Scaling a network with positive gains to a lossy or gainy network

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ABSTRACT

Necessary and sufficient conditions are presented under which it is possible to scale a network with positive gains to a lossy or a gainy network. A procedure to perform such a scaling operation is given.

INTRODUCTION

In this paper scaling a network with positive gains to a so called lossy network (a network with all gains positive and not greater than one) or to a gainy network (a network with all gains not smaller than one) is considered. Necessary and sufficient conditions will be stated and a procedure to perform such a scaling operation will be described. An obvious advantage of scaling is that algorithms available for lossy or gainy networks, such as [1] and [6], can be applied to a wider class of networks with gains.
Consider a directed and connected graph G(N,A). N is the set of nodes, A the set of arcs. Each arc $(i,j) \in A$ has a gain k_{ij}. Other functions, such as arc capacities may be defined on A as well. It is assumed that there is one source s and one sink t (if there are more sources and/or sinks one "super" source and one "super" sink can be constructed [3, p. 120]). Characteristic for a network with gains is the following set of equations:

(1) $$\sum_j x_{ij} - \sum_j k_{ji}\, x_{ji} = v_s \qquad i = s$$

(2) $$= 0 \qquad i \in N - \{s,t\}$$

(3) $$= -v_t \qquad i = t$$

in which v_s is the input flow, v_t is the output flow, x_{ij}, $(i,j) \in A$ are the flow variables. Equations (2) say that conservation of flow is maintained in intermediate nodes.

A circuit (directed cycle) is said to be absorbing (generating) if the product of gains along the circuit in the direction of the arcs is less (greater) than one.

SCALING GENERAL SETS OF LINEAR CONSTRAINTS

In this chapter the notion of scaling will be formally introduced. Let

(4) $$Bx = b$$

represent a general set of linear constraints.
B is a p $\times$ q matrix, x is a q vector and b is a p vector. Suppose the following transformations are carried out:

(a) $x = Cy$, in which C is a regular diagonal matrix of order q.
(b) both sides of (4) are pre multiplied by a regular diagonal matrix R of order p.

Then equations (4) are said to be scaled to:

(5) $$RBC\,y = Rb$$

The matrix $\bar{B}$, defined by:

(6) $$\bar{B} := RBC$$

is called the scaled matrix of B. In simple words scaling a matrix B means that every column j of B is multiplied by a column scale $c_j \neq 0$ and every row i of B is multiplied by a row scale $r_i \neq 0$. Therefore the elements $\bar{b}_{ij}$ of $\bar{B}$ are given by:

(7) $$\bar{b}_{ij} = r_i\, c_j\, b_{ij} \qquad i = 1,\dots,p;\ j = 1,\dots,q.$$

SCALING NETWORKS WITH GAINS

Necessary and sufficient conditions for scaling a network with gains to:

(a) a network with unit gains ($k_{ij} = 1$, $\forall(i,j) \in A$)
(b) a network with positive gains ($k_{ij} > 0$, $\forall(i,j) \in A$)

are provided by Glover and Klingman [2] for case (a) and Truemper [8] for both cases. In the sequel necessary and sufficient conditions will be given for scaling a network with positive gains to:

(c) a lossy network ($0 < k_{ij} \leq 1$, $\forall (i,j) \in A$)

(d) a gainy network ($k_{ij} \geq 1$, $\forall (i,j) \in A$)

Also a procedure will be described, which performs such a scaling operation. Because there is much resemblance with the work of Truemper [8], some of his results are given here:

THEOREM 1. A network with gains can be scaled to a network with unit gains (rep. positive gains) if and only if nonzero d_i ($\forall i \in N$) exist such that $\frac{d_i k_{ij}}{d_j} = 1$ (resp. $\frac{d_i k_{ij}}{d_j} > 0$) for all $(i,j) \in A$.

A scaling procedure for cases (a) and (b) is:

scaling procedure 1:

(α) Select an arbitrary spanning tree T in G(N,A).

(β) Put $d_u = 1$ for some $u \in N$ and determine d_i ($\forall i \in N$) such that $d_i k_{ij} = d_j$ for all $(i,j) \in T$.

(γ) New gains are given by $\bar{k}_{ij} := \frac{d_i k_{ij}}{d_j}$ for all $(i,j) \in A$.

SCALING TO LOSSY OR GAINY NETWORKS

In the rest of this paper the following assumptions hold:

1. there exists a directed path from s to all $i \in N$
2. $0 < k_{ij} < \infty$ (all $(i,j) \in A$)

Two theorems for lossy networks are stated and proved next:

THEOREM 2. A network with positive gains can be scaled to a lossy network if and only if nonzero d_i ($\forall i \in N$) exist such that $0 < \frac{d_i k_{ij}}{d_j} \leq 1$ for all $(i,j) \in A$.

Proof

(if): Let $d_i \neq 0$ ($\forall i \in N$) be given such that $0 < \frac{d_i k_{ij}}{d_j} \leq 1$. Substitute $x_{ij} = d_i y_{ij}$ ($\forall (i,j) \in A$) in equations (1) - (3) and divide each row i of these equations by d_i. The result is a network with gains $\bar{k}_{ij} := \frac{d_i k_{ij}}{d_j}$, with y_{ij} as variables (arcs). Because of the assumption we have: $0 < \bar{k}_{ij} \leq 1$ for all $(i,j) \in A$.

(only if): Let $^1/d_i$ $(\forall i \in N)$ be the row scales and c_{ij} $(\forall(i,j) \in A)$ be the column scales. Each coëfficient in the left hand side of equations (1) - (3) with value 1 must remain 1 after scaling. So according to (7) c_{ij} must be equal to d_i for all j, for which $(i,j) \in A$. Therefore scaling always results in a network with gains: $\bar{k}_{ij} := \frac{d_i k_{ij}}{d_j}$ and it is proved that $0 < \frac{d_i k_{ij}}{d_j} \leq 1$ is necessary for scaling to a lossy network. □

THEOREM 3. A network with positive gains can be scaled to a lossy network if and only if the network does not contain any generating circuit.

Proof.

(if): Define $d_{ij} := -\ln k_{ij}$ for all $(i,j) \in A$. For any circuit C the following property holds:

$$\prod_{(i,j)\in C} k_{ij} \leq 1, \text{ so:}$$

$$\sum_{(i,j)\in C} d_{ij} \geq 0, \tag{8}$$

If d_{ij} is considered as "length" of arc (i,j), constraint (8) says that there are no so called negative (directed) cycles in the network. Therefore, under assumptions 1 and 2, there is a finite shortest path between s and all other nodes in V. Let Δ_i be the shortest distance from node s to node $i \in N$ $(\Delta_s := 0)$ and define

$$d_i := e^{-\Delta_i}, \quad \forall i \in N.$$

Because Δ_i is the shortest distance from s to i, the following inequality holds:

(10) $$\Delta_i + d_{ij} \geq \Delta_j \quad , \quad \forall (i,j) \in A$$

That is to say:

(11) $$0 < \bar{k}_{ij} := \frac{d_i\ k_{ij}}{d_j} \leq 1, \quad \forall (i,j) \in A$$

since d_i is finite and positive for all $i \in N$.

(only if): Suppose the network contains a generating circuit C consisting of arcs $(1,2)$, $(2,3)$, ..., $(\ell-1,\ell)$, $(\ell,1) \in A$. Moreover suppose that nonzero $d_i (\forall i \in N)$ exist such that:

(12) $$0 < \bar{k}_{ij} := \frac{d_i\ k_{ij}}{d_j} \leq 1, \quad \forall (i,j) \in A$$

The following statement holds:

$$\prod_{(i,j)\in C} \bar{k}_{ij} = \frac{d_1\ k_{12}}{d_2} \cdot \frac{d_2\ k_{23}}{d_3} \cdot \ldots \cdot \frac{d_{\ell-1}\ k_{\ell-1,\ell}}{d_\ell} \cdot \frac{d_\ell\ k_{\ell 1}}{d_1}$$

$$= \prod_{(i,j)\in C} k_{ij} \ .$$

Now $\prod_{(i,j)\in C} \bar{k}_{ij} \leq 1$, because of (12), but $\prod_{(i,j)\in C} k_{ij} > 1$, because C is a generating circuit. From this contradiction the assertion follows. □

For gainy networks analogous theorems hold:

THEOREM 4. A network with positive gains can be scaled to a gainy network if and only if nonzero $d_i (\forall i \in N)$ exist such that $\frac{d_i\ k_{ij}}{d_j} \geq 1$ for all $(i,j) \in A$.

<u>Proof</u>: analogously as for theorem 2. □

THEOREM 5. A network with positive gains can be scaled to a gainy network if and only if the network does not contain any absorbing circuit.

Proof. Analogously as for theorem 3 with:

$$d_{ij} := \ln k_{ij} \qquad \forall (i,j) \in A$$
$$\Delta_i := e^{\Delta_i}. \qquad \forall i \in N \qquad \square$$

A procedure for scaling networks with positive gains to lossy (resp. gainy) networks is:

scaling procedure 2

(α) Define $d_{ij} := -\ln k_{ij}$ (resp. $d_{ij} := \ln k_{ij}$) as length of arc (i,j), $\forall (i,j) \in A$ and determine the shortest distance Δ_i from s to all $i \in N$ $(\Delta_s = 0)$.

(β) Put $d_i := e^{-\Delta_i}$ (resp. $d_i = e^{\Delta_i}$), $\forall i \in N$.

(γ) New gains are given by $\bar{k}_{ij} := \frac{d_i k_{ij}}{d_j}$ for all $(i,j) \in A$.

In order to solve the shortest path problem in step (α) an algorithm should be used which can handle negative arc lengths, such as [9]. Yen's algorithm [9] finds out whether negative (directed) cycles are present in the network. If not, it determines the shortest path from one node to all other nodes efficiently (in polynomial time).

It is well known that the union of shortest paths from one node to all other nodes in a network G is a spanning tree if G has no nonpositive cycles (and contains a spanning tree if G has no negative cycles, cf. [4, p. 66,67]). Therefore the main difference with Truempers scaling procedure [8] is that in stead of an arbitrary spanning tree a special spanning tree is determined in step (α).

Obviously there is a strong relationship between the one-to-all shortest path problem and the scaling problem in a network with gains.

In both problems a spanning tree plays an important role. Theorems 3 and 5 are comparable with a theorem of Nemhauser [5] for shortest path problems. He gives necessary and sufficient conditions under which a shortest path problem can be transformed to a shortest path problem with positive distances only.

It is remarked that an an acyclic network does not contain absorbing or generating circuits and can therefore be scaled to a lossy network and to a gainy network.

EXAMPLE.

Consider the network G of fig. 1, specified in the first two columns of table 1.

Arcs	k_{ij}	$\bar{k}_{ij}$
s1	2	1
s2	1	$\frac{1}{3}$
12	$\frac{3}{2}$	1
14	$\frac{2}{3}$	$\frac{2}{3}$
23	$\frac{1}{2}$	1
24	$\frac{2}{3}$	1
31	1	$\frac{3}{4}$
35	2	1
43	$\frac{1}{2}$	$\frac{2}{3}$
45	1	$\frac{2}{3}$

Table 1.

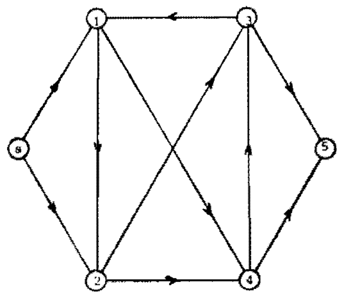

Fig. 1.

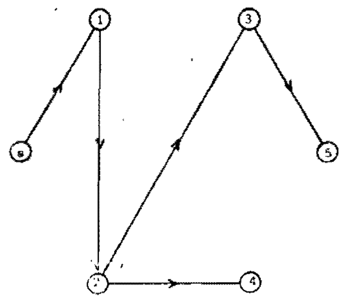

Fig. 2.

G contains no generating circuits and can be scaled to a lossy network. The (in this case unique) spanning tree of shortest paths is given in fig. 2.

$\Lambda_s = 0,\ \Lambda_1 = -\ln 2,\ \Lambda_2 = -\ln 3,\ \Lambda_3 = -\ln \frac{3}{2},\ \Lambda_4 = -\ln 2,\ \Lambda_5 = -\ln 3.$
$d_s = 1,\ d_1 = 2,\ d_2 = 3,\ d_3 = \frac{3}{2},\ d_4 = 2,\ d_5 = 3.$
The values of $\bar{k}_{ij}$ are denoted in the last column of table 1.

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