# An asymptotic problem in extremal processes

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**ABSTRACT**

If $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$(*) \qquad \int_{x}^{\infty} \exp[i\,\varphi(s) - s + x]\,ds \to 0 \quad (x \to \infty)$$

then

$$(**) \qquad \int_{-\infty}^{\infty} \exp[i\,\varphi(s+t)]\;F'(s)\,ds \to 0 \quad (t \to \infty)\;,$$

where $F \in \mathcal{F}$. The set $\mathcal{F}$ consists of all probability distributions $F$ on $\mathbb{R}$ satisfying the requirement $\int_{-\infty}^{\infty} |F''(x)|\,dx < \infty$. If $(**)$ holds for all $F \in \mathcal{F}$ then $(*)$ holds.

## INTRODUCTION

Let $(X_m)$ be a sequence of independent random variables with common distribution $F$. We define the sequence $(Z_n)$ by

$$Z_n = \max\{X_1, X_2, \ldots, X_n\} \quad (n \in \mathbb{N}) .$$

Let us denote the fractional part of a real number $\alpha$ by $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$, where $\lfloor\alpha\rfloor$ is the largest integer not exceding $\alpha$. The question is: for which distributions $F$ converges $\{Z_n\}$ in distribution to $U$, where $U$ is uniform distributed on $[0,1)$. In studying this question the following problem $(*)$ occurs: For which functions $H : \mathbb{R}^+ \to \mathbb{R}$ is it true that, for all $k \in \mathbb{Z}\backslash\{0\}$

$$\int_{-\log n}^{\infty} \exp[2\pi ik\, H(x + \log n)]\ \exp[-x - e^{-x}]\, dx \to 0 \quad (n \to \infty) .$$

In this paper this question will be partially answered.

$(*)$ The problem was posed by R. Wilms, Department of Mathematics and Computing Science, Eindhoven University of Technology, The Netherlands.

## PRELIMINARIES

Let $\mathcal{F}$ denote the set of all probability distributions $F$ on $\mathbb{R}$ with the property that $\int_{-\infty}^{\infty} |f'(x)|\, dx < \infty$, where $f := F'$ is the density. We remark that this condition implies that $f(x) \to 0$ if $|x| \to \infty$.
Let $\mathcal{G}$ denote the set of all functions $g$ with the properties: DOM $g \supset [x_g, \infty)$ for some real number $x_g$ (dependent on $g$), $g(x) > 0$ on $[x_g, \infty)$ and $g$ is continuously differentiable on $[x_g, \infty)$.

By $\mathcal{C}$ we denote the set of all real piecewise continuous functions $\varphi$ on $\mathbb{R}$. For $g \in \mathcal{G}$ we denote by $\mathcal{C}(g)$ (or $\mathcal{C}(g(x))$) the subset of all functions $\varphi \subset \mathcal{C}$ which satisfy (1) and (2).

$$\text{(1)} \qquad \int_{x_g}^{\infty} e^{i\varphi(x)}\, g(x)\, dx \ \text{ exists ,}$$

$$\text{(2)} \qquad [(g(x))^{-1} + |g'(x)|\, (g(x))^{-2}] \int_{x}^{\infty} e^{i\varphi(s)}\, g(s)\, ds \to 0 \quad (x \to \infty) .$$

The statement which we consider is

$$\text{(3)} \qquad I(t; F) := \int_{-\infty}^{\infty} e^{i\varphi(s+t)}\, F'(s)\, ds \to 0 \quad (t \to \infty) .$$

## RESULTS

The main results are formulated in Theorems (4) and (5).

(4) THEOREM
(3) holds for all $F \in \mathcal{F}$ if and only if $\varphi \in \mathcal{C}(e^{-x})$.

(5) THEOREM

$$\mathcal{C}(g(x)) \subset \mathcal{C}(e^{-x}) \quad (g \in \mathcal{G})$$

That $\varphi \in \mathcal{C}(g)$ can be a stronger condition then $\varphi \in \mathcal{C}(e^{-x})$ is illustrated in Theorem (6). The question for which $g$ we have $\mathcal{C}(g) = \mathcal{C}(e^{-x})$ is partially answered in Theorem (6).

(6) THEOREM

$$\mathcal{C}(e^{-x^2}) \neq \mathcal{C}(e^{-x})$$

$$\mathcal{C}(1) \neq \mathcal{C}(e^{-x})$$

$$\mathcal{C}(e^{-\alpha x}) = \mathcal{C}(e^{-x}) \quad (\alpha > 0) .$$

The question which functions are contained in $\mathcal{C}(e^{-x})$ is partially answered by Theorem (7) and (8).

(7) THEOREM
If $\varphi$ is twice continuously differentiable, $|\varphi'(x)| \to \infty \;\; (x \to \infty)$ and $\varphi''(x)\,(\varphi'(x))^{-2} \to 0 \;\; (x \to \infty)$ then $\varphi \in \mathcal{C}(e^{-x})$.

(8) THEOREM
Let $h : \mathbb{R} \to \mathbb{R}$ be continuous and, on some interval $[a, \infty)$ be twice continuously differentiable with $h'(x) \geq c > 0 \;\; (x \geq a)$. Moreover $h''(x) = O((h'(x))^2) \;\; (x \to \infty)$. Then $\varphi \in \mathcal{C}(e^{-x})$ implies $\varphi \circ h \in \mathcal{C}(e^{-x})$.

## EXAMPLES

$\mathcal{C}(1)$ consists of all functions $\varphi \in \mathcal{C}$ satisfying the condition that $\int_0^\infty e^{i\varphi(s)}\, ds$ exists.

$$\mathcal{C}(e^{-x^2}) = \{\varphi \in \mathcal{C} \mid x e^{x^2} \int_x^\infty \exp[i\,\varphi(s) - s^2]\, ds \to 0 \quad (x \to \infty)\}$$

$$\mathcal{C}(x^{-2}) = \{\varphi \in \mathcal{C} \mid x^2 \int_x^\infty e^{i\varphi(s)}\, s^{-2}\, ds \to 0 \quad (x \to \infty)\} .$$

The function $\varphi(x) = x$ does not belong to $\mathcal{C}(e^{-x})$. Nevertheless there are distributions $F \in \mathcal{F}$ for which (3) holds with $\varphi(x) = x$. An example is is given in the last section.

If $\varphi$ satisfies the conditions of Theorem (7) then also $k\varphi$ for every $k \in \mathbb{Z}\backslash\{0\}$.

REMARK
The author has not been able to prove or disprove the necessity of the condition $\varphi \in C(e^{-x})$ so that (3) holds in case $F(x) = \exp[-e^{-x}]$.

## PROOFS

In the sequel we shall use the following abbreviations:

$$\Phi(x; g) = \int_x^\infty e^{i\varphi(s)}\, g(x)\, ds\ ,$$

$$\Psi(x; g) = \sup\{[(g(y))^{-1} + |g'(y)|\,(g(y))^{-2}]\,|\Phi(y; g)| \mid y \geq x\}\ ,$$

$$\Phi_\alpha(x) := \Phi(x; e^{-\alpha x})\ ,\quad \Phi := \Phi_1\ ,$$

$$\Psi_\alpha(x) := \Psi(x; e^{-\alpha x})\ ,\quad \Psi := \Psi_1\ .$$

Clearly $\Psi_\alpha(x)$ exists if $\alpha > 0$ since $\Phi_\alpha(x) = O(e^{-\alpha x})\ \ (x \to \infty)$ .

PROOF OF THEOREM (4)
Let $\varphi \in C(e^{-x})$. We write

$$I(t; F) = \int_{-\infty}^{\infty} e^{i\varphi(s)}\, f(s-t)\, ds\ .$$

Let $x \in \mathbb{R}$. Then

$$\left|\int_{-\infty}^{x} e^{i\varphi(s)}\, f(s-t)\, ds\right| \leq \int_{-\infty}^{x} f(s-t)\, ds = F(x-t)\ .$$

Further, integrating by parts, we get

$$\int_x^\infty e^{i\varphi(s)}\, f(s-t)\, ds = -\int_x^\infty e^s\, f(s-t)\ d\Phi(s) =$$

$$e^x\, f(x-t)\ \Phi(x) + \int_x^\infty \Phi(s)\, e^s (f'(s-t) + f(s-t))\, ds\ ,$$

whence

$$|\int_x^\infty e^{i\varphi(s)}\, f(s-t)\,ds| \le f(x-t)\,\Psi(x) + \Psi(x)\int_x^\infty |f'(s-t)+f(s-t)|\,ds$$

$$\le \Psi(x)\,\{\int_{-\infty}^\infty |f'(s)|\,ds + 2\} =: c\,\Psi(x)\,.$$

It follows that

$$|\int_{-\infty}^\infty e^{i\varphi(s+t)}\, f(s)\,ds| \le F(x-t) + c\,\Psi(x)\,.$$

Taking $x = \frac{1}{2}t$ we see that (3) holds.

Let (3) hold for all $F \in \mathcal{F}$. We choose a sequence $(F_n)$ in $\mathcal{F}$ by $F_n'(x) = f_n(x) = e^{-x}$ on $[n^{-1}, \infty)$. Then

$$I(t; F_n) = R_n(t+1/n) + e^t\,\Phi(t+1/n)\,,$$

where

$$R_n(t) := \int_{-\infty}^t e^{i\varphi(s)}\, f_n(s-t)\,ds\,.$$

Clearly $|R_n(t+1/n)| \le 1 - e^{-1/n}$.
Let $\varepsilon > 0$ and $N = \lceil 4/\varepsilon \rceil + 1$. Let $T_N(\varepsilon)$ be such that $|I(t, F_N)| < \frac{1}{4}\varepsilon \quad (t > T_N(\varepsilon))$. Let $t > T_N(\varepsilon)$. Then

$$|e^{t+1/N}\,\Phi(t+1/N)| \le e^{1/N}\,|I(t; F_N)| + e^{1/N} - 1 < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon\,.$$

Hence

$$|\Psi(t)| < \varepsilon \quad (t \ge T_N(\varepsilon) + 1/N)\,.$$

It follows that $\varphi \in \mathcal{C}(e^{-x})$.

PROOF OF THEOREM (5)
Let $\varphi \in \mathcal{C}(g)$ for some $g \in \mathcal{G}$. Then

$$\Phi(x) = -\int_x^\infty e^{-s}\,(g(s))^{-1}\,d\,\Phi(s; g) =$$

$$e^{-x}(g(x))^{-1}\,\Phi(x; g) - \int_x^\infty \Phi(s; g)\,e^{-s}[(g(s))^{-1} + g'(s)\,(g(s))^{-2}]\,ds\,,$$

whence

$$|e^x \Phi(x)| \le (g(x))^{-1} |\Phi(x)| + \Psi(x;g) \le 2\Psi(x;g) \, .$$

Hence $\varphi \in \mathcal{C}(e^{-x})$.

PROOF OF THEOREM (6), $\mathcal{C}(e^{-x^2}) \neq \mathcal{C}(e^{-x})$.
Let $\varphi(x) = x^2$. Then $\varphi \in \mathcal{C}(e^{-x})$ by Theorem (5). But $\varphi \notin \mathcal{C}(e^{-x^2})$. For from

$$\int_x^\infty \exp[i\,s^2 - s^2]\,ds = \tfrac{1}{2}(1-i)^{-1}\,x^{-1}\,e^{(i-1)x^2} - (2i-2)^{-1}\int_x^\infty \exp[i\,s^2 - s^2]\,s^{-2}\,ds$$

and

$$|(2i-2)^{-1}\int_x^\infty \exp[i\,s^2 - s^2]\,ds| \le 2^{-3/2}\int_x^\infty e^{-s^2}\,s^{-2}\,ds = O(x^{-2}\,e^{-x^2}) \;\; (x\to\infty)$$

it follows that

$$|x\,e^{x^2}\int_x^\infty \exp[i\,s^2 - s^2]\,ds| \ge 2^{-3/2} + O(x^{-1}) \quad (x\to\infty) \, .$$

PROOF OF THEOREM (6), $\mathcal{C}(1) \neq \mathcal{C}(e^{-x})$.
Inspired by the fact that $\sum_2^\infty \frac{(-1)^n}{\sqrt{n}+(-1)^n}$ is divergent, we depart from the divergent integral $\int^\infty e^{iy}(\sqrt{y}+\sin\,y)^{-1}\,dy$ which originates from $\int^\infty e^{i\varphi(s)}\,ds$ by the substitution $\varphi(s) = y$, where $\varphi$ is a solution of $\varphi' = \sqrt{\varphi} + \sin\,\varphi$ which is positive on an interval of the kind $[a,\infty)$. The discrete analogue of $e^x\int_x^\infty e^{i\varphi(s)-s}\,ds$ resembles $e^{\sqrt{n}}\sum_{k=n}^\infty \frac{(-1)e^{-\sqrt{k}}}{\sqrt{k}+(-1)^k}$ which tends to zero for $n\to\infty$.

We take $\varphi(x) = \frac{1}{4}x^2 - 2x^{-1}\cos\,x^2$. This function satisfies the conditions in Theorem (7) whence $\varphi \in \mathcal{C}(e^{-x})$. We shall prove that $\varphi \notin \mathcal{C}(1)$. By elementary calculations we get

$$\varphi' = \sqrt{\varphi} + \sin\,\varphi + O(\varphi^{-1/2}) \quad (\varphi\to\infty) \, .$$

Then

$$\int_{\varphi(2\pi k)}^{\varphi(2\pi(k+1))} e^{i\varphi(s)}\,ds = \int_{2\pi k}^{2\pi(k+1)} e^{iy}(\sqrt{y} + \sin\,y + O(y^{-1/2}))^{-1}\,dy$$

$$\int_0^\pi e^{iy}\left[\left(\sqrt{2\pi k + y} + \sin\,y + O(k^{-1/2})\right)^{-1} - \left(\sqrt{2\pi k + \pi + y} - \sin\,y + O(k^{-1/2})\right)^{-1}\right]dy$$

$$= \int_0^\pi e^{iy}(2\pi k)^{-1} \left(-2 \sin y + O(k^{-1/2})\right) dy = -i(2k)^{-1} + O(k^{-3/2}) \, .$$

It follows that $\int^\infty e^{i\varphi(s)}\, ds$ diverges.

PROOF OF THEOREM (6), $\mathcal{C}(e^{-\alpha x}) = \mathcal{C}(e^{-x})$.
Let $\alpha > 0$, $\beta > 0$. Then

$$e^{\alpha t}\, \Phi_\alpha(t) = -e^{\alpha t} \int_t^\infty e^{(\beta-\alpha)s}\, d\,\Phi_\beta(s) =$$

$$e^{\beta t}\, \Phi_\beta(t) + (\beta - \alpha)\, e^{\alpha t} \int_t^\infty \Phi_\beta(s)\, e^{\beta s}\, e^{-\alpha s}\, ds \, ,$$

whence

$$|e^{\alpha t}\, \Phi_\alpha(t)| \leq (1 + |\beta - \alpha|\, \alpha^{-1})\ \Psi_\beta(t) \quad (t \in \mathbb{R}) \, .$$

Since $\Psi_\beta(t)$ is non increasing in $t$ we find that

$$\Psi_\alpha(t) \leq (1 + |\beta - \alpha|\, \alpha^{-1})\ \Psi_\beta(t) \, .$$

By taking $\alpha = 1$, replacing $\beta$ by $\alpha$ we find $\mathcal{C}(e^{-\alpha x}) \subset \mathcal{C}(e^{-x})$. By taking $\beta = 1$ we find $\mathcal{C}(e^{-x}) \subset \mathcal{C}(e^{-\alpha x})$.

PROOF OF THEOREM (7)
Let $\varphi$ satisfy the conditions in Theorem (7). Then

$$|e^x \int_x^\infty \exp[i\,\varphi(s) - s]\, ds| = |e^x \int_x^\infty (i\,\varphi'(s) - 1)^{-1}\, d \exp[i\,\varphi(s) - s]| =$$

$$|-(i\,\varphi'(x) - 1)^{-1} \exp[i\,\varphi(x)] - \int_x^\infty \exp[i\,\varphi(s) - s]\ i\,\varphi''(s)\ (i\,\varphi'(s) - 1)^{-2}\, ds|$$

$$\leq (|\varphi'(x)| - 1)^{-1} + \sup\{|\varphi''(s)|(\varphi'(s) - 1)^{-2} : s \geq x\} \to 0 \quad (x \to \infty) \, .$$

PROOF OF THEOREM (8)

$$e^t \int_t^\infty \exp[i\,\varphi(s) - s]\, ds \to 0 \quad (t \to \infty)$$

whence

$$e^{h(t)} \int_{h(t)}^{\infty} \exp[i\,\varphi(s) - s]\, ds \to 0 \quad (t \to \infty) .$$

Substituting $s = h(\sigma)$ we get

$$e^{h(t)} \int_{t}^{\infty} \exp[i\,\varphi(h(\sigma)) - h(\sigma)]\; h'(\sigma)\, d\sigma \to 0 \quad (t \to \infty) .$$

By the conditions on $h$ it follows that

$$e^{h(t)}\; [(h'(t))^{-1} + |h''(t)\; (h'(t))^{-2} - 1|] \int_{t}^{\infty} \exp[i\,\varphi(h(\sigma)) - h(\sigma)]\, h'(\sigma)\, d\sigma \to 0 \;\; (t \to \infty)$$

whence

$$\varphi \circ h \in \mathcal{C}(e^{-h(x)}\; h'(x)) .$$

By Theorem (5) it follows that $\varphi \circ h \in \mathcal{C}(e^{-x})$.

## MISCELLANEOUS

The condition $\varphi \in \mathcal{C}(e^{-x})$ is not necessary for some distributions $F \in \mathcal{F}$ in order for (3) to hold. For instance, the density $f$ defined by

$$f(x) = u(x) + u(x - \pi/2) ,$$

$$u(x) = \pi^{-1}\, x^{-2}\, \sin^2 x \quad (x \in \mathbb{R})$$

has the Fourier transform

$$U(\omega)\; (1 + \exp[-i\pi\omega/2]) ,$$

$$U(\omega) = (2\pi)^{-1}\; \max\{1 - |\omega|/2,\; 0\} .$$

Hence, with $\varphi(x) = \omega x$, we have

$$\int_{-\infty}^{\infty} e^{i\varphi(x+t)}\; f(x)\, dx = (2\pi)^{-1}\; (1 + \exp[i\pi\omega/2])\; e^{i\omega t}\; \max\{1 - |\omega|/2,\; 0\} .$$

It follows that (3) holds if $|\omega| \geq 2$ .

There are several unsolved problems.
Let $\varphi \in C(e^{-x})$. Does it follow that $\alpha\varphi \in C(e^{-x})$ if $\alpha \in \mathbb{R}$, $\alpha \neq 0$? Let also $\psi \in C(e^{-x})$. Does it follow that $\psi\varphi \in C(e^{-x})$? If we assume that $\varphi$ and $\psi$ satisfy the conditions of Theorem (7) then the answer is in the affirmative.