runs through some index set I of power $2^{\mathbf{m}}$), with the following property. If k is any positive integer, and $\alpha_1, \dots, \alpha_k$ are distinct elements of I , then the difference

$$(4) \quad S_{\alpha_1} \setminus \bigcup_{i=2}^k S_{\alpha_i}$$

has power \mathbf{m} . This can be derived from our theorem in the following way. Divide S into \mathbf{m} countable disjoint subsets $S^{(\mu)}$ (μ runs through an index set M of power \mathbf{m}). For every $\mu \in M$ we put $A_\mu = S^{(\mu)}$. A choice function f_α (defined on M , with $f_\alpha(\mu) \in A_\mu$) now defines a subset S_α of S , viz. the set $\{f_\alpha(\mu) \mid \mu \in M\}$. Taking our collection F^* , we get $2^{\mathbf{m}}$ subsets of S . These evidently satisfy the condition that (4) has power \mathbf{m} , as for \mathbf{m} μ 's $f_{\alpha_1}(\mu)$ is different from $f_{\alpha_2}(\mu), \dots, f_{\alpha_k}(\mu)$.

3. The part of our proof that starts from the special case (1), is independent of the axiom of choice. Therefore, in many special cases a set of $2^{\mathbf{m}}$ independent choice functions can be effectively constructed. For example, if M is the set of all positive integers, and if $A_\mu = \{1, 2, \dots, \mu\}$ ($\mu \in M$), then it is not difficult to give an explicit construction. That is, we can define a collection $\{f_\alpha\}$, where α runs through the collection of all subsets of the set of all positive integers, such that (i) the f_α 's are independent, and (ii) if μ is a given integer, then we can evaluate a number n , such that, for every α , the value of $f_\alpha(\mu)$ can be evaluated when the intersection of α and the set $\{1, 2, \dots, n\}$ is known.

4. In some algebraic systems the theorem can be used for the construction of large subsets whose elements are independent in some algebraic sense. In a subsequent paper we shall show that the group Σ_M of all permutations of an infinite set M contains a free subgroup of order $2^{|M|}$. And, more generally, if we have $2^{|M|}$ groups, each of which can be embedded into Σ_M , then their free product can be embedded into Σ_M . The same thing holds for their direct product.

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