

## Solution to problem 67-4 : A double sum

**Citation for published version (APA):**

Boersma, J. (1968). Solution to problem 67-4 : A double sum. *SIAM Review*, 10(1), 117-119.  
<https://doi.org/10.1137/1010028>

**DOI:**

[10.1137/1010028](https://doi.org/10.1137/1010028)

**Document status and date:**

Published: 01/01/1968

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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- The final published version features the final layout of the paper including the volume, issue and page numbers.

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$$F(x) = zI_0(x) + \sinh z \left[ I_0(x) \int_0^x \exp(t \cosh z) K_0(t) dt - K_0(x) \int_0^x \exp(t \cosh z) I_0(t) dt \right].$$

$F(x)$  can also be expressed as a series of modified Bessel functions by making use of the expansion

$$\exp(x \cosh \theta) = I_0(x) + 2 \sum_1^{\infty} I_n(x) \cosh n\theta,$$

giving

$$F(x) = zI_0(x) + 2 \sum_1^{\infty} n^{-1} I_n(x) \sinh nz.$$

Also solved by L. CARLITZ (Duke University), I. FARKAS (University of Toronto) and the proposer.

*Editorial note.* Carlitz notes that if we put

$$I(z) = \sum_{n=0}^{\infty} \frac{C_{2n} z^{2n}}{(2n)!},$$

it follows that

$$(2n + 1)C_{2n} = -na \operatorname{csch} z \sum_{j=1}^n \binom{2n-1}{2j-1} C_{2n-2j}, \quad n \geq 1.$$

The proposer notes that the singularities of  $I(z)$  appear to be isolated essential singularities at the points  $z = \pm n\pi i, n = 1, 2, 3, \dots$ .

*Problem 67-4, A Double Sum,* by L. CARLITZ (Duke University).

Show that

$$\sum_{r=0}^m \sum_{s=0}^n \binom{r+s}{r}^2 \binom{m+n-r-s}{m-r}^2 = \frac{1}{2} \binom{2m+2n+2}{2m+1}.$$

Solution by the proposer.

We have

$$\begin{aligned} \{(1-x-y)^2 - 4xy\}^{-1/2} &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r (1-x-y)^{-2r-1} \\ &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r \sum_{k=0}^{\infty} \binom{2r+k}{k} (x+y)^k \\ &= \sum_{r=0}^{\infty} \binom{2r}{r} x^r y^r \sum_{m,n=0}^{\infty} \frac{(2r+m+n)!}{(2r)!m!n!} x^m y^n \\ &= \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{(2r)!(m-r)!(n-r)!} \\ &= \sum_{m,n=0}^{\infty} \binom{m+n}{n} x^m y^n \sum_{r=0}^{\min(m,n)} \binom{m}{n} \binom{n}{r} \\ &= \sum_{m,n=0}^{\infty} \binom{m+n}{n}^2 x^m y^n. \end{aligned}$$

Similarly,

$$\begin{aligned}
 \{(1-x-y)^2 - 4xy\}^{-1} &= \sum_{r=0}^{\infty} 2^{2r} x^r y^r (1-x-y)^{-2r-2} \\
 (1) \qquad \qquad \qquad &= \sum_{r=0}^{\infty} 2^{2r} x^r y^r \sum_{m,n=0}^{\infty} \frac{(2r+m+n+1)!}{(2r+1)! m! n!} x^m y^n \\
 &= \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{m! n!} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{(2r+1)!}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{(2r+1)!} 2^{2r} &= \sum_{r=0}^{\min(m,n)} \frac{(-m)_r (-n)_r}{r!(3/2)_r} \\
 &= \frac{(3/2)_{m+n}}{(3/2)_m (3/2)_n} = \frac{(2m+2n+1)!}{(2m+1)!(2n+1)!} \frac{m! n!}{(m+n)!},
 \end{aligned}$$

so that

$$(2) \qquad \frac{1}{2} \sum_{m,n=0}^{\infty} \binom{2m+2n+2}{2m+1} x^m y^n = \{(1-x-y)^2 - 4xy\}^{-1}.$$

Comparing (1) and (2) we obtain the stated result.

*Remark.* In exactly the same way we can prove that

$$\sum_{m,n=0}^{\infty} C(m, n; \lambda) x^m y^n = \{(1-x-y)^2 - 4xy\}^{-\lambda},$$

where

$$C(m, n; \lambda) = \frac{(2\lambda)_{m+n}}{m! n!} \frac{(\lambda + \frac{1}{2})_{m+n}}{(\lambda + \frac{1}{2})_m (\lambda + \frac{1}{2})_n}.$$

This implies

$$\sum_{r=0}^m \sum_{s=0}^n C(r, s; \alpha) C(m-r, n-s; \beta) = C(m, n; \alpha + \beta).$$

J. BOERSMA (Technological University, Eindhoven, The Netherlands) obtained his solution by noting that the double series  $s_{mn}$  is the coefficient of  $x^m y^n$  in the expansion of the generating function

$$F(x, y) = \left\{ \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r+s}{r} x^r y^s \right\}^2 = \{F_4(1, 1, 1, 1; x, y)\}^2,$$

where  $F_4$  denotes a hypergeometric function of two variables. From [4, 5.7(a) and 5.10(b)],

$$F(x, y) = [(1-x-y)^2 - 4xy]^{-1}.$$

Then expanding out  $F(x, y)$  by the binomial theory, he showed that

$$\begin{aligned} s_{mn} &= \frac{2^{2m}(n+m+1)!}{(n-m)!(2m+1)!} F\left(-m - \frac{1}{2}, -m; n - m + 1; 1\right) \\ &= \frac{1}{2} \binom{2m+2n+2}{2m+1}. \end{aligned}$$

## REFERENCE

- [4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.