

An asymptotic problem in extremal processes

Citation for published version (APA):

Brands, J. J. A. M. (1991). *An asymptotic problem in extremal processes*. (RANA : reports on applied and numerical analysis; Vol. 9110). Eindhoven University of Technology.

Document status and date:

Published: 01/01/1991

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

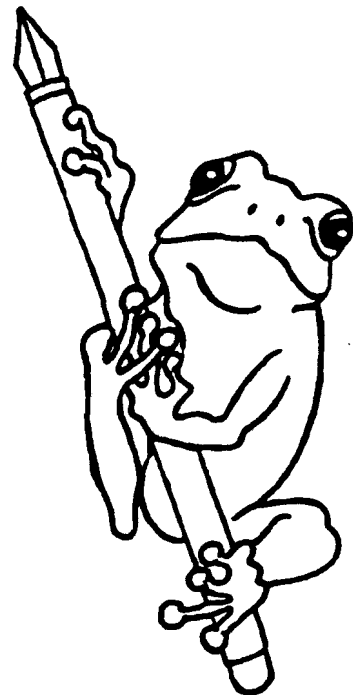
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 91-10
November 1991
AN ASYMPTOTIC PROBLEM
IN EXTREMAL PROCESSES
by
J.J.A.M. Brands



ISSN: 0926-4507
Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

AN ASYMPTOTIC PROBLEM IN EXTREMAL PROCESSES

by

J.J.A.M. Brands

Department of Mathematics and Computing Science

Eindhoven University of Technology

The Netherlands

ABSTRACT

If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the condition

$$(*) \quad \int_x^\infty \exp[i\varphi(s) - s + x] ds \rightarrow 0 \quad (x \rightarrow \infty)$$

then

$$(**) \quad \int_{-\infty}^\infty \exp[i\varphi(s+t)] F'(s) ds \rightarrow 0 \quad (t \rightarrow \infty),$$

where $F \in \mathcal{F}$. The set \mathcal{F} consists of all probability distributions F on \mathbb{R} satisfying the requirement $\int_{-\infty}^\infty |F''(x)| dx < \infty$. If $(**)$ holds for all $F \in \mathcal{F}$ then $(*)$ holds.

INTRODUCTION

Let (X_m) be a sequence of independent random variables with common distribution F . We define the sequence (Z_n) by

$$Z_n = \max \{X_1, X_2, \dots, X_n\} \quad (n \in \mathbb{N}).$$

Let us denote the fractional part of a real number α by $\{\alpha\} := \alpha - [\alpha]$, where $[\alpha]$ is the largest integer not exceeding α . The question is: for which distributions F converges $\{Z_n\}$ in distribution to U , where U is uniform distributed on $[0, 1)$. In studying this question the following problem (*) occurs: For which functions $H : \mathbb{R}^+ \rightarrow \mathbb{R}$ is it true that, for all $k \in \mathbb{Z} \setminus \{0\}$

$$\int_{-\log n}^{\infty} \exp[2\pi i k H(x + \log n)] \exp[-x - e^{-x}] dx \rightarrow 0 \quad (n \rightarrow \infty).$$

In this paper this question will be partially answered.

(*) The problem was posed by R. Wilms, Department of Mathematics and Computing Science, Eindhoven University of Technology, The Netherlands.

PRELIMINARIES

Let \mathcal{F} denote the set of all probability distributions F on \mathbb{R} with the property that $\int_{-\infty}^{\infty} |f'(x)| dx < \infty$, where $f := F'$ is the density. We remark that this condition implies that $\frac{f(x)}{x} \rightarrow 0$ if $|x| \rightarrow \infty$.

Let \mathcal{G} denote the set of all functions g with the properties: $\text{DOM } g \supset [x_g, \infty)$ for some real number x_g (dependent on g), $g(x) > 0$ on $[x_g, \infty)$ and g is continuously differentiable on $[x_g, \infty)$.

By \mathcal{C} we denote the set of all real piecewise continuous functions φ on \mathbb{R} . For $g \in \mathcal{G}$ we denote by $\mathcal{C}(g)$ (or $\mathcal{C}(g(x))$) the subset of all functions $\varphi \in \mathcal{C}$ which satisfy (1) and (2).

$$(1) \quad \int_{x_g}^{\infty} e^{i\varphi(x)} g(x) dx \text{ exists,}$$

$$(2) \quad [(g(x))^{-1} + |g'(x)| (g(x))^{-2}] \int_x^{\infty} e^{i\varphi(s)} g(s) ds \rightarrow 0 \quad (x \rightarrow \infty).$$

The statement which we consider is

$$(3) \quad I(t; F) := \int_{-\infty}^{\infty} e^{i\varphi(s+t)} F'(s) ds \rightarrow 0 \quad (t \rightarrow \infty).$$

RESULTS

The main results are formulated in Theorems (4) and (5).

(4) THEOREM

(3) holds for all $F \in \mathcal{F}$ if and only if $\varphi \in \mathcal{C}(e^{-x})$.

(5) THEOREM

$$\mathcal{C}(g(x)) \subset \mathcal{C}(e^{-x}) \quad (g \in \mathcal{G})$$

That $\varphi \in \mathcal{C}(g)$ can be a stronger condition than $\varphi \in \mathcal{C}(e^{-x})$ is illustrated in Theorem (6). The question for which g we have $\mathcal{C}(g) = \mathcal{C}(e^{-x})$ is partially answered in Theorem (6).

(6) THEOREM

$$\mathcal{C}(e^{-x^2}) \neq \mathcal{C}(e^{-x})$$

$$\mathcal{C}(1) \neq \mathcal{C}(e^{-x})$$

$$\mathcal{C}(e^{-\alpha x}) = \mathcal{C}(e^{-x}) \quad (\alpha > 0).$$

The question which functions are contained in $\mathcal{C}(e^{-x})$ is partially answered by Theorem (7) and (8).

(7) THEOREM

If φ is twice continuously differentiable, $|\varphi'(x)| \rightarrow \infty$ ($x \rightarrow \infty$) and $\varphi''(x) (\varphi'(x))^{-2} \rightarrow 0$ ($x \rightarrow \infty$) then $\varphi \in \mathcal{C}(e^{-x})$.

(8) THEOREM

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and, on some interval $[a, \infty)$ be twice continuously differentiable with $h'(x) \geq c > 0$ ($x \geq a$). Moreover $h''(x) = O((h'(x))^2)$ ($x \rightarrow \infty$). Then $\varphi \in \mathcal{C}(e^{-x})$ implies $\varphi \circ h \in \mathcal{C}(e^{-x})$.

EXAMPLES

$\mathcal{C}(1)$ consists of all functions $\varphi \in \mathcal{C}$ satisfying the condition that $\int_0^\infty e^{i\varphi(s)} ds$ exists.

$$\mathcal{C}(e^{-x^2}) = \left\{ \varphi \in \mathcal{C} \mid x e^{x^2} \int_x^\infty \exp[i\varphi(s) - s^2] ds \rightarrow 0 \quad (x \rightarrow \infty) \right\}$$

$$\mathcal{C}(x^{-2}) = \left\{ \varphi \in \mathcal{C} \mid x^2 \int_x^\infty e^{i\varphi(s)} s^{-2} ds \rightarrow 0 \quad (x \rightarrow \infty) \right\}.$$

The function $\varphi(x) = x$ does not belong to $\mathcal{C}(e^{-x})$. Nevertheless there are distributions $F \in \mathcal{F}$ for which (3) holds with $\varphi(x) = x$. An example is given in the last section.

If φ satisfies the conditions of Theorem (7) then also $k\varphi$ for every $k \in \mathbb{Z} \setminus \{0\}$.

REMARK

The author has not been able to prove or disprove the necessity of the condition $\varphi \in \mathcal{C}(e^{-x})$ so that (3) holds in case $F(x) = \exp[-e^{-x}]$.

PROOFS

In the sequel we shall use the following abbreviations:

$$\Phi(x; g) = \int_x^\infty e^{i\varphi(s)} g(x) ds ,$$

$$\Psi(x; g) = \sup \{ [(g(y))^{-1} + |g'(y)| (g(y))^{-2}] |\Phi(y; g)| \mid y \geq x \} ,$$

$$\Phi_\alpha(x) := \Phi(x; e^{-\alpha x}) , \quad \Phi := \Phi_1 ,$$

$$\Psi_\alpha(x) := \Psi(x; e^{-\alpha x}) , \quad \Psi := \Psi_1 .$$

Clearly $\Psi_\alpha(x)$ exists if $\alpha > 0$ since $\Phi_\alpha(x) = O(e^{-\alpha x})$ ($x \rightarrow \infty$).

PROOF OF THEOREM (4)

Let $\varphi \in \mathcal{C}(e^{-x})$. We write

$$I(t; F) = \int_{-\infty}^\infty e^{i\varphi(s)} f(s-t) ds .$$

Let $x \in \mathbb{R}$. Then

$$\left| \int_{-\infty}^x e^{i\varphi(s)} f(s-t) ds \right| \leq \int_{-\infty}^x f(s-t) ds = F(x-t) .$$

Further, integrating by parts, we get

$$\begin{aligned} \int_x^\infty e^{i\varphi(s)} f(s-t) ds &= - \int_x^\infty e^s f(s-t) d\Phi(s) = \\ &e^x f(x-t) \Phi(x) + \int_x^\infty \Phi(s) e^s (f'(s-t) + f(s-t)) ds , \end{aligned}$$

whence

$$\begin{aligned}
\left| \int_x^\infty e^{i\varphi(s)} f(s-t) ds \right| &\leq f(x-t) \Psi(x) + \Psi(x) \int_x^\infty |f'(s-t) + f(s-t)| ds \\
&\leq \Psi(x) \left\{ \int_{-\infty}^\infty |f'(s)| ds + 2 \right\} =: c \Psi(x).
\end{aligned}$$

It follows that

$$\left| \int_{-\infty}^\infty e^{i\varphi(s+t)} f(s) ds \right| \leq F(x-t) + c \Psi(x).$$

Taking $x = \frac{1}{2}t$ we see that (3) holds.

Let (3) hold for all $F \in \mathcal{F}$. We choose a sequence (F_n) in \mathcal{F} by $F'_n(x) = f_n(x) = e^{-x}$ on $[n^{-1}, \infty)$. Then

$$I(t; F_n) = R_n(t + 1/n) + e^t \Phi(t + 1/n),$$

where

$$R_n(t) := \int_{-\infty}^t e^{i\varphi(s)} f_n(s-t) ds.$$

Clearly $|R_n(t + 1/n)| \leq 1 - e^{-1/n}$.

Let $\varepsilon > 0$ and $N = \lceil 4/\varepsilon \rceil + 1$. Let $T_N(\varepsilon)$ be such that $|I(t, F_N)| < \frac{1}{4}\varepsilon$ ($t > T_N(\varepsilon)$). Let $t > T_N(\varepsilon)$. Then

$$|e^{t+1/N} \Phi(t + 1/N)| \leq e^{1/N} |I(t, F_N)| + e^{1/N} - 1 < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Hence

$$|\Psi(t)| < \varepsilon \quad (t \geq T_N(\varepsilon) + 1/N).$$

It follows that $\varphi \in \mathcal{C}(e^{-x})$.

PROOF OF THEOREM (5)

Let $\varphi \in \mathcal{C}(g)$ for some $g \in \mathcal{G}$. Then

$$\begin{aligned}
\Phi(x) &= - \int_x^\infty e^{-s} (g(s))^{-1} d\Phi(s; g) = \\
&= e^{-x} (g(x))^{-1} \Phi(x; g) - \int_x^\infty \Phi(s; g) e^{-s} [(g(s))^{-1} + g'(s) (g(s))^{-2}] ds,
\end{aligned}$$

whence

$$|e^x \Phi(x)| \leq (g(x))^{-1} |\Phi(x)| + \Psi(x; g) \leq 2 \Psi(x; g).$$

Hence $\varphi \in \mathcal{C}(e^{-x})$.

PROOF OF THEOREM (6), $\mathcal{C}(e^{-x^2}) \neq \mathcal{C}(e^{-x})$.

Let $\varphi(x) = x^2$. Then $\varphi \in \mathcal{C}(e^{-x})$ by Theorem (5). But $\varphi \notin \mathcal{C}(e^{-x^2})$. For from

$$\int_x^\infty \exp[i s^2 - s^2] ds = \frac{1}{2}(1-i)^{-1} x^{-1} e^{(i-1)x^2} - (2i-2)^{-1} \int_x^\infty \exp[i s^2 - s^2] s^{-2} ds$$

and

$$|(2i-2)^{-1} \int_x^\infty \exp[i s^2 - s^2] ds| \leq 2^{-3/2} \int_x^\infty e^{-s^2} s^{-2} ds = O(x^{-2} e^{-x^2}) \quad (x \rightarrow \infty)$$

it follows that

$$|x e^{x^2} \int_x^\infty \exp[i s^2 - s^2] ds| \geq 2^{-3/2} + O(x^{-1}) \quad (x \rightarrow \infty).$$

PROOF OF THEOREM (6), $\mathcal{C}(1) \neq \mathcal{C}(e^{-x})$.

Inspired by the fact that $\sum_2^\infty \frac{(-1)^n}{\sqrt{n+(-1)^n}}$ is divergent, we depart from the divergent integral $\int_0^\infty e^{iy} (\sqrt{y} + \sin y)^{-1} dy$ which originates from $\int_0^\infty e^{i\varphi(s)} ds$ by the substitution $\varphi(s) = y$, where φ is a solution of $\varphi' = \sqrt{\varphi} + \sin \varphi$ which is positive on an interval of the kind $[a, \infty)$. The discrete analogue of $e^x \int_x^\infty e^{i\varphi(s)-s} ds$ resembles $e^{\sqrt{n}} \sum_{k=n}^\infty \frac{(-1)^k e^{-\sqrt{k}}}{\sqrt{k+(-1)^k}}$ which tends to zero for $n \rightarrow \infty$.

We take $\varphi(x) = \frac{1}{4}x^2 - 2x^{-1} \cos x^2$. This function satisfies the conditions in Theorem (7) whence $\varphi \in \mathcal{C}(e^{-x})$. We shall prove that $\varphi \notin \mathcal{C}(1)$. By elementary calculations we get

$$\varphi' = \sqrt{\varphi} + \sin \varphi + O(\varphi^{-1/2}) \quad (\varphi \rightarrow \infty).$$

Then

$$\begin{aligned} \int_{\varphi(2\pi k)}^{\varphi(2\pi(k+1))} e^{i\varphi(s)} ds &= \int_{2\pi k}^{2\pi(k+1)} e^{iy} (\sqrt{y} + \sin y + O(y^{-1/2}))^{-1} dy \\ &= \int_0^\pi e^{iy} \left[(\sqrt{2\pi k + y} + \sin y + O(k^{-1/2}))^{-1} - (\sqrt{2\pi k + \pi + y} - \sin y + O(k^{-1/2}))^{-1} \right] dy \end{aligned}$$

$$= \int_0^{\pi} e^{iy} (2\pi k)^{-1} (-2 \sin y + O(k^{-1/2})) dy = -i(2k)^{-1} + O(k^{-3/2}).$$

It follows that $\int e^{i\varphi(s)} ds$ diverges.

PROOF OF THEOREM (6), $\mathcal{C}(e^{-\alpha x}) = \mathcal{C}(e^{-x})$.

Let $\alpha > 0$, $\beta > 0$. Then

$$e^{\alpha t} \Phi_{\alpha}(t) = -e^{\alpha t} \int_t^{\infty} e^{(\beta-\alpha)s} d\Phi_{\beta}(s) =$$

$$e^{\beta t} \Phi_{\beta}(t) + (\beta - \alpha) e^{\alpha t} \int_t^{\infty} \Phi_{\beta}(s) e^{\beta s} e^{-\alpha s} ds,$$

whence

$$|e^{\alpha t} \Phi_{\alpha}(t)| \leq (1 + |\beta - \alpha| \alpha^{-1}) \Psi_{\beta}(t) \quad (t \in \mathbb{R}).$$

Since $\Psi_{\beta}(t)$ is non increasing in t we find that

$$\Psi_{\alpha}(t) \leq (1 + |\beta - \alpha| \alpha^{-1}) \Psi_{\beta}(t).$$

By taking $\alpha = 1$, replacing β by α we find $\mathcal{C}(e^{-\alpha x}) \subset \mathcal{C}(e^{-x})$. By taking $\beta = 1$ we find $\mathcal{C}(e^{-x}) \subset \mathcal{C}(e^{-\alpha x})$.

PROOF OF THEOREM (7)

Let φ satisfy the conditions in Theorem (7). Then

$$|e^x \int_x^{\infty} \exp[i\varphi(s) - s] ds| = |e^x \int_x^{\infty} (i\varphi'(s) - 1)^{-1} d \exp[i\varphi(s) - s]| =$$

$$|-(i\varphi'(x) - 1)^{-1} \exp[i\varphi(x)] - \int_x^{\infty} \exp[i\varphi(s) - s] i\varphi''(s) (i\varphi'(s) - 1)^{-2} ds|$$

$$\leq (|\varphi'(x)| - 1)^{-1} + \sup\{|\varphi''(s)|(\varphi'(s) - 1)^{-2} : s \geq x\} \rightarrow 0 \quad (x \rightarrow \infty).$$

PROOF OF THEOREM (8)

$$e^t \int_t^{\infty} \exp[i\varphi(s) - s] ds \rightarrow 0 \quad (t \rightarrow \infty)$$

whence

$$e^{h(t)} \int_{h(t)}^{\infty} \exp[i\varphi(s) - s] ds \rightarrow 0 \quad (t \rightarrow \infty).$$

Substituting $s = h(\sigma)$ we get

$$e^{h(t)} \int_t^{\infty} \exp[i\varphi(h(\sigma)) - h(\sigma)] h'(\sigma) d\sigma \rightarrow 0 \quad (t \rightarrow \infty).$$

By the conditions on h it follows that

$$e^{h(t)} [(h'(t))^{-1} + |h''(t) (h'(t))^{-2} - 1|] \int_t^{\infty} \exp[i\varphi(h(\sigma)) - h(\sigma)] h'(\sigma) d\sigma \rightarrow 0 \quad (t \rightarrow \infty)$$

whence

$$\varphi \circ h \in \mathcal{C}(e^{-h(x)} h'(x)).$$

By Theorem (5) it follows that $\varphi \circ h \in \mathcal{C}(e^{-x})$.

MISCELLANEOUS

The condition $\varphi \in \mathcal{C}(e^{-x})$ is not necessary for some distributions $F \in \mathcal{F}$ in order for (3) to hold. For instance, the density f defined by

$$\begin{aligned} f(x) &= u(x) + u(x - \pi/2), \\ u(x) &= \pi^{-1} x^{-2} \sin^2 x \quad (x \in \mathbb{R}) \end{aligned}$$

has the Fourier transform

$$\begin{aligned} U(\omega) &= (1 + \exp[-i\pi\omega/2]), \\ U(\omega) &= (2\pi)^{-1} \max\{1 - |\omega|/2, 0\}. \end{aligned}$$

Hence, with $\varphi(x) = \omega x$, we have

$$\int_{-\infty}^{\infty} e^{i\varphi(x+t)} f(x) dx = (2\pi)^{-1} (1 + \exp[i\pi\omega/2]) e^{i\omega t} \max\{1 - |\omega|/2, 0\}.$$

It follows that (3) holds if $|\omega| \geq 2$.

There are several unsolved problems.

Let $\varphi \in \mathcal{C}(e^{-x})$. Does it follow that $\alpha\varphi \in \mathcal{C}(e^{-x})$ if $\alpha \in \mathbb{R}$, $\alpha \neq 0$? Let also $\psi \in \mathcal{C}(e^{-x})$. Does it follow that $\psi\varphi \in \mathcal{C}(e^{-x})$? If we assume that φ and ψ satisfy the conditions of Theorem (7) then the answer is in the affirmative.