

Solution to problem 67-4 : A double sum

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$$F(x) = zI_0(x) + \sinh z \left[I_0(x) \int_0^x \exp(t \cosh z) K_0(t) dt - K_0(x) \int_0^x \exp(t \cosh z) I_0(t) dt \right].$$

$F(x)$ can also be expressed as a series of modified Bessel functions by making use of the expansion

$$\exp(x \cosh \theta) = I_0(x) + 2 \sum_1^{\infty} I_n(x) \cosh n\theta,$$

giving

$$F(x) = zI_0(x) + 2 \sum_1^{\infty} n^{-1} I_n(x) \sinh nz.$$

Also solved by L. CARLITZ (Duke University), I. FARKAS (University of Toronto) and the proposer.

Editorial note. Carlitz notes that if we put

$$I(z) = \sum_{n=0}^{\infty} \frac{C_{2n} z^{2n}}{(2n)!},$$

it follows that

$$(2n + 1)C_{2n} = -na \operatorname{csch} z \sum_{j=1}^n \binom{2n-1}{2j-1} C_{2n-2j}, \quad n \geq 1.$$

The proposer notes that the singularities of $I(z)$ appear to be isolated essential singularities at the points $z = \pm n\pi i, n = 1, 2, 3, \dots$.

Problem 67-4, A Double Sum, by L. CARLITZ (Duke University).

Show that

$$\sum_{r=0}^m \sum_{s=0}^n \binom{r+s}{r}^2 \binom{m+n-r-s}{m-r}^2 = \frac{1}{2} \binom{2m+2n+2}{2m+1}.$$

Solution by the proposer.

We have

$$\begin{aligned} \{(1-x-y)^2 - 4xy\}^{-1/2} &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r (1-x-y)^{-2r-1} \\ &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r \sum_{k=0}^{\infty} \binom{2r+k}{k} (x+y)^k \\ &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r \sum_{m,n=0}^{\infty} \frac{(2r+m+n)!}{(2r)!m!n!} x^m y^n \\ &= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{(2r)!(m-r)!(n-r)!} \\ &= \sum_{m,n=0}^{\infty} \binom{m+n}{n} x^m y^n \sum_{r=0}^{\min(m,n)} \binom{m}{n} \binom{n}{r} \\ &= \sum_{m,n=0}^{\infty} \binom{m+n}{n}^2 x^m y^n. \end{aligned}$$

Similarly,

$$\begin{aligned}
 \{(1-x-y)^2 - 4xy\}^{-1} &= \sum_{r=0}^{\infty} 2^{2r} x^r y^r (1-x-y)^{-2r-2} \\
 (1) \qquad \qquad \qquad &= \sum_{r=0}^{\infty} 2^{2r} x^r y^r \sum_{m,n=0}^{\infty} \frac{(2r+m+n+1)!}{(2r+1)! m! n!} x^m y^n \\
 &= \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{m! n!} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{(2r+1)!}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{(2r+1)!} 2^{2r} &= \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{r!(3/2)_r} \\
 &= \frac{(3/2)_{m+n}}{(3/2)_m (3/2)_n} = \frac{(2m+2n+1)!}{(2m+1)!(2n+1)!} \frac{m! n!}{(m+n)!},
 \end{aligned}$$

so that

$$(2) \qquad \frac{1}{2} \sum_{m,n=0}^{\infty} \binom{2m+2n+2}{2m+1} x^m y^n = \{(1-x-y)^2 - 4xy\}^{-1}.$$

Comparing (1) and (2) we obtain the stated result.

Remark. In exactly the same way we can prove that

$$\sum_{m,n=0}^{\infty} C(m, n; \lambda) x^m y^n = \{(1-x-y)^2 - 4xy\}^{-\lambda},$$

where

$$C(m, n; \lambda) = \frac{(2\lambda)_{m+n}}{m! n!} \frac{(\lambda + \frac{1}{2})_{m+n}}{(\lambda + \frac{1}{2})_m (\lambda + \frac{1}{2})_n}.$$

This implies

$$\sum_{r=0}^m \sum_{s=0}^n C(r, s; \alpha) C(m-r, n-s; \beta) = C(m, n; \alpha + \beta).$$

J. BOERSMA (Technological University, Eindhoven, The Netherlands) obtained his solution by noting that the double series s_{mn} is the coefficient of $x^m y^n$ in the expansion of the generating function

$$F(x, y) = \left\{ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r+s}{r} x^r y^s \right\}^2 = \{F_4(1, 1, 1, 1; x, y)\}^2,$$

where F_4 denotes a hypergeometric function of two variables. From [4, 5.7(a) and 5.10(b)],

$$F(x, y) = [(1-x-y)^2 - 4xy]^{-1}.$$

Then expanding out $F(x, y)$ by the binomial theory, he showed that

$$\begin{aligned} s_{mn} &= \frac{2^{2m}(n+m+1)!}{(n-m)!(2m+1)!} F\left(-m - \frac{1}{2}, -m; n - m + 1; 1\right) \\ &= \frac{1}{2} \binom{2m+2n+2}{2m+1}. \end{aligned}$$

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