Nonlinear disturbance decoupling and linearization : a partial interpretation of integral feedback

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Nonlinear disturbance decoupling and linearization: a partial interpretation of integral feedback

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Abstract

The relation between the solvability of the disturbance decoupling problem for a nonlinear system and its linearization around a working point is investigated. It turns out that generically the solvability of the disturbance decoupling via regular dynamic state feedback is preserved under linearization. This result gives a partial interpretation of introducing integral action in classical PID-control applied to nonlinear systems. The theory is illustrated by means of a worked example.

1. Introduction

Like in the linear geometric theory, [16], one of the first "structural" synthesis problems that has been posed and has been solved locally in a nonlinear context, is the so called disturbance decoupling problem (DDP) for a nonlinear system. This problem may be stated as follows. Consider the nonlinear control system Σ_q, described by

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u + p(x)q \\ y & = & h(x) \end{array} \right. \tag{1}$$

where $x = \mathrm{col}(x_1, \cdots, x_n) \in \mathbb{R}^n$ are local coordinates for the state space manifold $\mathcal{X}$, $u \in \mathbb{R}^m$ denotes the controls, $q \in \mathbb{R}^r$ the disturbances and $y \in \mathbb{R}^p$ the outputs. Let $g_1, \cdots, g_m$ denote the columns of the matrix g and let $p_1, \cdots, p_r$ denote the columns of the matrix p. All data in (1), i.e., the vector fields f, $g_1, \cdots, g_m$ and $p_1, \cdots, p_r$ as well as the function h, will be assumed to be analytic in this paper. In the DDP one is asked to design, if possible, a static state feedback

$$Q_s: \ u = \alpha(x) + \beta(x)v \tag{2}$$

with $\alpha(x)$ and $\beta(x)$ respectively an m-vector and an (m,m)-matrix depending analytically on x, and where $v \in \mathbb{R}^m$ denotes a new control vector, such that in the closed loop system $\Sigma_q \circ Q_s$ the output y is unaffected by the disturbances q, no matter how v is chosen. Usually the DDP is considered under the assumption that in the static feedback law (2) the matrix $\beta(x)$ is nonsingular for all v, this in order to keep as much control on the system as possible, while at the same time disturbance decoupling is achieved. Define the distributions $\mathcal{G} := \mathrm{span}\{g_1, \cdots, g_m\}$, $\mathcal{P} := \mathrm{span}\{p_1, \cdots, p_r\}$ and let Δ^* be the maximal locally controlled invariant distribution in Kerdh for (1). If the distributions $\mathcal{G}$, Δ^*, $\Delta^* \cap \mathcal{G}$ have constant dimension, the DDP is locally solvable for (1) if and only if $\mathcal{P} \subset \Delta^*$ (cf. [11],[12]).

Another version of the disturbance decoupling problem is the so called dynamic disturbance decoupling problem, abbreviated as DDDP, in which the disturbance decoupling is done via a dynamic state feedback. In other words, define a dynamic state feedback as

$$Q_d \left\{ \begin{array}{rcl} \dot{z} & = & \alpha(x,z) + \beta(x,z)v \\ u & = & \gamma(x,z) + \delta(x,z)v \end{array} \right. \tag{3}$$

with the compensator state $z \in \mathbb{R}^\nu$ and $v \in \mathbb{R}^m$ is again a new control vector. We assume the system (3) to be *regular*, which implies invertibility between the old controls u and the new controls v of this system (see Section 3 for a more specific definition). In the DDDP one seeks a dynamic state feedback (3) such that in the closed loop system $\Sigma_q \circ Q_d$ the disturbances q do not influence the outputs. The DDDP has been posed and solved in a local fashion in [5],[6] (see also [13]) and one of the remarkable conclusions is that for nonlinear systems the DDDP might be solvable for systems for which the (regular) DDP is not. The latter statement is, as is known (cf. [1]), in contrast with the linear theory, since for linear systems for which the DDP is not solvable, also the DDDP is not solvable.

The purpose of the present paper is to make a further step in exploiting the idea of using dynamic feedback in achieving disturbance decoupling while at the same time trying to relate this to one of the basic approaches in control engineering practice, namely *linearization*. Assume for the moment that (1) is considered around some working point x_0, so $f(x_0) = 0$, and let the (Jacobian) linearization $L\Sigma_q$ of (1) around x_0 be given by

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & F\xi + Gu + Pq \\ \eta & = & H\xi \end{array} \right. \tag{4}$$

A first elementary engineering approach to tackle any synthesis problem for (1) would be to address the same design goal for its linearization (4) and use the linear solution as an approximate solution for the nonlinear system.

We will show that for a large (generic) class of nonlinear systems (1) this approach indeed makes sense in case one allows *regular dynamic state feedbacks* in the solution of the disturbance decoupling problem for the linear system (4). As mentioned before, for the linear system (4) itself

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it would be enough to limit ourselves to static state feedbacks for solving the DDP (if the problem is solvable at all), but we will show explicitly that *only the solvability of the DDDP is preserved under linearization.*

The solution of the DDDP we propose in our papers [5],[6],[7] is very much based on a special class of dynamic state feedbacks, that we call Singh compensators. A particular feature of such a compensator is that some (but not all) of the controls u_i are integrated a certain number of times. Thus we encounter in a Singh compensator schemes as $\dot{z}_i = v_i$, $u_i = z_i$, which is the same as allowing $\dot{u}_i = v_i$. Since this type of dynamic compensation will naturally arise in our solution of the DDDP, one could view our results as a partial justification of introducing integral action in classical PID-control applied to nonlinear systems.

The paper is organized as follows. In Section 2 we derive a connection between the solvability of the static disturbance decoupling problem for a nonlinear system and its linearization around a working point. Since this result is only included to be used in Section 3, we will not give it in its full generality but only for the specific case that the decoupling matrix of the nonlinear system under consideration is invertible. In Section 3 we first introduce some algebraic preliminaries and a special sort of regular dynamic state feedback, the Singh compensator. After this, we derive our main result. In Section 4 a worked example is given. Finally, in Section 5 some conclusions are drawn. Throughout the paper we restrict ourselves to square systems, i.e., $m = p$ in (1). All the results in the paper can be easily extended to nonsquare systems.

2. Static disturbance decoupling and linearization

In this section we investigate the connection between the solvability of the DDP for Σ_q around an equilibrium point and the solvability of the problem for the linearization of Σ_q around this equilibrium point. We restrict ourselves to the case that the decoupling matrix of Σ_q has full rank at the equilibrium point.

The decoupling matrix of Σ_q may be defined in the following way. Let Σ_0 denote the system Σ_q without disturbances, i.e., $q \equiv 0$. For Σ_0, we define inductively (with $y^{(0)} := y$)

$$y^{(k+1)}(x, u, \cdots, u^{(k)}) = \frac{\partial y^{(k)}}{\partial x}[f(x) + g(x)u] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)} \tag{5}$$

The *relative degree* r_i of y_i $(i = 1, \cdots, m)$ is defined by

$$r_i = \min\{k \in I\!N \mid \frac{\partial y_i^{(k)}}{\partial u} \not\equiv 0\} \tag{6}$$

Assume that all relative degrees of Σ_q are finite. Then we may define the *decoupling matrix* $A(x)$ ([11],[12]) of Σ_q by

$$A(x) = \left(\frac{\partial y_i^{(r_i)}}{\partial u_j}(x) \right)_{i,j=1,\cdots,m} \tag{7}$$

Let $x_0 \in \mathcal{X}$ be an equilibrium point of Σ_q, i.e., $f(x_0) = 0$. Assume (without loss of generality) that $h(x_0) = 0$. Let $L\Sigma_q$, the linearization of Σ_q around x_0, be given by (4). We make the following assumptions.

Assumption 2.1 The decoupling matrix $A(x)$ of Σ_q is invertible at $x = x_0$.

Assumption 2.2 The distributions $\mathcal{P}$ and $\Delta^* \cap \mathcal{P}$ have constant dimension on a neighborhood of x_0.

We then have the following result (see [10]).

Theorem 2.3 *Consider the system Σ_q and let $x_0 \in \mathcal{X}$ be an equilibrium point of Σ_q satisfying $h(x_0) = 0$ and Assumptions 2.1 and 2.2. Then the DDP for Σ_q is locally solvable around x_0 if and only if it is solvable for $L\Sigma_q$.* ∎

Assumption 2.2 and Theorem 2.3 may be interpreted as follows. Consider (5) and the system Σ_q. Then we may define, analogously to (6), *disturbance relative degrees s_i* $(i = 1, \cdots, m)$ for Σ_q by

$$s_i = \min\{k \in I\!N \mid \frac{\partial y_i^{(k-1)}}{\partial x} p(x) \not\equiv 0\} \quad (i = 1, \cdots, m) \tag{8}$$

Thus s_i is the smallest time derivative of y_i that explicitly depends on the disturbances. It is well known (see [12]) that if Assumption 2.1 holds, the DDP is solvable for Σ_q if and only if $r_i < s_i$ $(i = 1, \cdots, m)$. In case Assumption 2.1 does not hold, this is only a necessary condition for solvability of the DDP (see [12]).

Denote the relative degrees of $L\Sigma_q$ by r_i^ℓ $(i = 1, \cdots, m)$ and its disturbance relative degrees by s_i^ℓ $(i = 1, \cdots, m)$. Furthermore, let the decoupling matrix of $L\Sigma_q$ be denoted by A^ℓ. It can then be shown that $r_i^\ell \geq r_i$, $s_i^\ell \geq s_i$ $(i = 1, \cdots, m)$. Assuming that A^ℓ is invertible, we have that the DDP is solvable for $L\Sigma_q$ if and only if $r_i^\ell < s_i^\ell$ $(i = 1, \cdots, m)$. It is then clear that from this we can conclude solvability of the DDP for Σ_q *if* $r_i^\ell = r_i$, $s_i^\ell = s_i$ $(i = 1, \cdots, m)$. Now, if Assumption 2.1 holds, it follows immediately from $r_i^\ell = s_i^\ell$ that $r_i = s_i$ if Assumption 2.2 is satisfied. Thus Assumption 2.2 is posed in order to guarantee the coincidence of the (disturbance) relative degrees of Σ_q and $L\Sigma_q$.

3. Dynamic disturbance decoupling and linearization

In this section we generalize the results from Section 2 to the case of disturbance decoupling via regular dynamic state feedback. We restrict ourselves to systems of full rank. From the discussion at the end of Section 2 it follows that if we can find a dynamic state feedback for the system under consideration such that the decoupling matrix of the compensated system is invertible and the relative degrees of the compensated system are smaller than the disturbance relative degrees of the compensated system, then the DDP for the compensated system is solvable and so the DDDP for the original system is solvable. So in fact what we need to do in order to solve the DDDP, is to find a dynamic state feedback that satisfies the first requirement and at the same time assures that the relative degrees of the compensated system remain as small as possible. In Section 3.1 such a special dynamic state feedback, the Singh compensator, is introduced. It has the property that it is a dynamic state feedback of minimal dimension (ν in (3) is as small as possible) that satisfies the first requirement above. The relative degrees of the compensated system are intrinsically defined: they are the so called essential orders ([3]) of the original system (cf. [7]). The connection between the solvability of the DDDP for the original system and its linearization around an equilibrium point is then established in Section 3.2 by spelling out assumptions that guarantee that the (disturbance) relative degrees of a system plus Singh compensator equal the (disturbance) relative degrees of the linearization of system plus Singh compensator around an equilibrium point. Here we use the special properties of a Singh compensator with respect to linearization that were reported in [9].

3.1 Mathematical preliminaries and Singh compensator

We start with some algebraic concepts that were introduced in [2]. Consider the nonlinear system Σ_0, i.e., the system derived from Σ_q by setting $q \equiv 0$. Recall that a meromorphic function η is a function of the form $\eta = \pi/\theta$, where π and θ are analytic functions. Assume that the control functions $u(t)$ are n times continuously differentiable. Then define $u^{(0)} := u$, $u^{(i+1)} := (d/dt)u^{(i)}$. View $x, u, \cdots, u^{(n-1)}$ as variables and let $\mathcal{K}$ denote the field consisting of the set of rational functions of $(u, \cdots, u^{(n-1)})$ with coefficients that are meromorphic in x. Note that $y, \dot{y}, \cdots, y^{(n)}$ defined defined by (5) have components in the field $\mathcal{K}$. Let $\mathcal{E}$ denote the vector space over $\mathcal{K}$ spanned by $\{dx, d\dot{y}, \cdots, dy^{(k)}\}$. Define subspaces $\mathcal{E}_0, \cdots, \mathcal{E}_n$ of $\mathcal{E}$ by

$$\mathcal{E}_k = \operatorname{span}_{\mathcal{K}}\{dx, d\dot{y}, \cdots, dy^{(k)}\} \tag{9}$$

Then the *rank* $\rho^*(\Sigma_0)$ of Σ_0 is defined by (see [2]) $\rho^*(\Sigma_0) = \dim_{\mathcal{K}} \mathcal{E}_n - \dim_{\mathcal{K}} \mathcal{E}_{n-1}$. Note that we always have $\rho^*(\Sigma_0) \leq m$. Σ_0 is said to be of *full rank* if $\rho^*(\Sigma_0) = m$.

Now consider a dynamic state feedback Q_d for Σ_q of the form (3). Q_d is said to be a *regular* dynamic state feedback for Σ_q if the system

$$\left\{ \begin{array}{lcl} \dot{x} & = & f(x) + g(x)u \\ \dot{z} & = & \alpha(x,z) + \beta(x,z)v \\ u & = & \gamma(x,z) + \delta(x,z)v \end{array} \right. \tag{10}$$

with controls v and outputs u has full rank (see [2]).

Next we present a special sort of regular dynamic state feedback for Σ_q, that we call a Singh compensator (see [14],[5]). Here we restrict ourselves to a system Σ_q with the property that Σ_0 is of full rank. The results can easily be extended to systems that do not have full rank (see e.g. [6],[4]). So, consider a nonlinear control system Σ of full rank. For $r, s \in \mathbb{N}$, introduce the notation $\mathcal{I}_{rs} := \{r, \cdots, s\}$. Using e.g. Singh's algorithm ([14],[2]), we can then find a permutation of the outputs and positive integers $\gamma_1, \cdots, \gamma_m$ satisfying $\gamma_1 \leq \cdots \leq \gamma_m \leq n$, such that for $k = 0, \cdots, n$

$$\{dx, \{dy_i^{(j)} \mid \gamma_i \leq k, j \in \mathcal{I}_{\gamma_i k}\}\} \tag{11}$$

forms a basis for $\mathcal{E}_k$. Denoting $\tilde{y}_k = \operatorname{col}(y_i \mid \gamma_i = k)$, $\hat{y}_k = \operatorname{col}(y_i \mid \gamma_i > k)$, this means that we may write for $k = 1, \cdots, n$

$$\begin{aligned} \tilde{y}_k^{(k)} = \; & \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik}\}) + \\ & \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik-1}\})u \\ \hat{y}_k^{(k)} = \; & \hat{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik}\}) \end{aligned} \tag{12}$$

where the matrices $\tilde{B}_k := (\tilde{b}_1^T \; \cdots \; \tilde{b}_k^T)^T$ have full row rank over $\mathcal{K}$ (cf. [2],[7]). Define $\tilde{Y}_n := (\dot{\tilde{y}}_1^T \; \cdots \; \tilde{y}_n^{(n)^T})^T$ and $\tilde{A} := (\tilde{a}_1^T \; \cdots \; \tilde{a}_n^T)^T$. Then (12) yields in particular:

$$\begin{aligned} \tilde{Y}_n = \; & \tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in}\}) + \\ & \tilde{B}_n(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in-1}\})u \end{aligned} \tag{13}$$

We call a pair $(x, y) = (x_0, 0)$ a ***strongly regular point*** for Σ_q if for every possible permutation of the outputs as described above, the matrix $\tilde{B}_n$ in (13) has full row rank over $\mathbb{R}$, when evaluated at $(x_0, 0)$. If the pair $(x_0, 0)$ is a strongly regular point for Σ_q, we know that for (13) there exists a neighborhood $U \subset \mathcal{X}$ of x_0 and a neighborhood $Y_0 \subset \mathbb{R}^{nm}$ of $(\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in-1}) = 0$ such that $\tilde{B}_n$ is invertible on $U \times Y_0$. Then on $U \times Y_0$ we obtain from (13):

$$u = \tilde{B}_n^{-1}[\tilde{Y}_n - \tilde{A}_n] \tag{14}$$

Clearly, γ_i is the lowest time-derivative of y_i appearing in the right hand side of (14). Let δ_i be the highest time-derivative of y_i appearing in the right hand side of (14). It can be shown that the δ_i and $\sum_{i=1}^m \gamma_i$ are intrinsic, i.e., independent of the permutation of the outputs that is chosen (cf. [7]). In fact, the δ_i are just the essential orders ([3]) of Σ. Hence also $\sigma := \sum_{i=1}^m (\delta_i - \gamma_i)$ is intrinsic. Moreover, the right hand side of (14) is affine in $y_i^{(\delta_i)}$. Thus we may rewrite (14) as

$$\begin{aligned} u \; = \; & \phi_1(x, \{y_i^{(j)} \mid i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{\gamma_i \delta_i - 1}\}) + \\ & \sum_{i=1}^m \phi_{2i}(x, \{ y_i^{(j)} \mid i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{\gamma_i \delta_i - 1}\}) y_i^{(\delta_i)} \end{aligned}$$

for certain vector-valued functions ϕ_1, ϕ_{2i} $(i = 1, \cdots, m)$. Let z_i $(i = 1, \cdots, m)$ be a vector of dimension $\delta_i - \gamma_i$ and consider the system:

$$\left\{ \begin{array}{lcl} \dot{z}_i & = & A_i z_i + B_i v_i \quad (i = 1, \cdots, m) \\ u & = & \phi_1(x, z_1, \cdots, z_m) + \\ & & \displaystyle\sum_{i=1}^m \phi_{2i}(x, z_1, \cdots, z_m) v_i \end{array} \right. \tag{15}$$

with (A_i, B_i) in Brunovsky canonical form. Then (15) is called a *Singh compensator* for Σ around x_0.

The Singh compensator has the following properties (see [4],[7],[9]).

Proposition 3.4 *Consider the nonlinear system Σ_q and assume that $\rho^*(\Sigma_0) = m$. Let $x_0 \in \mathcal{X}$ be an equilibrium point of Σ_q. Assume that $h(x_0) = 0$ and that $(x, y) = (x_0, 0)$ is a strongly regular point for Σ_q. Let Q be a Singh compensator for Σ_q. Then* *(i) Q is a regular dynamic state feedback for Σ_q.*

(ii) Q is a minimal order decoupling compensator for Σ_q.

(iii) *a. The point $(x, z) = (x_0, 0)$ is an equilibrium point for $\Sigma_q \circ Q$.*

b. Denote by $L(\Sigma_q \circ Q)$ the linearization of $\Sigma_q \circ Q$ around $(x_0, 0)$. Then $L(\Sigma_q \circ Q) = L\Sigma_q \circ LQ$, where LQ, the linearization of Q around $(x_0, 0)$, is a Singh compensator for $L\Sigma_q$.

c. Conversely, if R is a Singh compensator for $L\Sigma_q$, then there is a Singh compensator Q for Σ_q such that $LQ = R$ and $L(\Sigma_q \circ Q) = L\Sigma_q \circ R$.

3.2 Dynamic disturbance decoupling and linearization

The disturbance decoupling problem via regular dynamic state feedback is defined below.

Definition 3.5 Disturbance decoupling problem via regular dynamic state feedback (DDDP) Consider a nonlinear system Σ_q and let a point $x_0 \in \mathcal{X}$ be given. The DDDP is said to be locally solvable around x_0 if there exist a regular dynamic state feedback Q_d for Σ_q of the form (3), a neighborhood $U \subset \mathcal{X}$ of x_0 and an open subset $\mathcal{Z} \subset I\!R^{\nu}$ such that the outputs of the composed system $\Sigma_q \circ Q_d$ restricted to $U \times \mathcal{Z}$ are independent of the disturbances. ∎

The following theorem, which can be found in [5],[6], gives a local solution of the DDDP. In the statement of the theorem we employ the following notation. In (12) for Σ_0, the $\hat{y}_k^{(k)}$ $(k = 0, \cdots, n; \hat{y}_0 = y)$ can be viewed as functions on $\mathcal{X}_e := \mathcal{X} \times I\!R^{\sigma}$. By the same token, $\mathrm{Ker}d\hat{y}_k^{(k)}$ $(k = 0, \cdots, n)$ defines a distribution on $\mathcal{X}_e$. Define the distributions $\mathcal{G}_e, \mathcal{P}_e$ on $\mathcal{X}_e$ by $\mathcal{G}_e := \mathcal{G} \times \{0\}$, $\mathcal{P}_e := \mathcal{P} \times \{0\}$. For a particular permutation of the outputs of Σ_0 (as described in Subsection 3.1), define $\bar{\Delta}_e := \cap_{k=0}^{n} \mathrm{Ker}d\hat{y}_k^{(k)}$.

Theorem 3.6 *Consider the nonlinear system Σ_q and let $x_0 \in \mathcal{X}$ be such that $(x_0, 0)$ is a strongly regular point for Σ_q. Then the DDDP is locally solvable around x_0 if and only if for every permutation of the outputs for Σ_0 (as described in Subsection 3.1) we have*

$$\mathcal{P}_e \subset \bar{\Delta}_e \tag{16}$$

Moreover, if the DDDP is locally solvable around x_0, every Singh compensator for Σ_q around x_0 solves the DDDP for Σ_q. ∎

Remark 3.7 Another way of solving the DDDP can be found in [13].

Consider an equilibrium point $x_0 \in \mathcal{X}$ of Σ_q, satisfying $h(x_0) = 0$, and the linearization $L\Sigma_q$ of Σ_q around x_0. We investigate the connection between the solvability of the DDDP for Σ_q and $L\Sigma_q$. The following assumptions are made.

Assumption 3.8 $(x, y) = (x_0, 0)$ is a strongly regular point for Σ_q.

Assumption 3.9 For every permutation of the outputs of Σ_0 as described in Section 2, $\mathcal{P}_e$ and $\bar{\Delta}_e \cap \mathcal{P}_e$ have constant dimension on a neighborhood of $(x_0, 0)$ in $\mathcal{X}_e$

We now come to the statement of our main result.

Theorem 3.10 *Consider the nonlinear system Σ_q, where $\rho^*(\Sigma_0) = m$. Let $x_0 \in I\!R^n$ be an equilibrium point of Σ_q satisfying $h(x_0) = 0$ and Assumptions 3.8 and 3.9. Then the DDDP for Σ_q is locally solvable around x_0 if and only if it is solvable for $L\Sigma_q$.*

Proof *(necessity)* Assume that the DDDP for Σ_q is locally solvable around x_0. Let Q be a Singh compensator that solves the problem around x_0. Then we have in particular that the DDP is solvable for $\Sigma_q \circ Q$. It can easily be checked (see e.g. [7]) that the decoupling matrix of $\Sigma_q \circ Q$ is the (m, m)-identity matrix. Hence by Theorem 2.3 the DDP is solvable for $L(\Sigma_q \circ Q)$, the linearization of $\Sigma_q \circ Q$ around $(x_0, 0)$. By Proposition 3.4 we have that $L(\Sigma_q \circ Q) = L\Sigma_q \circ R$, where R is a Singh compensator for $L\Sigma_q$. Since by Proposition 3.4 LQ is a regular dynamic state feedback for $L\Sigma_q$, this means that the DDDP is solvable for $L\Sigma_q$.

(sufficiency) Assume that Assumptions 3.8 and 3.9 hold and that the DDDP is solvable for $L\Sigma_q$ via a Singh compensator R. By Proposition 3.4, there is a Singh compensator Q for Σ_q such that $(x_0, 0)$ is an equilibrium point of $\Sigma_q \circ Q$ and such that $L(\Sigma_q \circ Q)$, the linearization of $\Sigma_q \circ Q$ around $(x_0, 0)$, satisfies $L(\Sigma_q \circ Q) = L\Sigma_q \circ R$. As above, we have that the decoupling matrix of $\Sigma_q \circ Q$ is the (m, m)-identity matrix. Then it follows that Δ_e^*, the maximal locally controlled invariant distribution in $\mathrm{Ker}\, dh$ for $\Sigma_q \circ Q$, is given by (see [11],[12])

$$\Delta_e^* = \bar{\Delta}_e \cap \mathrm{Ker}\, dz \tag{17}$$

This implies that Assumption 3.9 for Σ_q is the same as Assumption 2.2 for $\Sigma_q \circ Q$. Hence by Theorem 2.3 the solvability of the DDDP for $L\Sigma_q$ implies the solvability of the DDP for $\Sigma_q \circ Q$. Since by Proposition 3.4 the Singh compensator Q is a regular dynamic state feedback for Σ_q, this implies on its turn that the DDDP is solvable for Σ_q. ∎

From Theorem 3.10 it follows that if Assumptions 3.8 and 3.9 hold, the solvability of the DDP for $L\Sigma_q$ implies solvability of the DDDP for Σ_q, but not necessarily solvability of the DDP for Σ_q (for a counter example, see the following section). If indeed the DDDP, but not the DDP, is solvable for Σ_q, *no* static state feedback that solves the DDP for $L\Sigma_q$ will be a first order approximation of a feedback that solves the DDDP for Σ_q. As a result of this such a static state feedback will in general not result in a satisfactory disturbance attenuation when applied to Σ_q. At the same time the remedy is clear: one should look for a *dynamic* state feedback that solves the DDDP for $L\Sigma_q$ and that at the same time is the linearization of a dynamic state feedback that solves the DDDP for Σ_q. By Proposition 3.4 and Theorem 3.10, any Singh compensator for $L\Sigma_q$ will do this job (provided the DDP is solvable for $L\Sigma_q$). In other words, one should incorporate *integral action* to some of the controls

4. Example

Consider a nonlinear system in $\mathbb{R}^6$ of the form

$$\Sigma_q \begin{cases} \dot{x}_1 = (x_2+1)u_1 \\ \dot{x}_2 = x_5 \\ \dot{x}_3 = -x_2 - x_3 + x_4 + (x_4 - 1)u_1 \\ \dot{x}_4 = u_2 \\ \dot{x}_5 = -15x_5 + q \\ \dot{x}_6 = 8x_2 - 8x_4 - 16x_6 \\ y_1 = x_1 \\ y_2 = x_3, \end{cases}$$

with $u \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, q an unknown disturbance and equilibrium point $x_0 = 0$. For this system we have $\Delta^* = \operatorname{span}\{\partial/\partial x_6\}$, and since $\mathcal{P} = \operatorname{span}\{\partial/\partial x_5\} \not\subset \Delta^*$, the DDP is not solvable for Σ_q.

The linearization of Σ_q around x_0 is given by

$$L\Sigma_q \begin{cases} \dot{\xi}_1 = \tilde{u}_1 \\ \dot{\xi}_2 = \xi_5 \\ \dot{\xi}_3 = -\xi_2 - \xi_3 + \xi_4 - \tilde{u}_1 \\ \dot{\xi}_4 = \tilde{u}_2 \\ \dot{\xi}_5 = -15\xi_5 + \tilde{q} \\ \dot{\xi}_6 = 8\xi_2 - 8\xi_4 - 16\xi_6 \\ \tilde{y}_1 = \xi_1 \\ \tilde{y}_2 = \xi_3, \end{cases}$$

with equilibrium point $\xi_0 = 0$. After some calculations we find that for $L\Sigma_q$ we have $\mathcal{V}^* = \operatorname{span}\{e_2 + e_4, , e_5, e_6\}$, where e_i denotes the i-th basis vector of the standard basis of $\mathbb{R}^6$. The DDP for $L\Sigma_q$ is solvable because $\operatorname{Im} P = \operatorname{span}\{e_5\} \subset \mathcal{V}^*$. One regular static state feedback which solves the DDP for $L\Sigma_q$ is

$$\begin{aligned} \tilde{u}_1 &= -4\xi_1 + \tilde{v}_1 \\ \tilde{u}_2 &= -\xi_2 - \xi_3 + \xi_4 + \xi_5 + \tilde{v}_2, \end{aligned} \tag{18}$$

where $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)^T$ are the new controls. We remark at this point that the regular static state feedback (18) does not correspond to a first order approximation of a static state feedback which approximately solves the local DDP for the system Σ_q, since, we have seen, the DDP for Σ_q is not solvable. An additional evidence of this fact is provided by the following numerical simulations. In these simulations we apply the linear static state feedback (18) to Σ_q with the new controls $\tilde{v}_1$, $\tilde{v}_2$ designed to provide closed loop asymptotic stability,

$$\begin{aligned} \tilde{v}_1 &= 0 \\ \tilde{v}_2 &= 8\xi_2 - 8\xi_4 - \xi_6 \end{aligned} \tag{19}$$

The initial conditions are chosen to be $x(0) = (0.2, 0.1, 0, 0, 0.6, 0)^T$ and the disturbance q is selected as a sinusoidal function of time: $q(t) = 50 \sin t$ $(0 \leq t \leq 40)$, Figures 1 and 2 show how the outputs of the nonlinear system Σ_q are influenced by the disturbance $q(t)$ Now, some calculations will show that Σ_q satisfies Assumptions 3.6 and 3.7 and that $\rho^*(\Sigma_0) = m = 2$. Hence, the DDDP for Σ_q is locally solvable around x_0 and thus it is also solvable for $L\Sigma_q$. A Singh compensator that solves the DDDP for $L\Sigma_q$ is

$$LQ \begin{cases} \dot{\tilde{z}} = \tilde{v}_1 \\ \tilde{u}_1 = \tilde{z} \\ \tilde{u}_2 = -\xi_2 - \xi_3 + \xi_4 + \xi_5 - \tilde{z} + \tilde{v}_1 + \tilde{v}_2. \end{cases}$$

In order to have an asymptotically stable closed loop system $L\Sigma_q \circ LQ$ we will set

$$\begin{aligned} \tilde{v}_1 &= -4\xi_1 - \tilde{z} + \bar{v}_1 \\ \tilde{v}_2 &= -\xi_4 + \xi_2 + 8\xi_6 + \bar{v}_2, \end{aligned} \tag{20}$$

where $\bar{v}_1$, $\bar{v}_2$ are the new controls. It can be easily checked that the system $LQ \circ L\Sigma_q$ together with the additional feedback (20) remains disturbance decoupled. Since the DDDP is solvable for Σ_q, we can construct for the system Σ_q, a Singh compensator to obtain a solution for the DDDP. A Singh compensator is given by

$$Q \begin{cases} \dot{z} = v_1 \\ u_1 = \frac{z}{x_2+1} \\ u_2 = (-x_2 - x_3 + x_4 + x_5 + \frac{x_4 z}{x_2+1} - \frac{z}{x_2+1} \\ \qquad + \frac{x_4 v_1}{x_2+1} - \frac{x_4 x_5 z}{x_2+1} + \frac{v_1}{x_2+1} - \frac{x_5 z}{x_2+1})(\frac{x_2+1}{x_2+z+1}) \end{cases}$$

with $v = (v_1, v_2)^T$ the new control vector. Note that the linearization of Q around the equilibrium yields the linear compensator LQ. With respect to the new control vector v in Q there is no obstacle to choose it as in the linear case, eq. (20), that is,

$$\begin{aligned} v_1 &= -4x_1 - z + \bar{v}_1 \\ v_2 &= 4x_2 - 4x_4 + 8x_6 + \bar{v}_2, \end{aligned} \tag{21}$$

with $\bar{v}_1$, $\bar{v}_2$, the new controls.

Figures 3 and 4 illustrate numerical results of the composed system $\Sigma_q \circ LQ$. Here LQ is as described above with $\tilde{v}_1$ and $\tilde{v}_2$ specified in (20) with $\bar{v}_1 = \bar{v}_2 = 0$, and the initial condition of Σ_q is $x(0) = (0.2, 0.5, 0, 0, 0.6, 0)^T$.

5. Conclusions

The purpose of this paper was to study the disturbance decoupling problem for a nonlinear system in relation to the same problem for its linearization. The main result, see Theorem 3.10, states that under generic conditions - which is a mathematical phrasing for almost always- the problem is solvable in the nonlinear case via *dynamic state feedback* if and only if the linear problem is solvable. Since it is known that if for a linear system the disturbance decoupling problem is solvable, then it is solvable via a (linear) static state feedback (cf. [16]), we arrive at the remarkable conclusion that the nonlinear disturbance decoupling problem is solvable via a dynamic state feedback if and only if the linear(ized) disturbance decoupling problem is solvable via a static state feedback.

The above result induces an interesting way of obtaining an approximate solution of the nonlinear disturbance decoupling problem, namely by taking a suitably defined linear dynamic compensator that achieves disturbance decoupling. The dynamic compensator we use is of a specific nature and arises as the linearization of a nonlinear decoupling compensator. One of the specific features of the considered dynamic compensator is the introduction of extra integral actions on a part of the input channels. This explains the use of adding integral action in achieving disturbance attenuation. This idea of providing an approximation for a solution of the nonlinear dynamic disturbance decoupling problem was illustrated on a mathematical simulation example.

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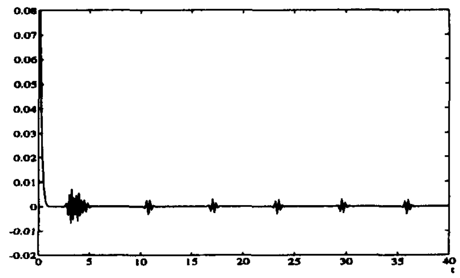

Figure 1: The output y_1 of the system Σ_q in closed loop with the linear static state feedback (46).

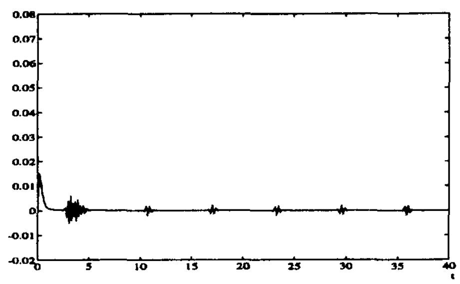

Figure 2: The output y_2 of the system Σ_q in closed loop with the linear static state feedback (46).

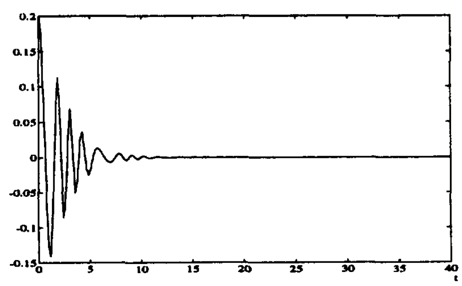

Figure 3: The output y_1 of the system $\Sigma_q \circ LQ$.

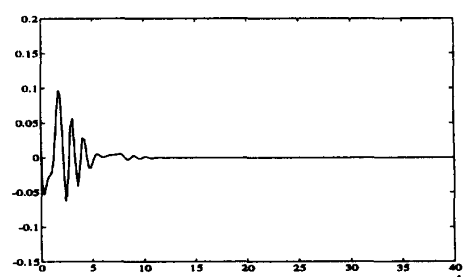

Figure 4: The output y_2 of the system $\Sigma_q \circ LQ$.