Crack Propagation Analysis

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1 Introduction

Engineering structures are designed to withstand the loads they are expected to be subject to while in service. Large stress concentrations are avoided, and a reasonable margin of security is taken to ensure that values close to the maximum admissible stress are never attained. However, material imperfections which arise at the time of production or usage of the material are unavoidable, and hence must be taken into account. Indeed even microscopic flaws may cause structures which are assumed to be safe to fail, as they grow over time.

In the past, when a component of some structure exhibited a crack, it was either repaired or simply retired from service. Such precautions are nowadays in many cases deemed unnecessary, not possible to enforce, or may prove too costly. In fact, on one hand, the safety margins assigned to structures have to be smaller, due to increasing demands for energy and material conservation. On the other hand, the detection of a flaw in a structure does not automatically mean that it is not safe to use anymore. This is particularly relevant in the case of expensive materials or components of structures whose usage it would be inconvenient to interrupt.

In this setting fracture mechanics plays a central role, as it provides useful tools which allow for an analysis of materials which exhibit cracks. The goal is to predict whether and in which manner failure might occur.

Historically, in the western world, the origins of this branch of science seem to go back to the days of Leonardo da Vinci (15th-16th centuries). According to some authors [18, 20] records show that the renown scientist underwent the study of fracture strength of materials, using a device described on the Codex Atlanticus [21]. He looked into the variation of failure strength in different lengths of iron wire of the same diameter. The conclusion was that short wires, with less probability of containing a defect, were apparently stronger than long wires.

Centuries later, at the time of the first World War, the English aeronautical engineer Alan Griffith was able to theorize on the failure of brittle materials [8]. He used a thermodynamic approach to analyze the centrally cracked glass plate present in an earlier work of Inglis [10]. Note that Griffith’s theory is strictly restricted to elastic brittle materials like glass, in which virtually no plastic deformation near the tip of the crack occurs. However, extensions which account for such a deformation and further extend this
theory have been suggested, for example in [11, 12, 13, 14, 16].

In this work, we shall focus on the brittle fracture of elastic materials. For that we will perform a numerical analysis of a cracked plate in a plane stress situation. This requires three distinct problems to be solved. Firstly, numerical methods to determine the stress and displacement fields around the crack must be available. The second problem consists of the numerical computation of the fracture parameters, such as the stress intensity factors, the J integral, the energy release rate or another. Finally one needs to decide on criteria to determine under which conditions the crack will propagate, as well as the direction of propagation.

We conclude with an overview of this report. It is divided in four sections, including this introduction, Section 1.

In Section 2, we present the main ideas of linear elastic fracture mechanics. Bearing that in mind, we start by writing down the equations of elasticity in 2.1. Next, we look into how a cracked plate can be loaded and distinguish three modes of loading in 2.2, modes $I$, $II$ and $III$. From these we will only consider the in-plane loads, which excludes mode $III$. Subsequently, in 2.3, we describe the behaviour of the stresses and displacements in the vicinity of a crack. The stress intensity factors, which play a fundamental role in this area, are introduced. These are well known for some geometries, as can be seen in 2.4. There we also give an example of a static fracture analysis, which consists of computing the stress intensity factor for a mode $I$ situation and comparing it with the value predicted by another author. We conclude this section, in 2.5, by describing criteria for crack growth, both for a mode $I$ and a mixed mode situations. In general, that implies not only having an equation to decide when does crack propagation begin, but also in which direction the crack grows.

Section 3 is dedicated to a a quasi-static fracture analysis. Given a cracked plate in a mixed mode loading situation, we set up an algorithm to predict the path a growing crack will follow.

Finally, in Section 4, we describe some extensions to the theory we had presented earlier, as well as other concepts and techniques that are frequently used to handle crack growth. Namely, we address dynamic fracture in 4.1 and define the energy release rate and the J integral in 4.2.
2 Elastic Fracture

2.1 Governing equations

As is well known, the dynamic behaviour of a linear elastic material which occupies the domain $\Omega$ is modeled by the following equations for the stresses $\sigma$, the strains $\epsilon$ and the displacements $u$, which hold for every point in $\Omega$

$$\nabla \cdot \sigma + b = \rho \ddot{u},$$  \hspace{1cm} (1)

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$ \hspace{1cm} (2)

$$\sigma = C\epsilon = \lambda(\text{tr}\epsilon)I + 2\mu\epsilon,$$ \hspace{1cm} (3)

where the double-dot notation refers to a second order derivative with respect to time. Besides these equations, we consider the boundary conditions on $\Gamma_D$ and $\Gamma_N$, respectively the Dirichlet and Neumann parts of $\Omega$

$$u = f \text{ on } \Gamma_D,$$ \hspace{1cm} (4)

$$\sigma \cdot n = g \text{ on } \Gamma_N.$$ \hspace{1cm} (5)

Finally, the formulation of the problem is made complete by the choice of some initial conditions for the displacements and their derivatives.

In the previous equations, $f$ and $g$ are given functions, $n$ is the outward normal to $\partial\Omega$, $b$ represents the body force and $\rho$ the material density. As for $\lambda$ and $\mu$, these are called Lamé moduli. They are related to the parameters $E$ and $\nu$, respectively the material Young’s modulus and Poisson’s ratio, by

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}$$ \hspace{1cm} (6)

and

$$\nu = \frac{\lambda}{2(\mu + \lambda)}.$$ \hspace{1cm} (7)

Now, in this report we assume that $f = 0$, and focus on the particular static problem of elasticity in the absence of body forces. This is formulated by replacing the first equation in the general problem (1)-(6) by

$$\nabla \cdot \sigma = 0.$$ \hspace{1cm} (8)
Then, the weak formulation of the problem at hand, consisting of equations (2)-(5) and (8), is given by the following equation for $u$

$$a(u, v) = <l, v>, \forall \ v \in V. \quad (9)$$

Here $a(\cdot, \cdot)$ and $<\cdot, \cdot>$ are defined by

$$a(u, v) = \int_{\Omega} C\epsilon(u) : \epsilon(v) dx, \quad (10)$$

$$<l, v> = \int_{\Omega_y} g v ds, \quad (11)$$

and the space $V$ by

$$V = \{v = (v_i) : v_i \in H^1(\Omega), v_i = 0 \text{ on } \Gamma_D, i = 1, 2\}. \quad (12)$$

### 2.2 Modes of fracture

Consider a cracked plate. We can distinguish several manners in which a force may be applied to the plate which might enable the crack to propagate. Irwin [13] proposed a classification corresponding to the three situations represented in Figure 1.

![Figure 1 - a) Mode I fracture. b) Mode II fracture. c) Mode III fracture.](image)

Accordingly, we consider three distinct modes: mode I, mode II and mode III.
In the mode I, or opening mode, the body is loaded by tensile forces, such that the crack surfaces are pulled apart in the $y$ direction. The deformations are then symmetric with respect to the planes perpendicular to the $y$ axis and the $z$ axis.

In the mode II, or sliding mode, the body is loaded by shear forces parallel to the crack surfaces, which slide over each other in the $x$ direction. The deformations are then symmetric with respect to the plane perpendicular to the $z$ axis and skew symmetric with respect to the plane perpendicular to the $y$ axis.

Finally, in the mode III, or tearing mode, the body is loaded by shear forces parallel to the crack front the crack surfaces, and the crack surfaces slide over each other in the $z$ direction. The deformations are then skew-symmetric with respect to the plane perpendicular to the $z$ and the $y$ axis.

For each of these modes, crack extension may only take place in the direction of the $x$ axis, the original orientation of the crack. In a more general situation, we typically find a mixed mode situation, where there is a superposition of the modes. In a linear elastic mixed mode problem, the principle of stress superposition states that the individual contributions to a given stress component are additive, so that if $\sigma_{ij}^I$, $\sigma_{ij}^{II}$ and $\sigma_{ij}^{III}$ are respectively the stress components associated to the modes $I$, $II$ and $III$, then the stress component $\sigma_{ij}$ is given by

$$\sigma_{ij} = \sigma_{ij}^I + \sigma_{ij}^{II} + \sigma_{ij}^{III},$$

(13)

for $i, j = x, y$.

Now consider any of the three modes we have introduced. Within the scope of the theory of linear elasticity, a crack introduces a discontinuity in the elastic body such that the stresses tend to infinity as one approaches the crack tip. Using the semi-inverse method of Westerngaard [23], Irwin [11, 12] related the singular behaviour of the stress components to the distance to the crack tip $r$. The relation he obtained can be written in a simplified form as

$$\sigma \simeq \frac{K}{\sqrt{2\pi r}}.$$  

(14)

The parameter $K$, the stress intensity factor, plays a fundamental role in fracture mechanics, as it characterizes the stress field in this region.
Henceforth, we will consider the problem of a cracked plate in a plane stress situation, which means that mode III situations (pure or mixed) will be disregarded.

### 2.3 Concepts of fracture mechanics

Consider a static crack in a plate which is in a plane stress situation. Assume that the crack surfaces are free of stress and that the crack is positioned along the negative \( x \)-axis, as in Figure 2.

![Figure 2: Crack tip coordinates.](image)

Then the distribution of the stresses in the region near the tip of the crack may be derived as an interior asymptotic expansion. In polar coordinates, we have - see for example [5, 6, 7, 13] -

\[
\sigma_{ij}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} f_{ij}^I(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{II}(\theta) + \sigma_{ij}^0 + O(\sqrt{r}),
\]

for \( r \to 0 \) and \( i, j = x, y \), and where \( \sigma_{ij}^0 \) indicates the finite stresses at the crack tip. Here the normalizing constants for the symmetric and anti-symmetric parts of the stress field \( K_I \) and \( K_{II} \) represent respectively the stress intensity factors for the corresponding modes I and II, and are defined by

\[
K_I = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{yy}(r, 0),
\]

\[
K_{II} = \lim_{r \to 0} \sqrt{2\pi r} \sigma_{xy}(r, 0).
\]

The angular variation functions for mode I are given respectively by
\begin{align*}
    f_{xx}^I(\theta) &= \cos(\frac{1}{2} \theta) \left( 1 - \sin(\frac{1}{2} \theta) \sin(\frac{3}{2} \theta) \right), \quad (18) \\
    f_{yy}^I(\theta) &= \cos(\frac{1}{2} \theta) \left( 1 + \sin(\frac{1}{2} \theta) \sin(\frac{3}{2} \theta) \right), \quad (19) \\
    f_{xy}^I(\theta) &= \cos(\frac{1}{2} \theta) \sin(\frac{1}{2} \theta) \sin(\frac{3}{2} \theta), \quad (20)
\end{align*}
while the equivalent functions for mode II are
\begin{align*}
    f_{xx}^{II}(\theta) &= -\sin(\frac{1}{2} \theta) \left( 2 + \cos(\frac{1}{2} \theta) \cos(\frac{3}{2} \theta) \right), \quad (21) \\
    f_{yy}^{II}(\theta) &= \cos(\frac{1}{2} \theta) \sin(\frac{1}{2} \theta) \sin(\frac{3}{2} \theta), \quad (22) \\
    f_{xy}^{II}(\theta) &= \cos(\frac{1}{2} \theta) \left( 1 - \sin(\frac{1}{2} \theta) \sin(\frac{3}{2} \theta) \right). \quad (23)
\end{align*}

Again, we can obtain equations for the corresponding displacement field near the crack tip, which is discontinuous over the crack. This is given - as can be seen in [6, 13, 15] - as follows
\begin{align*}
    u_i(r, \theta) &= u_i^0 + \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \tilde{f}_i^I(\theta) + \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \tilde{f}_i^{II}(\theta) + O(\sqrt{r}), \quad (24)
\end{align*}
where \( i = x, y \) and for \( r \to 0 \). Here, \( u_i^0 \) are the crack tip displacements. The angular variation functions here are now given by
\begin{align*}
    \tilde{f}_x^I(\theta) &= \cos(\frac{1}{2} \theta) \left( \frac{1 - \nu}{1 + \nu} + \sin^2(\frac{1}{2} \theta) \right), \quad (25) \\
    \tilde{f}_y^I(\theta) &= \sin(\frac{1}{2} \theta) \left( \frac{2}{1 + \nu} - \cos^2(\frac{1}{2} \theta) \right), \quad (26) \\
    \tilde{f}_x^{II}(\theta) &= \sin(\frac{1}{2} \theta) \left( \frac{2}{1 + \nu} + \cos^2(\frac{1}{2} \theta) \right), \quad (27) \\
    \tilde{f}_x^{II}(\theta) &= \cos(\frac{1}{2} \theta) \left( -\frac{1 - \nu}{1 + \nu} + \sin^2(\frac{1}{2} \theta) \right). \quad (28)
\end{align*}

The formulas we have presented allow us to have a characterization of the stresses and the displacements in the vicinity of a crack tip.
2.4 A static problem

The concept of stress intensity factor plays a central role in fracture mechanics. We now refer to Tada [19] to present some classical examples of cracked geometries - represented in Figure 3 - for which the stress intensity factor has been computed or approximated explicitly. It is assumed that crack propagation may not occur, i.e., the problem is static.

![Figure 3 - a) Infinite plate with a center through crack under tension.](image)

![Figure 3 - b) Semi-Infinite plate with a center through crack under tension.](image)

![Figure 3 - c) Infinite stripe with a center through crack under tension.](image)

![Figure 3 - d) Infinite stripe with an edge through crack under tension.](image)

The stress intensity values for these geometries are as follows, where the letters a) - d) used to identify the formulas are in correspondence with those...
of the pictures in Figure 3:

a) \( K_I = \sigma \sqrt{\pi a}; \)

b) \( K_I = 1.1215\sigma \sqrt{\pi a}; \)

c) \( K_I = \sigma \sqrt{\pi a}(1 - 0.025(\frac{a}{h})^2 + 0.06(\frac{a}{h})^4)\sqrt{\sec \frac{\pi a}{2b}}, \)

d) \( K_I = \sigma \sqrt{\pi a}(\sqrt{\frac{2b}{\pi a}} \tan \frac{\pi a}{2b} \cdot \frac{0.752+2.02(\frac{a}{h})+0.37(1-\sin \frac{\pi a}{2b})^3}{\cos \frac{\pi a}{2b}}). \)

The previous examples involved geometries of infinite dimensions. Aliabadi [1] computed the stress intensity factor for the finite geometry represented in Figure 4.

![Figure 4: Finite plate with a center through crack under tension.](image)

He considered a rectangular plate, of height \( 2h \), width \( 2b \), with a central through crack of length \( 2a \), which was loaded from its upper and lower edges by a uniform tensile stress \( \sigma \). For this particular geometry, he estimated
\[ K_I = \sigma \sqrt{\pi a} (1 + 0.043\left(\frac{a}{b}\right) + 0.491\left(\frac{a}{b}\right)^2 + 7.125\left(\frac{a}{b}\right)^3 - 28.403\left(\frac{a}{b}\right)^4 + 59.583\left(\frac{a}{b}\right)^5 - 65.278\left(\frac{a}{b}\right)^6 + 29.762\left(\frac{a}{b}\right)^7). \]  

Using the numerical package Abaqus, we also determined the value of the stress intensity factor \( K_I \) for the same geometry. This was computed using finite elements on a mesh with quadratic triangular elements on the vicinity of the crack tip, and quadratic rectangular elements everywhere else. Quarter point elements, formed by placing the mid-side node near the crack tip at the quarter point, were used to account for the crack singularity. More details can be seen in [4]. In Table 1 we display some values of \( K_I/K_0 \), where \( K_0 = \sigma \sqrt{\pi a}, \) up to two significant digits. It can be seen that our results, identified by FEM (finite element method), are in line with those predicted by Aliabadi, and even more so for smaller values of \( a/b \). We further illustrate this analysis in Figure 5, for which more data points were taken.

Table 1: Stress intensity factors.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( a/b = 0.1 )</th>
<th>( a/b = 0.2 )</th>
<th>( a/b = 0.4 )</th>
<th>( a/b = 0.6 )</th>
<th>( a/b = 0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aliabadi</td>
<td>1.01</td>
<td>1.06</td>
<td>1.22</td>
<td>1.48</td>
<td>2.02</td>
</tr>
<tr>
<td>FEM</td>
<td>1.01</td>
<td>1.06</td>
<td>1.22</td>
<td>1.48</td>
<td>1.99</td>
</tr>
</tbody>
</table>

Figure 5: Compared results.

As we had already mentioned, the stress intensity factor depends on the geometry of the plate we are considering. In particular, it depends on the
ratio $h/b$. On Table 2 we display the values of $K_I/K_0$, determined again using Abaqus, for different geometries.

<table>
<thead>
<tr>
<th></th>
<th>$a/b = 0.1$</th>
<th>$a/b = 0.2$</th>
<th>$a/b = 0.4$</th>
<th>$a/b = 0.6$</th>
<th>$a/b = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h/b = 0.25$</td>
<td>1.17</td>
<td>1.57</td>
<td>2.65</td>
<td>4.03</td>
<td>7.44</td>
</tr>
<tr>
<td>$h/b = 0.5$</td>
<td>1.04</td>
<td>1.17</td>
<td>1.63</td>
<td>2.42</td>
<td>3.77</td>
</tr>
<tr>
<td>$h/b = 1$</td>
<td>1.01</td>
<td>1.06</td>
<td>1.22</td>
<td>1.48</td>
<td>1.99</td>
</tr>
<tr>
<td>$h/b = 2$</td>
<td>1.01</td>
<td>1.02</td>
<td>1.11</td>
<td>1.30</td>
<td>1.81</td>
</tr>
<tr>
<td>$h/b = \infty$</td>
<td>1.01</td>
<td>1.02</td>
<td>1.11</td>
<td>1.30</td>
<td>1.81</td>
</tr>
</tbody>
</table>

We note that as the value of $h/b$ increases, the values of $K_I/K_0$ tend to the values of the last line ($h/b = \infty$), which refers to values that we would expect for an infinite stripe with a center through crack under tension, as in Figure 3-c). We have computed these using the values of $K_I$ from the formula c). To better illustrate this idea, we conclude with the graphical representations of the values of Table 2 on Figure 6.

![Figure 6: Plots of $K_I/K_0$.](image)

2.5 Fracture criteria

Consider a plate which exhibits a pre-existent crack. As we said before, in this work, we focus on elastic brittle fracture. We will then make use of the equations of elasticity for problems of plane stress. Besides these, and due
to the presence of the crack, an extra equation to serve as fracture criteria is also required - see, for example, [5, 6]. This will allow us to decide if and when the crack will propagate, and in which direction.

We start by considering a stationary semi-infinite line crack, loaded in a mode $I$ situation. The symmetry of the deformations implies that the crack may only propagate in a direction perpendicular to the loading. All that is required then is a necessary condition for the crack growth.

In the region surrounding the tip of the crack, the singular stresses are characterized by the stress intensity factor $K_I$. It is postulated that crack growth will occur when the equality

$$K_I = K_{I_c}$$

holds. As for $K_{I_c}$, which behaves as a threshold value for $K_I$, it is called the critical stress intensity factor. It is a material parameter, also known as mode $I$ fracture toughness. It may be determined experimentally. We include in Table 3 some examples of experimental data for the fracture toughness of some materials, as taken from [3].

<table>
<thead>
<tr>
<th>Material</th>
<th>Fracture toughness ($MN/m^{3/2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mild steel</td>
<td>140</td>
</tr>
<tr>
<td>Titanium alloys</td>
<td>55 – 120</td>
</tr>
<tr>
<td>High carbon steel</td>
<td>30</td>
</tr>
<tr>
<td>Nickel, copper</td>
<td>&gt; 100</td>
</tr>
<tr>
<td>Nickel, copper</td>
<td>&gt; 100</td>
</tr>
<tr>
<td>Concrete (steel reinforced)</td>
<td>10 – 15</td>
</tr>
<tr>
<td>Concrete (unreinforced)</td>
<td>0.2</td>
</tr>
<tr>
<td>Glasses, rocks</td>
<td>1</td>
</tr>
<tr>
<td>Ceramics (Alumina, SiC)</td>
<td>3 – 5</td>
</tr>
<tr>
<td>Nylon</td>
<td>3</td>
</tr>
<tr>
<td>Polyester</td>
<td>0.5</td>
</tr>
</tbody>
</table>

We now turn our attention to the more general situation when the loading is a combination of modes $I$ and $II$. Unlike the mode $I$ loading situation, where the direction of the crack growth is trivially determined, criteria on whether the crack will propagate but also on which direction it will do so
must be decided upon. This will be based on the circumferential tensile stress \(\sigma_{\theta\theta}\), which is obtained by rewriting the asymptotic expansion (15) into local polar coordinates [5, 6]. Doing so yields

\[
\sigma_{\theta\theta}(r, \theta) = \frac{K_{\theta\theta}(\theta)}{\sqrt{2\pi r}} + O(\sqrt{r}),
\]

where

\[
K_{\theta\theta}(\theta) = K_1 \cos^3\left(\frac{1}{2}\theta\right) - 3K_{II} \sin\left(\frac{1}{2}\theta\right) \cos^2\left(\frac{1}{2}\theta\right)
\]

is the effective stress intensity factor. It is postulated that crack growth will occur when

\[
\max_{\theta} K_{\theta\theta}(\theta) = K_c,
\]

which can be seen as a generalisation of (30). The direction of propagation is given by the angle \(\theta_p^{(\mathcal{K})}\) which maximizes \(K_{\theta\theta}(\theta)\),

\[
\theta_p^{(\mathcal{K})} = 2 \arctan \left( \frac{K_I - \sqrt{K_I^2 + 8K_{II}^2}}{4K_{II}} \right).
\]

This formula can be used to rewrite the condition for crack extension (33). Indeed, its left hand term is obtained by inserting the value of the angle given by (34) in the circumferential tensile stress expressed in (32). We thus obtain

\[
\frac{4\sqrt{2}K_{II}^3(K_I + 3\sqrt{K_I^2 + 8K_{II}^2})}{(K_I^2 + 12K_{II}^2 - K_I\sqrt{K_I^2 + 8K_{II}^2})^{3/2}} = K_c.
\]

### 3 Simulation of crack growth

In Section 2.4, a static fracture analysis was performed, where the goal was merely to compute the stress intensity factors. Now, with the fracture criteria introduced in Section 2.5, we are ready for a quasi-static analysis, which means we now look into the actual propagation of the crack. We observe that this is not a dynamic analysis in the sense that dynamic effects such as wave propagation are not taken into account.
Consider the linear elastic plate represented in Figure 7. Its dimensions are such that the ratio between its height $2h$ and its width $2b$ equals the unity. On the other hand, the ratio between the length of the crack measured along the $x$-axis and the width of the plate is such that $a/b = \sqrt{10}/25$. In order to consider a mixed mode situation, the initial crack was considered to be at a $\alpha = 45^\circ$ angle with the horizontal axis. Without loss of generality, let $b = 1$.

We assume that the uniform tensile stress $\sigma$, which was applied at the lower and upper horizontal boundaries of the plate, is large enough so that the fracture criterion (35) holds. In this way, we do know that the initial crack will propagate. The remaining question is: in which direction will it do so? Our answer is given in the last section by formula (34).

We are now ready to set up an algorithm for the prediction of the crack
path. The thing we have to do is to follow the position of the crack tip, as the propagation occurs. We use a step by step process. For that, we consider an increment $\Delta a$, which is the distance between two consecutive points of the crack tip. Once $\Delta a$ has been chosen, a possible algorithm will be as follows

**Given:** a step size $\Delta a$, the initial crack tip coordinates $x_{\text{tip}}^{(l)}$ for $l = 0$.

1- Increment $l$;
2- Compute the values of the stress intensity factors, $K_I$ and $K_{II}$;
3- Determine the angle of propagation $\theta^{(K)}_p$ using equation (34);
4- Update the crack tip coordinates

$$x_{\text{tip}}^{(l+1)} = x_{\text{tip}}^{(l)} + \Delta a (\cos(\theta^{(K)}_p), \sin(\theta^{(K)}_p)),$$

(36)
5- Back to step 1.

We use *Abaqus* and *Matlab* to implement this algorithm with $\Delta a = 0.025$. We thus obtain a set of consecutive crack tip coordinates, as displayed in Table 4. These constitute a discrete approximation for the crack path.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$x_{\text{tip}}$</th>
<th>$y_{\text{tip}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>1</td>
<td>0.13</td>
<td>0.095</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>0.090</td>
</tr>
<tr>
<td>3</td>
<td>0.18</td>
<td>0.088</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>0.086</td>
</tr>
<tr>
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<tr>
<td>14</td>
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<td>0.089</td>
</tr>
</tbody>
</table>

Like in Section 2.4, again we used a quadratic rectangular elements everywhere except near the crack tip, where quadratic triangular elements were used. Due to the crack tip singularity quarter point elements were used. We illustrate the discretisation elements of the crack tip region in Figure 8. The crack is represented in black.

The crack path can be seen in Figure 9. In a) we represent the initial crack, and in b) and c) the crack as it was determined after 7 and 15 steps, respectively.
Finally Figure 10 represents the deformed state of the plate for the initial crack, and again after 7 and 15 steps.

From these figures it seems that there is a tendency for the crack propagation
to occur mainly in mode I, during continued fracture. This agrees with the predictions of [6, 9].

4 Some extensions

4.1 Dynamic fracture

Criteria to determine whether crack growth will occur, as well as estimates for the direction in which it may do so, were presented in Section 2.5. Here we turn our attention to the case of a propagating crack. The velocity of the crack tip $c$ must then be considered, as it plays a fundamental role. We look into how it may be determined, at each time instant.

As can be seen in the work of Vroonhoven [22], dynamic stress intensity factors can be defined, which depend not only on the stresses $\sigma$ applied to the plate, its geometry and the crack length $a$, but also on the speed $c$ at which the crack tip propagates. These parameters may be written as

$$K_i^d = K_i(\sigma, a, c),$$  \hspace{1cm} (37)

for $i = I, II$. Here, we do not make the dependence on the geometry of the plate explicit, for the sake of clarity. It is shown in [7] that the dynamic stress intensity factors $K_i^d(\sigma, a, c)$ and the static stress intensity factors

$$K_i^s = K_i(\sigma, a, 0),$$  \hspace{1cm} (38)

as introduced in Section 2.3, are related by

$$K_i^d = k_i(c)K_i^s,$$  \hspace{1cm} (39)

where $k_i(c)$ depends in a non-linear way on the velocity. Approximations for the functions $k_i(c)$ for mode I and mode II can be found in [7, 22].

For mode I fracture, we recall (30), where we now interpret $K_I$ as the dynamical stress intensity factor

$$K_I^d = K_I(c).$$  \hspace{1cm} (40)

Our goal is to find an equation for the velocity. We then take the previous equation together with (39) to obtain the desired equation

$$k_{I}(c)K_I^s(\sigma, a, 0) = K_I(c).$$  \hspace{1cm} (41)
As for a mixed mode situation, we can then use the criterion (35), which, in terms of the static stress intensity factors $K_d^I$ and $K_d^{II}$ reads

$$\frac{4\sqrt{2}(K_d^{II})^3(K_d^I + 3\sqrt{(K_d^I)^2 + 8(K_d^{II})^2})}{((K_d^I)^2 + 12(K_d^{II})^2 - K_d^I\sqrt{(K_d^I)^2 + 8(K_d^{II})^2})^2} = K_{I_c}. \quad (42)$$

At each instant, the propagation angle is given by

$$\theta^{(K)}_p = 2\arctan \left( \frac{K_d^I - \sqrt{(K_d^I)^2 + 8(K_d^{II})^2}}{4K_d^{II}} \right). \quad (43)$$

### 4.2 Energy approach

Up to now, we have used a local approach to fracture, by focusing on the vicinity of the crack tip. This was not the case of the original approach of Griffith, who postulated that crack propagation will occur if the energy release rate during crack growth exceeds a critical level, given by the rate of increase in surface energy associated with the formation of new crack surfaces.

We define the energy release rate $G$ as the energy released per unit of crack area extension. In the specific case of thin plates, an alternative definition is possible, recurring to integration over the thickness and considering the unit length of the crack. In this way $G$ is seen to be the energy supplied per unit length along the crack edge, and used in creating the new fracture surface.

For a propagating crack it is convenient to consider the local coordinate system \( \{x(t), y(t)\} \) located at the crack tip of a thin plate of thickness $h$, such that the crack is a semi-infinite slit is positioned along the negative $x$ axis. This is the same situation as was depicted in Figure 2, but where now $x$, $y$, $r$ and $\theta$ are functions of time.

When a crack grows, the work that must be done to create the new crack surfaces equals the energy released. Using the corresponding polar coordinates \( \{r(t), \theta(t)\} \) such that $x = r \cos \theta$ and $y = r \sin \theta$, we define the energy release rate, as can be seen in [7], by

$$G = \lim_{\Delta a \to 0} \int_{-h/2}^{h/2} \int_0^{\Delta a} \sigma_{iy}(r, 0)[u_i(\Delta a - r, \pi) - u_i(\Delta a - r, -\pi)]drdz, \quad (44)$$

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where $\Delta a$ is the length of the crack extension and $i = x, y, z$.

An alternative method for the calculation of the energy release rate, based on path-independent contours, is the \textit{J-integral} that Rice [17] applied to crack problems. In the elastodynamic case the J-integral is defined in terms of the elastic and kinetic energy densities, respectively $W = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$ and $T = \frac{1}{2}p\dddot{u}_i\dddot{u}_i$, where $\dddot{u}_i$ is the time derivative of $u_i$.

\begin{equation}
J_k = \lim_{C \to 0} \int_{-h/2}^{h/2} \int_C ((W + T)n_k - \sigma_{ij}n_ju_{i,k})dsdz.
\end{equation}

Here $i, j = x, y, z$ and $u_{i,k}$ is the derivative of the $i$th component of the displacement vector with respect to the spatial coordinate $k = x, y$. The term $n_i$ represents the $i$th component of the outward normal. As for the domain $C$, it is a section of a circumference which starts and ends on the lower and upper crack surfaces respectively, and which is centered at the tip of the crack - Figure 11. In the limit $C \to 0$, as the radius of the circle tends to zero, $C$ reduces to the crack tip.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{integration_path.png}
\caption{Integration path.}
\end{figure}

Actually the designation of the \textit{J}-integral as a path-independent contour is only valid for the stationary case, in which we could replace $C$ by any other curve, with the same starting and ending points as $C$, and the result in (45) would be unaltered. In the dynamic case, only $J_x$ is independent on the shape of $C$.

Atluri [2] proved that

\begin{equation}
J_x = G,
\end{equation}

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thus providing us with another way to compute the energy release rate.

Formulas for the J-integral better suited for numerical computations can be found in [22].

We have thus presented a global approach to fracture, that differs from the local approach of Section 2. The energy release rate or the J integral here introduced can be used as fracture parameters and, like the stress intensity factors, be used to define fracture criteria - see for example [9, 22].
References


