

Spatial bandwidth: a heuristic calculation with applications

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SPATIAL BANDWIDTH: A HEURISTIC CALCULATION WITH APPLICATIONS

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BACKGROUND AND INTRODUCTION

About $2WT$ real numbers are needed to fully characterize a signal $s(t)$ of effective duration T and bandwidth W ; we then say that the approximate dimension of the signal space is $2WT$ [1]. In a more general case, and in addition to time t , $s(\mathbf{r}, t)$ also depends on the position in space \mathbf{r} . Such a model appears, for instance, in the electromagnetic analysis of waveguides or optical fibres [2]. The boundary conditions for $s(\mathbf{r}, t)$ are stated in terms of feasible regions \mathcal{A} and directions Ω : the field in a waveguide must be confined to its interior, and there is a limited range of directions (angles) that remain guided and therefore do not radiate. We then speak of spatial modes, which are a set of orthonormal functions onto which the signal can be uniquely decomposed.

Past tentative applications of this spatial analysis to communication theory were made by Gabor [3] and Levitin and Lebedev [4]. Recent work by Poon, Brodersen, Tse [5] on multiple-antenna channels combines electromagnetism and antenna theory to extend the $2WT$ -theorem and include the spatial dimensions. In the present work we analyze the dimension of the signal space with fundamental methods based on the uncertainty principle, thus providing an alternative to the counting method in [5]. As application, we also estimate the dimension of the signal space for optical transmission and storage and for satellite links.

DEFINITIONS AND NOTATION

We consider the reception of information via electromagnetic radiation¹ in a volume of space and time \mathcal{A}_T . Instead of Maxwell's equations, we adopt a very simple quantum-mechanical approach. Following Feynman [6], radiation is

¹For the sake of simplicity, we ignore propagation effects. This makes the analysis of (physical) transmission and reception equivalent, by a simple inversion of the radiation direction.

carried by discrete units, the photons, whose behaviour is described in terms of a complex-valued, square-integrable wave function $\psi(\mathbf{r}, t) \in L^2$. We denote the spatial and temporal coordinates by \mathbf{r} and t respectively, and associate to (\mathbf{r}, t) a 4-vector \mathbf{x} , in the sense of special relativity². Photons are observed in a region of space-time \mathcal{A}_T , i. e., $\mathbf{r} \in \mathcal{A}$, and with no real loss of generality $0 \leq t \leq T$. A dual 4-vector $\boldsymbol{\xi}$, including the frequency ν (energy) and wavevector \mathbf{k} (momentum)³ of the photons [6], allows us to link $\psi(\mathbf{x})$ and its dual $\hat{\psi}(\boldsymbol{\xi})$ via the Fourier transform:

$$\hat{\psi}(\boldsymbol{\xi}) = \int \psi(\mathbf{x}) e^{j2\pi\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x}, \quad \hat{\psi}(\mathbf{x}) = \int \hat{\psi}(\boldsymbol{\xi}) e^{-j2\pi\mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}, \quad (1)$$

where the integrals are taken over the relevant 4 dimensions⁴. Dual constraints, denoted by $\boldsymbol{\xi} \in \Omega_W$, are a range of available frequencies $\nu_{\min} \leq \nu \leq \nu_{\max}$, with $W = \nu_{\max} - \nu_{\min}$ and of feasible directions $\mathbf{k}/|\mathbf{k}| \in \Omega$.

The function $\psi(\mathbf{r}, t)$ admits two possible interpretations:

- In a quantum analysis ψ gives the expected number of photons in a space-time interval \mathcal{A}_T .
- In a classical analysis, ψ is related to the electromagnetic potential [6] and, when many photons are present, it describes a real-valued field $w(\mathbf{r}, t)$ as $w(\mathbf{r}, t) = \text{Re}(\psi(\mathbf{r}, t))$. This implies that $\psi(\mathbf{r}, t)$ is closely related to Gabor's analytic signal [3], a link we shall not explore further here.

Let us denote the characteristic functions of the space-time and momentum-energy regions by $\chi_{\mathcal{A}_T}$ and χ_{Ω_W} respectively. We define two orthogonal projection operators, P_T and P_W , and let them act on ψ as $P_T\psi = \psi\chi_{\mathcal{A}_T}$ and $(P_W\psi) = \hat{\psi}\chi_{\Omega_W}$. In general, we shall say that $\psi \in L^2$ is ϵ -concentrated on the set \mathcal{S} , with $\epsilon > 0$, if $\|\psi - \psi\chi_{\mathcal{S}}\| \leq \epsilon\|\psi\|$.

The effect of the constraints is modelled [1] by the operator $P_T Q_W P_T$; the dimension of the Hilbert space in which $P_T Q_W P_T \psi$ lie is reduced to a finite value, n_m . Denoting the elements of a basis of that subspace by ψ_i we have

$$\psi = \sum_{i=1}^{n_m} c_i \psi_i, \quad c_i = \langle \psi | \psi_i \rangle, \quad (2)$$

² $\mathbf{x} = (x, y, z, jct)$, $\mathbf{r} = (x, y, z)$ are the spatial components, c is the speed of light, and $j = \sqrt{-1}$. Furthermore, ψ is a two-dimensional vector, as corresponds to the two different polarization states of the photon.

³Momentum \mathbf{p} and energy E are related to the wavevector \mathbf{k} and the frequency ν as $\mathbf{p} = \frac{h}{2\pi}\mathbf{k}$ and ν as $E = h\nu$ respectively. $\boldsymbol{\xi} = h^{-1}(p_x, p_y, p_z, jE/c)$, h Planck's constant.

⁴We thus recover the expected Fourier relationship $\hat{\psi}(\mathbf{k}, \nu) = \int \psi(\mathbf{r}, t) e^{-j(2\pi\nu t - \mathbf{k}\cdot\mathbf{r})} d\mathbf{r} dt$.

where c_i are the coefficients in the expansion, calculated from the inner product in the Hilbert space. By definition, the functions ψ_i are orthonormal in space and in time, and thus generalize the concept of bandwidth to encompass the spatial dimensions. We indistinctly refer to n_m as the number of degrees of freedom, or as the dimension of the signal space.

THE UNCERTAINTY PRINCIPLE

A heuristic counting rule used by physicists is to give the number of degrees of freedom as the volume in phase space $(\mathbf{x}, \boldsymbol{\xi})$. It can be traced back at least to Bose's derivation in 1925 of the blackbody radiation formula in terms of particles. A similar heuristic application can be found in Gabor [3], who was the first to apply Heisenberg's uncertainty principle to problems in signal theory. Efforts to set the results in a more formal framework fructified in the 1980's, when Donoho and Stark [7], and Folland and Sitaram [8], proved

Theorem 1 *Let \mathcal{A}_T and Ω_W have finite measure. If there is a nonzero $f \in L^2$ such that f is ϵ -concentrated on T and \hat{f} is δ -concentrated on W , then*

$$(1 - \epsilon - \delta) \leq |P_T Q_W|, \quad (3)$$

where $|P_T Q_W|$ is the norm of $P_T Q_W$ as an operator on L^2 . Furthermore,

$$(1 - \epsilon - \delta)^2 \leq |T||W|. \quad (4)$$

We now apply this theorem to wave packets ψ_i , concentrated ($\epsilon = \delta = 0$) in small intervals $d\mathbf{x}$, $d\boldsymbol{\xi}$ around a central point $(\mathbf{x}_0, \boldsymbol{\xi}_0)$; the wave packet is best understood as an essentially different position that a photon can occupy. Then

$$|d\mathbf{x}||d\boldsymbol{\xi}| \geq (1 - \epsilon - \delta)^2 = 1, \quad (5)$$

a formula which supports the heuristic rule of giving the number of dimensions by the number of boxes in the joint space $(\mathbf{x}, \boldsymbol{\xi})$ of volume 1, i. e.,

$$\mathcal{B} = \{(\mathbf{x}, \boldsymbol{\xi}) \mid |d\mathbf{x}| = |\mathbf{x} - \mathbf{x}_0| < \Delta, |d\boldsymbol{\xi}| = |\boldsymbol{\xi} - \boldsymbol{\xi}_0| < \Delta^{-1}\}. \quad (6)$$

The boxes \mathcal{B} tile the phase space $(\mathbf{x}, \boldsymbol{\xi})$, each of them contributing with one orthogonal basis function ψ_i , and simultaneously with a small element $d\mathbf{x}d\boldsymbol{\xi}$,

$$n_m \simeq \sum_i \mathcal{B}_i = \iint \chi_{\mathcal{A}_T} \chi_{\Omega_W} d\mathbf{x} d\boldsymbol{\xi}, \quad (7)$$

where we have identified the sum of boxes \mathcal{B}_i as the total volume in phase space.

Remark 1 There are 4 variables at dx and $d\xi$, but they are coupled: special relativity considerations fix $dx^2 + dy^2 + dz^2 = c^2 dt^2$, and similarly for the energy-momentum, $dE^2 = c^2 |d\mathbf{p}|^2 = c^2 (dp_x^2 + dp_y^2 + dp_z^2)$. We assume that radiation propagates along direction z , and we choose the 3 remaining variables to be (x, y, t) and (p_x, p_y, p_z) ; we also use that $E = c|\mathbf{p}|$. The element $dx dy$ is then an infinitesimal element of surface $d\sigma$ orthogonal to the main direction of propagation.

Remark 2 A possible variation of the set of allowed directions/frequencies along the position/time can be neatly accounted for by making use of the dependence of dx and $d\xi$ on $\mathbf{x}_0 = (\mathbf{r}_0, t_0)$.

Remark 3 In case any of the 3 available dimensions, say i , is not used for communication, the corresponding pair of variables x_i and ξ_i is set to $dx_i d\xi_i = 1$.

APPLICATION: 1-DIMENSIONAL LINKS

For one-dimensional, or point-to-point, links the last remark suggests that integration over \mathbf{r} and \mathbf{k} is not required (that is, it gives one cell), and

$$n_m^{1D} = \iint \chi_{A_T} \chi_{\Omega_W} d\nu dt = WT, \quad (8)$$

as it should. The extra factor 2 in the $2WT$ -theorem comes from the two real components implicit in the complex vector.

APPLICATION: 3-DIMENSIONAL LINKS

For 3-dimensional transmissions, we use Remark 1 to consider only a flat surface orthogonal to the direction of propagation and write $dx = d\sigma dt$. We also assume no time variations. The use of spherical coordinates for ξ yields

$$c(\mathbf{r}_0) d\xi = \left(\frac{\nu}{c(\mathbf{r}_0)} \right)^2 d\nu d\Omega = \Omega(\mathbf{r}_0) \left(\frac{\nu}{c(\mathbf{r}_0)} \right)^2 d\nu, \quad (9)$$

where \mathbf{r}_0 runs through the transmitter surface, in which case the local solid angle $\Omega(\mathbf{r}_0)$ and speed of light $c(\mathbf{r}_0)$ are function of the position. Eq. (7) becomes

$$n_m^{3D} = \left(\int \chi_A \Omega(\mathbf{r}_0) \left(\frac{1}{c(\mathbf{r}_0)} \right)^2 d\sigma \right) \left(T \int \chi_W \nu^2 d\nu \right) \quad (10)$$

$$= \left(\int_{\mathbf{r}_0 \in A} \Omega(\mathbf{r}_0) \left(\frac{1}{c(\mathbf{r}_0)} \right)^2 d\sigma \right) \left(\frac{T}{3} (\nu_{\max}^3 - \nu_{\min}^3) \right) \quad (11)$$

$$\simeq WT \int_{\mathbf{r}_0 \in A} \Omega(\mathbf{r}_0) \left(\frac{\nu_0^2}{c(\mathbf{r}_0)} \right)^2 d\sigma. \quad (12)$$

In the last equality we have assumed that the bandwidth W is relatively small compared to the frequencies ν_{\max} and ν_{\min} . We also denote the central frequency by ν_0 and the corresponding wavelength by $\lambda_0 = \frac{c}{\nu_0}$. Note that the spatial and temporal modes are neatly separated, and that the "total" bandwidth thus consists of a temporal and a spatial component.

If there are no variations in the surface we recover Poon's formula⁵ [5]

$$n_m^{3D} = A\Omega WT \frac{1}{\lambda_0^2}. \quad (13)$$

APPLICATIONS: OPTICAL FIBRE

An optical fibre has circular symmetry, which allows us to write the integral Eq. (12) in polar coordinates. Assuming narrowband communications,

$$n_m^{\text{fibre}} = 2\pi WT \nu_0^2 \int_0^a \Omega(\rho) \left(\frac{1}{c(\rho)} \right)^2 \rho d\rho, \quad (14)$$

where a is the radius of the fibre core. The solid angle is $\Omega(\rho) = 2\pi(1 - \cos\theta(\rho))$, where $\theta(\rho)$ is the half-angle of the cone defined by the numerical aperture of the fibre. The largest angle θ for which rays propagate is given by Snell's law [2] $\cos\theta = \frac{n_{cl}}{n_c(\rho)}$, where n_{cl} is the index at the fibre cladding. $c(\rho) = \frac{c}{n_c(\rho)}$, where c is the speed of light in vacuum and $n_c(\rho)$ is the refractive index. Therefore

$$n_m^{\text{fibre}} = 2\pi WT \nu_0^2 \int_0^a \Omega(\rho) \left(\frac{1}{c(\rho)} \right)^2 \rho d\rho, \quad (15)$$

$$= 4\pi^2 WT \frac{\nu_0^2}{c^2} \int_0^a n_c(\rho)^2 \rho \left(1 - \frac{n_{cl}}{n_c(\rho)} \right) d\rho. \quad (16)$$

Let us now assume a power-law index profile with parameter α , that is, $n_c(\rho) = n_0 \sqrt{1 - 2\Delta \left(\frac{\rho}{a} \right)^\alpha} \simeq n_0 \left(1 - \Delta \left(\frac{\rho}{a} \right)^\alpha \right)$; we use here the weakly guiding approximation, that is $\Delta \ll 1$, and keep only the terms of order Δ . Replacing this in Eq. (16), and after some algebra we obtain

$$n_m^{\text{fibre}} = \frac{4\pi^2 WT}{\lambda_0^2} n_0^2 \Delta \frac{a^2}{2} \frac{\alpha}{\alpha + 2}. \quad (17)$$

⁵Note that photon polarization doubles the number of degrees of freedom and the caveat that their analysis gives the number of real numbers required to represent the signal, giving an extra factor of 2.

The number of spatial modes coincides with the value of the electromagnetic analysis [2], providing an alternative derivation and interpretation of the formula. Note also that the condition for single-mode guiding [2], applied to Eq. (17) gives just 1.4 modes, a very good approximation.

APPLICATIONS: OPTICAL STORAGE

In this case information is purely stored in and retrieved from space, and using Remark 3 we consider only the spatial modes, i. e., $dvd t = 1$. As an example, let us take Blue Disk/TwoDOS. The wavelength in vacuum is $\lambda_0 = 405$ nm, the numerical aperture is 0.85, and the disk diameter is 12 cm. Denoting by n_c the refractive index in the region above the disk, we have

$$n_m^{\text{disk}} = \frac{A_{\text{disk}}}{\lambda_0^2} \Omega n_c^2 \simeq 6.8 n_c^2 \text{ GB.} \quad (18)$$

Here the numerical aperture is related to the half-angle θ of the acceptance cone by $\text{NA} = \sin 2\theta$, and $\Omega = 2\pi(1 - \cos \theta) \simeq 0.8$ sterad. Table 1 compares the results for some existing systems with their theoretical limits. The discrepancies between the reported values and the predictions of the formula are likely to be related to the exact value of the refractive index. It should be noted that the analysis points at the tuning of the refractive index as a possible way of increasing the capacity of such a storage medium.

System	λ (nm)	NA	Ω (sterad)	Storage (GB)	
				Real	Z_{max}
CD	780	0.45	0.17	0.7	$0.4 n_c^2$
DVD	650	0.60	0.32	4.7	$1.1 n_c^2$
Blue Disk	405	0.85	0.79	25.0	$6.8 n_c^2$

Table 1: Number of Degrees of Freedom for Optical Storage

APPLICATIONS: SATELLITE LINKS (BEAMFORMING)

Let us study a single satellite on geostationary orbit, at about $d_{\text{sat}} = 36000$ km from Earth. The illuminated land surface S subtends a solid angle from the satellite very closely given by $\gamma S d_{\text{sat}}^{-2}$, where γ is a correction factor taking into

account the latitude ⁶ φ :

$$\gamma = \cos(\varphi + \theta), \quad \theta = \arctan \frac{r_E \sin \varphi}{r_E(1 - \cos \varphi) + d_{\text{sat}}}, \quad (19)$$

here r_E is the radius of the Earth, about 6400 km. For simplicity, and with no real loss of precision, we ignore the variations of latitude and satellite distance along the land surface, as well as the precise shape of the landmass. This is a consequence of the large distance between the satellite and the Earth surface.

For relatively narrowband systems of central wavelength λ_0 and, in absence of spatial variations along the surface of the satellite, the total number of independent degrees of freedom is

$$n_m^{\text{satellite}} = \frac{A_{\text{sat}}}{\lambda_0^2} \Omega W T = A_{\text{sat}} \gamma S \frac{\nu_0^2}{c^2 d_{\text{sat}}^2} W T. \quad (20)$$

Here c is the light speed in air. For $\nu_0 = 12$ GHz, a usual frequency for satellite TV, and measuring S in millions of square kilometers km^2 and A_{sat} in square meters, the number of spatial channels is given by $n_{\text{spat}} = n_m^{\text{satellite}} / W T = 1.25 A_{\text{sat}} \gamma S$.

Table 2 gives the number of spatial channels per polarization for several regions in the world and for several satellite dimensions. It can be seen that the satellite must be rather large in order to accommodate many users per region; in this line it should be remembered that areas of 3-4 m^2 are the largest feasible today at this frequency. For the Benelux, it is therefore impossible to simultaneously point at two positions using the same polarization and frequency.

Region (latitude)	Land Area (km^2)	$\gamma(\varphi)$	Number of Channels		
			$A_{\text{sat}} = 1 \text{ m}^2$	10 m^2	100 m^2
Benelux ($\varphi = 52$)	60,000	0.51	< 0.1	0.4	4
Europe ($\varphi = 48$)	2,500,000	0.57	1.8	17.9	179
USA/China ($\varphi = 30$)	10,000,000	0.82	10.2	102	1025

Table 2: Number of Simultaneous Channels per Polarization per Satellite

CONCLUSIONS

In this work we have estimated the dimension of the signal space, taking into account the spatial and temporal aspects in a unified manner. Removing

⁶If the latitude is larger than $\varphi_{\text{max}} = r_E(r_E + d_{\text{sat}})^{-1}$, there is no visibility of the satellite from the Earth, and $\gamma = 0$.

electromagnetic considerations to a minimum allows us to cast the problem in terms of how the number of essentially different positions and directions a photon can take along the transmitting surface. This problem has been tackled with methods based on the uncertainty principle, and shows that the total bandwidth consist of both a temporal and a spatial component.

We have estimated the total bandwidth of several cases, including optical transmission and storage and satellite links. These examples show that the dimension of the signal space provides a good figure of merit for the analysis and design of communication systems.

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