Solution to problem 67-4 : A double sum

Citation for published version (APA):
Boersma, J. (1968). Solution to problem 67-4 : A double sum. *SIAM Review*, *10*(1), 117-119.
https://doi.org/10.1137/1010028

DOI:
10.1137/1010028

Document status and date:
Published: 01/01/1968

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

$$F(x) = zI_0(x) + \sinh z \left[I_0(x) \int_0^x \exp(t \cosh z) K_0(t)\, dt - K_0(x) \int_0^x \exp(t \cosh z) I_0(t)\, dt \right].$$

$F(x)$ can also be expressed as a series of modified Bessel functions by making use of the expansion

$$\exp(x \cosh \theta) = I_0(x) + 2 \sum_1^\infty I_n(x) \cosh n\theta,$$

giving

$$F(x) = zI_0(x) + 2 \sum_1^\infty n^{-1} I_n(x) \sinh nz.$$

Also solved by L. CARLITZ (Duke University), I. FARKAS (University of Toronto) and the proposer.

Editorial note. Carlitz notes that if we put

$$I(z) = \sum_{n=0}^\infty \frac{C_{2n} z^{2n}}{(2n)!},$$

it follows that

$$(2n + 1)C_{2n} = -na \operatorname{csch} z \sum_{j=1}^n \binom{2n-1}{2j-1} C_{2n-2j}, \qquad n \geqq 1.$$

The proposer notes that the singularities of $I(z)$ appear to be isolated essential singularities at the points $z = \pm n\pi i$, $n = 1, 2, 3, \cdots$.

Problem 67–4, *A Double Sum*, by L. CARLITZ (Duke University).

Show that

$$\sum_{r=0}^m \sum_{s=0}^n \binom{r+s}{r}^2 \binom{m+n-r-s}{m-r}^2 = \frac{1}{2} \binom{2m+2n+2}{2m+1}.$$

Solution by the proposer.

We have

$$\begin{aligned}
\{(1-x-y)^2 - 4xy\}^{-1/2} &= \sum_{r=0}^\infty \binom{2r}{r} x^r y^r (1-x-y)^{-2r-1} \\
&= \sum_{r=0}^\infty \binom{2r}{r} x^r y^r \sum_{k=0}^\infty \binom{2r+k}{k} (x+y)^k \\
&= \sum_{r=0}^\infty \binom{2r}{r} x^r y^r \sum_{m,n=0}^\infty \frac{(2r+m+n)!}{(2r)!\,m!\,n!} x^m y^n \\
&= \sum_{m,n=0}^\infty x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(m+n)!}{(2r)!(m-r)!(n-r)!} \\
&= \sum_{m,n=0}^\infty \binom{m+n}{n} x^m y^n \sum_{r=0}^{\min(m,n)} \binom{m}{n}\binom{n}{r} \\
&= \sum_{m,n=0}^\infty \binom{m+n}{n}^2 x^m y^n.
\end{aligned}$$

Similarly,

$$\{(1-x-y)^2 - 4xy\}^{-1} = \sum_{r=0}^{\infty} 2^{2r} x^r y^r (1-x-y)^{-2r-2}$$

$$(1) \qquad = \sum_{r=0}^{\infty} 2^{2r} x^r y^r \sum_{m,n=0}^{\infty} \frac{(2r+m+n+1)!}{(2r+1)!\,m!\,n!} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(m+n+1)!}{m!\,n!} x^m y^n \sum_{r=0}^{\min(m,n)} \frac{(-m)_r(-n)_r}{(2r+1)!}.$$

Now

$$\sum_{r=0}^{\min(m,n)} \frac{(-m)_r(-n)_r}{(2r+1)!} 2^{2r} = \sum_{r=0}^{\min(m,n)} \frac{(-m)_r(-n)_r}{r!(3/2)_r}$$

$$= \frac{(3/2)_{m+n}}{(3/2)_m(3/2)_m} = \frac{(2m+2n+1)!}{(2m+1)!(2n+1)!} \frac{m!\,n!}{(m+n)!},$$

so that

$$(2) \qquad \frac{1}{2} \sum_{m,n=0}^{\infty} \binom{2m+2n+2}{2m+1} x^m y^n = \{(1-x-y)^2 - 4xy\}^{-1}.$$

Comparing (1) and (2) we obtain the stated result.

Remark. In exactly the same way we can prove that

$$\sum_{m,n=0}^{\infty} C(m,n;\lambda) x^m y^n = \{(1-x-y)^2 - 4xy\}^{-\lambda},$$

where

$$C(m,n;\lambda) = \frac{(2\lambda)_{m+n}}{m!\,n!} \frac{(\lambda+\frac{1}{2})_{m+n}}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_n}.$$

This implies

$$\sum_{r=0}^{m} \sum_{s=0}^{n} C(r,s;\alpha) C(m-r,n-s;\beta) = C(m,n;\alpha+\beta).$$

J. Boersma (Technological University, Eindhoven, The Netherlands) obtained his solution by noting that the double series s_{mn} is the coefficient of $x^m y^n$ in the expansion of the generating function

$$F(x,y) = \left\{\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{r+s}{r}^2 x^r y^s\right\}^2 = \{F_4(1,1,1,1;x,y)\}^2,$$

where F_4 denotes a hypergeometric function of two variables. From [4, 5.7(a) and 5.10(b)],

$$F(x,y) = [(1-x-y)^2 - 4xy]^{-1}.$$

Then expanding out $F(x,y)$ by the binomial theory, he showed that

$$s_{mn} = \frac{2^{2m}(n+m+1)!}{(n-m)!(2m+1)!} F\left(-m-\frac{1}{2}, -m; n-m+1; 1\right)$$
$$= \frac{1}{2}\binom{2m+2n+2}{2m+1}.$$

REFERENCE

[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.