Counterexamples to Robertson's conjecture

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It is shown that, contrary to a conjecture by Robertson [4], some of the coefficients $d_{nj}(\frac{1}{2})$ in $\{((1+z)^x/(1-z)^x-1)/(2zx)\}^{\frac{1}{2}} = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} d_{nj}(\frac{1}{2})\, x^j$ are negative. Results from probability theory turn out to be useful.

1. Introduction

In [7], among other things, Todorov considers the Taylor expansion around $z=0$ of the function $d(z) = d(z; x, y)$ defined by $d(0) = 1$ and

$$d(z) = \left\{ \frac{(\frac{1+z}{1-z})^x - 1}{2zx} \right\}^y = \sum_{n=0}^{n} z^n \sum_{j=0}^{n} d_{nj}(y)\, x^j \quad (|z| < 1)\ . \tag{1}$$

Especially, he examines the conjecture in Robertson [4] that $d_{nj}(\frac{1}{2}) \geq 0$ for all n and j, and obtains some supporting evidence. In the present note, however, this conjecture is shown to be false.

The proofs depend on well-known and fairly elementary results on the divisibility of probability distributions, which are briefly discussed in the Appendix. For further information we refer to [1] and [6].

2. The sign of $d_{nj}(1/N)$

We rewrite $d(z)$ in (1) as

$$2^y d(z) = (R(z))^y \left(\frac{e^u - 1}{u}\right)^y \quad , \tag{2}$$

where $u = zx\, R(z)$ with

$$R(z) = z^{-1} \log \frac{1+z}{1-z} = 2 \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1} \quad .$$

Now let $C(u) = (e^u - 1)/u$, and for $y > 0$

$$(C(u))^y = \left(\frac{e^u - 1}{u}\right)^y = \sum_{j=0}^{\infty} c_j(y)\, u^j \quad , \tag{3}$$

and for $p > 0$

$$(R(z))^p = \sum_{m=0}^{\infty} r_m(p)\, z^m \quad . \tag{4}$$

Then from (2) we obtain the following expansion.

$$\begin{aligned} 2^y d(z) &= \sum_{j=0}^{\infty} c_j(y)\, (zx)^j (R(z))^{j+y} \\ &= \sum_{j=0}^{\infty} c_j(y)\, (zx)^j \sum_{m=0}^{\infty} r_m(j+y)\, z^m \\ &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} c_j(y)\, r_{n-j}(j+y)\, x^j \quad . \end{aligned}$$

Hence by (1),

$$2^y d_{nj}(y) = c_j(y)\, r_{n-j}(j+y) \quad . \tag{5}$$

We now need three lemmas giving information about c_j and r_{n-j}; most proofs are deferred to the Appendix.

Lemma 1. Let $r_m(p)$ be defined by (4). Then

$$r_m(p) > 0 \quad (p > 0; m = 0, 2, \cdots) \quad .$$

Lemma 2. Let $y = 1/N$, where N is an integer, $N \geq 2$, and let $c_j(y)$ be defined by (3). Then there exists an integer $j > 0$ such that $c_j(y) < 0$.

Corollary. For $y = 1/N$ with N an integer, $N \geq 2$, not all $d_{nj}(y)$ in (1) (cf. (5)) are nonnegative.

The next lemma leads to explicit counterexamples to Robertson's conjecture (13 turns out to be the unlucky number).

Lemma 3. Let $c_j(y)$ be defined by (3). Then

$$c_j(\tfrac{1}{2}) > 0 \ \text{ for } \ i = 1, 2, \cdots, 12; \ c_{13}(\tfrac{1}{2}) < 0 \ .$$

Proof. Since $(\sum_{j=0}^{\infty} c_j(\frac{1}{2})\, u^j)^2 = (e^u - 1)/u$, the $c_j(\frac{1}{2})$ can be computed recursively from $c_0(\frac{1}{2}) = 1$ and

$$c_{j+1}(\tfrac{1}{2}) = \tfrac{1}{2}\left(\frac{1}{(j+2)!} - \sum_{k=1}^{j} c_k(\tfrac{1}{2})\, c_{j+1-k}(\tfrac{1}{2})\right) \quad (j = 0, 1, \cdots) \ .$$

It turns out that $c_j(\frac{1}{2}) > 0$ for $j = 0, 1, \cdots, 12$ and $c_{13}(\frac{1}{2}) = -4.6235\ 10^{-13}$ (rounded to the last decimal shown).

Corollary.

$$d_{nj}(\tfrac{1}{2}) > 0 \ \text{ for } \ n = 0, 1, \cdots, 12; \ j = 0, 1, \cdots, n; \ n - j \text{ even} \ ,$$

$$d_{n,\,13}(\tfrac{1}{2}) < 0 \ \text{ for } \ n \geq 13; \ n \text{ odd} \ .$$

Remark. Computations indicate that $c_j(\frac{1}{2}) < 0$ if $j = 13 + 4k$ and $j = 14 + 4k$, and $c_j(\frac{1}{2}) > 0$ if $j = 15 + 4k$ and $j = 16 + 4k$ $(k = 0, 1, \cdots)$; this sign pattern is similar to that of $((e^u - 1)/u)^{-\frac{1}{2}} = (C(u))^{-\frac{1}{2}}$ (cf. Jordon [2]).

3. Appendix

Since the sequence $((2n + 1)^{-1})$ is log-convex, Lemma 1 is a special case of the following result, which is well-known for probability distributions on $\{0, 1, \cdots\}$.

Proposition 1. Let $(p_n)_0^{\infty}$ be a strictly log-convex sequence of positive numbers, i.e.

$$p_{n+1} p_{n-1} > p_n^2 \quad (n = 1, 2, \cdots) \ .$$

Let $p(z) = \sum_{n=0}^{\infty} p_n z^n$ be the (possibly formal) power series generating (p_n), and let $y > 0$. Then

$$(P(z))^y = \sum_{n=0}^{\infty} p_n(y)\, z^n \ ,$$

with $p_n(y) > 0 \ \ (n = 0, 1, \cdots)$.

Proof. The proof for probability distributions as given in [5, p.137] is easily adapted to more general sequences, e.g. by considering $\alpha^n p_n$, for a suitable $\alpha > 0$, instead of p_n. See also [1, Vol. I, p.289] for general information about infinitely divisible distributions on $\{0, 1, \cdots\}$.

The following proposition is equivalent to Lemma 2.

Proposition 2. Let $C(u) = (e^u - 1)/u$ and let N be an integer, $N \geq 2$. Then

$$(C(u))^{1/N} = \sum_{j=0}^{\infty} c_j(1/N)\, u^j \ , \tag{6}$$

where some $c_j(1/N)$ are negative.

Proof. Clearly, all coefficients in (6) are real. As in Lemma 3 we have, writing c_j for $c_j(1/N)$,

$$c_{n+1} = \frac{1}{N}\left(\frac{1}{(n+2)!} - \Sigma c_{k_1}\, c_{k_2} \cdots c_{k_N}\right) \ ,$$

where the sum extends over all k_j with $1 \leq k_j \leq n$ and $k_1 + \cdots + k_N = n+1$.
Now if all c_n were nonnegative, then we would have

$$0 \leq c_{n+1} \leq \frac{1}{N}\,\frac{1}{(n+2)!} \ .$$

This would imply that $C^{1/N}$ has infinite radius of convergence, which it has not; since $2\pi i$ is a branch point, the radius of convergence is 2π.

Remark 1. A similar argument shows that some of the $c_j(y)$ are negative if y is an arbitrary, non-integer, rational number.

Remark 2. Proposition 2 is strongly suggested by the following simple fact in probability theory. If X is a random variable with an uniform distribution on $(0, 1)$, and N is an integer, $N \geq 2$, then X cannot be divided as follows:

$$X = Y_1 + \cdots + Y_N \ ,$$

where the Y_j are independent and have the same distribution (on $(0, \frac{1}{N})$). For $N \geq 3$ this is most easily seen by taking variances:

$$\frac{1}{12} = \mathrm{var}\, X = N\ \mathrm{var}\, Y_1 < N \frac{1}{4N^2} = \frac{1}{4N} ,$$

since var $Y < 1/(4a^2)$ if Y is restricted to the open interval $(0, a)$.

Now, the uniform density has Laplace transform $C(-s)$, and so it follows that $(C(-s))^{1/N}$ is not completely monotone (cf. [1, Vol. II, p.439]), and hence that $(C(u))^{1/N}$ is not absolutely monotone, i.e., does not have all its derivatives nonnegative. For $N = 2$ see [3].

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