MASTER

Short-time asymptotics of a Dynamical-Variational Transport cost on finite graphs
In pursuit of curvature

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Award date:
2020

EINDHOVEN UNIVERSITY OF TECHNOLOGY

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Short-time asymptotics of a Dynamical-Variational Transport cost on finite graphs

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by

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July, 2020

Abstract

In this thesis we define a sequence of rescaled action functionals, whose minimal values define a sequence of rescaled Dynamical-Variational Transport (DVT) costs on graphs, and show existence of a short-time limiting functional in the framework of Γ-convergence. Via an additional lower bound, we obtain convergence of the sequence of rescaled DVT-costs to the minimal value of the Γ-limit. Via the continuity equation we obtain an inequality of the 1-Wasserstein distance and the limiting DVT-cost.

Furthermore, we attempt to characterize minimizers of the DVT-cost on graphs consisting of two points via Euler–Lagrange equations. We show that strong solutions of some of these equations exist, and that the entropy is convex along these solutions.

This work can be considered a first step in the search for a characterization of lower bounds of Y. Ollivier's notion of curvature on graphs in terms of the heat flow on graphs as a contractive gradient flow.

Preface

This thesis is the final step in completing the Master's program Industrial and Applied Mathematics at Eindhoven University of Technology, and thereby my academic education.

First of all, I would like to thank my supervisors Oliver Tse and Jasper Hoeksema for guiding me through this project. I have really enjoyed our weekly discussions, which were always inspirational and helped me a great deal. In the later stages of writing, the quality of this work has improved tremendously because of their eye for detail. Also, I would like to thank Mark Peletier and Michiel Hochstenbach for being part of my assessment committee, and for the extremely detailed comments on this report.

Furthermore, I would like to thank my parents and family for their support and for the nice working environment at home, especially in this period ruled by COVID-19.

I would also like to thank my friends for supporting me and for providing the occasional distractions. Special thanks to Guus Bauwens for his help with some simulations, to Nikolaj d'Hooghe for commenting on my introduction from the perspective of a non-mathematician, and to Adri Dieleman for trying to understand a less than optimal proof of mine.

The template of this thesis was created by Joos Buijs and Thijs Nugteren.

Contents

Chapter 1

Introduction

In the digital era that we live in, data is everywhere. It is gathered in huge amounts in all kinds of businesses, which try to extract valuable information for their benefit using for instance 'Big Data' analysis. Furthermore, Machine Learning and Artificial Intelligence are ubiquitous and are becoming ever more important. This increasing importance of data in society demands a good understanding of the structure of data and the invention of methods for data analysis.

Many datasets can be cast in the form of graphs. This object consists of a set of vertices and a set of connections between vertices, called edges, which are endowed with weights. We call this object a weighted graph. If all weights are the same, then we call the graph unweighted. A typical example of a dataset that can be given the structure of a graph, is a set of vertices consisting of cities, a set of edges consisting of roads between these cities, endowed with weights that are the lengths of these roads.

A notion that encodes much information about the structure of a graph, is that of Ollivier-Ricci curvature, introduced in by Yann Ollivier in [11]. For example, Ollivier shows that a global positive lower bound of the Ollivier-curvature implies an upper bound of the diameter of the graph. Futhermore, it is shown by Jost and Liu [8] that a region with high curvature implies that this region is highly interconnected. Because of this property, Ollivier-curvature is interesting for the field of network analysis, which has many applications in all fields of science.

The purpose of this thesis

This thesis is concerned with getting closer to a gradient flow characterization of Ollivier-curvature. This effort consists of two main parts. Firstly, we establish a connection between the 1-Wasserstein distance W^1, a central object of Ollivier-curvature, and a Dynamical-Variational Transport (DVT) cost, a central object in the gradient flow framework of Peletier et al [12]. This connection arises through the convergence of a sequence of rescaled action functionals in the framework of Γ-convergence under a certain topology on the domain of the functionals. Via this Γ-limit we establish a connection between the rescaled DVT-cost and W^1 in the form of an inequality.

Secondly, we attempt to characterize the minimizers of the DVT-cost between Dirac measures on graphs consisting of two vertices via its associated Euler-Lagrange equation. Although, we do not succeed, we do show that the candidate minimizers enjoy nice properties, such as symmetry and convexity of the entropy in particular cases.

These two matters can be considered a first step in the direction of a gradient flow formulation of Ollivier's notion of curvature.

From Ricci curvature on Riemannian manifolds to Ollivier curvature on graphs
We give an intuitive introduction of Ollivier curvature on graphs in analogy with the celebrated notion Ricci curvature for Riemannian manifolds and state in more detail some things we have already mentioned.

According to the notion of Ricci curvature, a ball is positively curved, Euclidean space itself has curvature zero - we call this a flat manifold - and a satellite dish is negatively curved. The Ricci curvature, although it is a local notion, provides a lot of (global) information about the geometric object. For example, the Bonnet–Meyers Theorem gives an upper bound on the diameter of the object if the curvature is bounded from below by a positive constant.

In Riemannian geometry one studies differentiable manifolds M ('smooth' objects that locally look like Euclidean space), equipped with a Riemannian metric $(g_p)_{p\in M}$ – i.e., g_p is an inner product on the tangent space at the point $p \in M$ – and a volume measure. This enables us to define a notion of the length of a path between two points. Consequently, we can define on M a metric space structure, where the distance d between two points on M is defined by the length of the shortest path between these points. We call this the geodesic metric.

The Ricci curvature of M manifests itself in the following way. Let $p, q \in M$ and consider two geodesic balls $B(p)$ and $B(q)$ with the same radius. Here $B(q)$ is obtained by transporting $B(p)$ along geodesics. Thus every $p' \in B(p)$ has a unique counterpart $q' \in B(q)$. If M is positively curved, then the average of the distances $d(p', q')$ over all pairs (p', q') is smaller than $d(p, q)$. If M is negatively curved, we replace 'smaller' by 'larger' in the previous sentence. If M is flat, then the average distance is exactly the same as $d(p, q)$. Accordingly, if one puts a load of mass 1 on $B(p)$ and one transports it to $B(q)$, then, relative to the flat situation, the effort this takes is lower if M is positively curved and higher if it is negatively curved. More precisely, the Ricci curvature has the same sign as the expression

$$1 - \frac{\text{Average distance between } B(p) \text{ and } B(q)}{d(p,q)}. \tag{1.1}$$

The notion of Ollivier-Ricci curvature is based on exactly this expression, and replaces the average distance by a transport distance of measures. For simplicity, let (V, E) be a connected, unweighted, undirected graph, where V denoted the set of vertices and E the set of edges. An edge between two vertices $x, y \in V$ is denoted by $xy \in E$. We equip the graph with the combinatorial graph distance d_G. That is, the distance between $x, y \in V$ is equal to the smallest number of edges in a path from x to y. Thus, $((V, E), d_G)$ is a metric space.

Let $xy \in E$ and $\varepsilon \in [0,1]$. We replace the average distance in (1.1) by the 1-Wasserstein distance $W^1(m_\varepsilon^x, m_\varepsilon^y)$ between two probability measures m_ε^x and m_ε^y with $\operatorname{supp} m_\varepsilon^x \subset B_{d_G}(x;1)$, $\operatorname{supp} m_\varepsilon^y \subset B_{d_G}(y;1)$, the unit balls with respect to d_G centered at x and y, respectively. Lin et al. [9] modify Ollivier's notion of curvature and introduce notion of ε-Ollivier curvature $K_\varepsilon(x,y)$ for any $x,y \in V$ is defined as

$$K_\varepsilon(x,y) := 1 - \frac{W^1(m_\varepsilon^x, m_\varepsilon^y)}{d_G(x,y)},$$

where m_ε^z for any $z \in V$ is defined by

$$m_\varepsilon^z(v) := \begin{cases} \varepsilon & \text{if } v = z\,, \\ (1-\varepsilon)/\deg(z) & \text{if } vz \in E\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(z)$ is the number of neighbours of z. Furthermore, the modified Ollivier-curvature K is defined by

$$K(x,y) := \lim_{\varepsilon \uparrow 1} \frac{K_\varepsilon(x,y)}{1-\varepsilon},$$

for every $x,y \in V$.

As mentioned before, analogous with the Bonnet–Meyers Theorem in Riemannian geometry, it is shown in [11] that a global positive lower bound of the Ollivier-curvature implies Bonnet–Meyers type theorems.

Furthermore, for neighbours $x,y \in V$, it is shown by Jost and Liu [8] that $K_0(x,y)\max\{\deg(x), \deg(y)\}$ is a lower bound of the number of neighbours that x and y have in common. Hence, a region of high curvature implies a high degree of interconnectivity.

Curvature and gradient flows

We motivate why to expect a connection between Ollivier-curvature and gradient flows in the first place, as well as the aforementioned two main parts of the thesis, again using the analogy with the Riemannian geometry.

Von Renesse and Sturm [15] have shown that the Ricci curvature of M being bounded from below by $k \in \mathbb{R}$ is equivalent to the heat flow on M being a k-contractive gradient flow in the metric space $(\mathcal{P}^2(M), W^2)$ with respect to the entropy relative to the volume measure on M. The space $(\mathcal{P}^2(M), W^2)$ is the space of probability measures with finite second moment equipped with the 2-Wasserstein distance.

This notion of gradient flow is a generalization to the setting of metric spaces of the notion gradient flows on $\mathbb{R}^n$. In latter case a gradient flow with respect to a functional $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a solution $u : [0, t_+) \longrightarrow \mathbb{R}^n$ of the Initial Value Problem given by

$$\dot{u}(t) = -\nabla\phi(u(t))\,, \qquad u(0) = u_0\,.$$

This equivalence of lower curvature bounds in the Riemannian setting gives rise to the expectation that a lower bound on the Ollivier-curvature can be characterized via gradient flows. However, the discrete nature of V prevents using arguments of the 'connected' Riemannian setting.

In [12] Peletier et al. introduce a framework for gradient flows that is suitable for discrete spaces and which admits a gradient flow formulation of the heat flow on graphs. A central object in this framework is the Dynamical-Variational Transport (DVT) cost. Such a cost between two measures μ and ν minimizes an action functional (Variational) over all solutions of a boundary value problem with boundary conditions μ and ν (Dynamical). This is precisely the framework of gradient flows that we expect to be suitable for establishing a gradient flow characterization of Ollivier curvature.

To further motivate the use of this framework, in [10] Léonard shows that rescaled entropy functionals converge in the framework of Γ-convergence to the Monge–Kantorovich optimal transport functional, of which the minimal value is defined as W^1. Since the action functionals, which are used to define the DVT-cost, allow for functionals similar to the entropy functionals, we expect that these rescaled action functionals also converges to the Monge–Kantorovich functional, connecting the DVT-cost and W^1, and thereby getting closer to a gradient flow characterization of Ollivier curvature. This motivates the first part of the thesis.

Von Renesse and Sturm have also shown that the Ricci curvature of M being bounded from below by $k \in \mathbb{R}$ is equivalent to the entropy being k-convex along geodesics. The minimizers of the DVT-cost play a similar role in the framework of Peletier et al. as geodesics in the Riemannian case. Therefore, we study the minimizers of the DVT-cost and study their entropy in particular. This motivates the second part of the thesis.

Outline

In Chapter 2 we rigorously define the DVT-cost in setting that is less general than in [12], which we introduce in Section 2.1. Then, we introduce the different ingredients for the definition of the DVT-cost: in Section 2.2 we define dissipation potential and in Section 2.3 we discuss the continuity equation and the concept of solutions to this equation, and show a multitude of constructions to obtain new solutions from given ones. These concepts are combined in the definition of the DVT-cost in Section 2.4.

In Chapter 3 we show convergence of the aforementioned rescaled action functionals under a certain topology on their domain, in the framework of Γ-convergence. The proof consist of two major parts, the *liminf* and *limsup inequality*, which we prove in separate Sections 3.3 and 3.4, respectively. Along the way we derive a lower bound of the rescaled functionals, and combining this with the Γ-convergence, we prove convergence of minimal values of these functionals in Section 3.5. Via the Γ-limit we establish a connection between the rescaled DVT-cost and W^1 in the form of an inequality in Section 3.6.

Chapter 4 contains an attempt to characterize the minimizers of the DVT-cost between two Dirac measures on graphs consisting of two vertices. Although we do not succeed in giving a complete characterization, we do obtain candidate minimizers via the Euler–

Lagrange equations.

In Section 4.1 we show the equivalence of the minimization problem for the DVT-cost and an alternative minimization problem. Subsequently, we derive the Euler–Lagrange equation of this alternative problem. In Section 4.2 we consider initial value problems (IVPs) associated to the Euler–Lagrange equation. We derive several properties of the IVP solutions. In particular, solutions of certain IVPs are solutions of the Euler–Lagrange equation. In Section 4.3 we show that the entropy along certain solutions of the IVPs is convex.

The appendices A, B and C treat topics in analysis, measure theory and the framework of Γ-convergence, respectively, which are not familiar to most master-level students.

We assume that the reader is familiar with measure theory on the level of the course *Measure and Integration*, topology as taught in *Introduction to Topology* and ODE theory as in *Differential equations*, all taught in the Bachelor's program at Utrecht University.

Chapter 2

The DVT-cost

In this section we define the Dynamical-Variational Transport (DVT) cost $\mathcal{W}^\tau(\mu, \nu)$ between two finite measures μ and ν on a space V equipped with a stationary measure π and a reversible jump process $(\kappa(x, \cdot))_{x \in V}$ as in [12]. In the case that V is a weighted, undirected, finite graph, we could for example have that κ is the 'matrix' of jump probabilities induced by the weights on the edges, and π is the stationary distribution of the associated Markov chain.

In Section 2.4, we introduce the DVT-cost, which is of the form

$$\mathcal{W}^\tau(\mu, \nu) = \inf \left\{ \int_0^\tau \mathcal{R}(\rho_t, j_t) dt : \rho_0 = \mu,\ \rho_\tau = \nu \right\} .$$

Here,

(i) the infimum is taken over all pairs of measure-valued curves $t \mapsto (\rho_t, j_t)$ with fixed boundary values $\rho_0 = \mu$, $\rho_\tau = \nu$, satisfying the condition that $(\rho_t)_{t \in [0,\tau]}$ is a solution of the continuity equation with flux $(j_t)_{t \in [0,\tau]}$. We introduce the continuity equation and the concept of solutions in Section 2.3;

(ii) $\mathcal{R}$ is called the dissipation potential, which is introduced in Section 2.2. For its definition, we need a dissipation density Ψ and a flux density α. The dissipation potential takes into account the jump process κ. In the case that V is a graph, this means that $\mathcal{R}$ takes the value $+\infty$ if there is flux along a pair $(x, y) \in V \times V$ with $\kappa(x, y) = 0$.

We can interpret the DVT-cost between two measures as follows. In physics, when we integrate a potential along the trajectory of a particle, we obtain the energy it takes to have the particle follow this trajectory from its starting point to its end point. Analogously, the DVT-cost measures the minimum amount of energy of all 'trajectories' with starting point μ and end point ν. Here, we demand that the 'trajectories' behave in a certain way, namely they satisfy the continuity equation.

Before we can introduce the dissipation potential $\mathcal{R}$ and the continuity equation, we specify the assumptions on (V, π, κ) and (Ψ, α) in the next section.

2.1 The setting

We begin by stating the assumptions on (V, π, κ). We denote by $\mathcal{E} := V \times V$ the 'edge' space and by $\mathcal{M}^+(V)$ and $\mathcal{M}(\mathcal{E})$ the set of non-negative finite Borel measures on V, and the set of Borel measures of finite total variation on $\mathcal{E}$, respectively. For the definition of the narrow topology, see Definition A.13.

Assumptions (V, π, κ)**.** The sets of 'vertices' (V, π) is a locally compact, metrizable, separable measure space with a non-negative finite Radon measure $\pi \in \mathcal{M}^+(V)$.

The kernel κ is given by a collection $(\kappa(x, \cdot))_{x \in V}$ of Radon measures in $\mathcal{M}^+(V)$, depending measurably on x. That is, the mapping $x \mapsto \kappa(x, A)$ is measurable for any Borel set $A \subset V$. We also assume that

$$c_\kappa := \sup_{x \in V} \kappa(x, V) < +\infty \,,$$

and that κ satisfies the Feller property in the narrow topology. That is, the map

$$x \mapsto \int_{y \in V} \varphi(y) \kappa(x, dy)$$

is continuous for every $\varphi \in C_b(V)$.

We define the measure $\vartheta_\pi \in \mathcal{M}(\mathcal{E})$ by

$$\vartheta_\pi(A \times B) := \int_A \kappa(x, B) \pi(dx) \,.$$

Indeed, by Lemma B.5 this defines a measure. We also write this measure as $\vartheta_\pi(dx, dy) = \kappa(x, dy)\pi(dy)$. In the case that V is a finite set, we say that $x, y \in V$ are neighbours if $\vartheta_\pi(x, y) \neq 0$. An important subset of $\mathcal{E}$ is $\mathcal{E}_N := \{(x, y) \in \mathcal{E} : \vartheta_\pi(x, y) \neq 0\}$, the set of edges between neighbouring vertices.

We denote by $\mathsf{s} : \mathcal{E} \longrightarrow \mathcal{E}$ the symmetry map defined by $\mathsf{s}(x, y) := (y, x)$, and by $\mathsf{s}_* : \mathcal{M}(\mathcal{E}) \longrightarrow \mathcal{M}(\mathcal{E})$ the push-forward mapping, see Definition B.2. Another assumption is the following reversibility condition.

Detailed Balance Condition. We assume that $\vartheta_\pi \in \mathcal{M}(\mathcal{E})$ satisfies

$$\vartheta_\pi(A \times B) = \int_A \kappa(x, B)\pi(dx) = \int_B \kappa(y, A)\pi(dy) = \vartheta_\pi(B \times A) \,,$$

or equivalently that $\mathsf{s}_* \vartheta_\pi = \vartheta_\pi$.

The assumptions on the *dissipation density* Ψ and the *flux density* α are the following.

Assumptions Ψ and α. The dissipation density $\Psi : \mathbb{R} \longrightarrow [0,\infty)$ is even and satisfies

$$\Psi(0) = 0\,, \text{ and } 0 < \Psi(\xi) < +\infty \text{ for all } \xi \in \mathbb{R} \setminus \{0\}\,,$$
$$\Psi \text{ is strictly convex, strictly increasing on } (0,\infty), \text{ and superlinear}\,,$$

cf. [12, Lemma 3.1].

The flux density $\alpha : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is concave and satisfies the following properties:

(i) $\alpha(u,v) = 0$ if and only if $u = 0$ or $v = 0$;

(ii) α is positively 1-homogeneous, i.e.,

$$\alpha(\lambda u, \lambda v) = \lambda \alpha(u,v)\,, \qquad \text{for all } \lambda, u, v \in [0,\infty)\,;$$

(iii) α is symmetric, i.e.,

$$\alpha(u,v) = \alpha(u,v)\,, \qquad \text{for all } u, v \in [0,\infty)\,.$$

Remark. Here, by superlinearity we mean that

$$\lim_{\xi \to \pm\infty} \frac{\Psi(\xi)}{|\xi|} = +\infty\,.$$

Remark. The flux density α is automatically upper semicontinuous. Namely, due to concavity, α is continuous on $(0,\infty) \times (0,\infty)$, and thus upper semicontinuous. On the boundary of its domain, α assumes its minimal value. Hence, α is upper semicontinuous on the boundary.

2.2 The dissipation potential

In this section we give a rigorous definition of the dissipation potential $\mathcal{R}$ according to [12, Section 4.2].

We first consider the lower semicontinuous extension of the mapping

$$(0,\infty) \times (0,\infty) \times \mathbb{R} \ni (u_1, u_2, w) \mapsto \Psi\left(\frac{w}{\alpha(u_1,u_2)}\right) \alpha(u_1,u_2)\,.$$

We denote the Legendre dual of Ψ by $\Psi^* : \mathbb{R} \longrightarrow \mathbb{R}$, which is defined by $\Psi^*(w) = \sup_{\xi \in \mathbb{R}}\{w\xi - \Psi(\xi)\}$.

Definition 2.2.1. We define the *action density function* $\Upsilon : [0,\infty) \times [0,\infty) \times \mathbb{R} \longrightarrow [0,+\infty]$ by

$$\Upsilon(u,v,w) := \sup_{\xi \in \mathbb{R}}\{w\xi - \alpha(u,v)\Psi^*(\xi)\} = \begin{cases} \Psi\left(\frac{w}{\alpha(u,v)}\right)\alpha(u,v) & \text{if } uv > 0\,, \\ 0 & \text{if } w = 0 \text{ and } uv = 0\,, \\ +\infty & \text{if } w \neq 0 \text{ and } uv = 0\,. \end{cases}$$

Remark. The second inequality follows straightforwardly by working out the supremum for the three separate cases of the right hand side.

Lemma 2.2.2 ([12, Lemma 4.7])**.** *The action density function Υ is convex, positively 1-homogeneous and lower semicontinuous.*

Proof. We define for $\xi \in \mathbb{R}$ the map $f_\xi : [0,\infty) \times [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\xi(u,v,w) := w\xi - \alpha(u,v)\Psi^*(\xi)$, where Ψ^* is the Legendre dual of Ψ. Then by definition of Υ we have

$$\Upsilon(u,v,w) = \sup_{\xi \in \mathbb{R}} f_\xi(u,v,w)\,.$$

Remark that f_ξ is convex, because $(u,v,w) \mapsto w$ is linear and $-\alpha$ is convex. Therefore, the pointwise supremum Υ is convex.

For positive 1-homogeneity, let $\lambda \geq 0$. Then by positive 1-homogeneity of α, it follows that f_ξ is positively 1-homogeneous. Therefore, the pointwise supremum Υ is again positively 1-homogeneous.

For lower semicontinuity, remark that by Lemma A.4 $-\alpha$ is lower semicontinuous, because α is upper semicontinuous. By Lemma A.5 the sum f_ξ is lower semicontinuous. Then by Lemma A.6 the pointwise supremum Υ is lower semicontinuous. This concludes the proof.

□

For the definition of weak*-topology, see Appendix A.1. For measures $(\mu_n)_{n\in\mathbb{N}}$ and μ_0 we denote the situation that (μ_n) converges to μ_0 w.r.t. the weak*-topology as $n \to \infty$ by

$$\mu_n \overset{*}{\rightharpoonup} \mu_0\,, \qquad \text{as } n \to \infty\,.$$

Lemma 2.2.3. *The functional $\mathcal{I} : \mathcal{M}(\mathcal{E};[0,\infty)^2 \times \mathbb{R}) \longrightarrow [0,+\infty]$ defined by*

$$\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3) \mapsto \iint_{\mathcal{E}} \Upsilon\left(\frac{d\sigma_1}{d|\boldsymbol{\sigma}|}, \frac{d\sigma_2}{d|\boldsymbol{\sigma}|}, \frac{d\sigma_3}{d|\boldsymbol{\sigma}|}\right) d|\boldsymbol{\sigma}|\,,$$

is positively 1-homogeneous, convex and sequentially lower semicontinuous with respect to the weak-topology on $\mathcal{M}(\mathcal{E};[0,\infty)^2 \times \mathbb{R})$.*

Proof. The statement follows by Lemma B.8. □

Let $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3) \in \mathcal{M}(\mathcal{E};\mathbb{R}^3)$. With a slight abuse of notation, we will write

$$\frac{d\boldsymbol{\sigma}}{d|\boldsymbol{\sigma}|} = \left(\frac{d\sigma_1}{d|\boldsymbol{\sigma}|}, \frac{d\sigma_2}{d|\boldsymbol{\sigma}|}, \frac{d\sigma_3}{d|\boldsymbol{\sigma}|}\right)\,.$$

Now we have all the ingredients to define the dissipation potential.

Definition 2.2.4 ([12, Definition 4.9]). We define the dissipation potential $\mathcal{R} : \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E}) \longrightarrow [0, +\infty]$ by

$$\mathcal{R}(\rho, j) := \iint_{\mathcal{E}} \Upsilon\left(\frac{d\boldsymbol{\sigma}_{\rho,j}}{d|\boldsymbol{\sigma}_{\rho,j}|}\right) d|\boldsymbol{\sigma}_{\rho,j}|\,,$$

where $\boldsymbol{\sigma}_{\rho,j} = (\vartheta_\rho^-, \vartheta_\rho^+, j)$ with $\vartheta_\rho^-(dx, dy) = \kappa(x, dy)\rho(dx)$ and $\vartheta_\rho^+ = \mathsf{s}_*\vartheta_\rho^-$.

Remark. By definition $\mathcal{R}$ does not depend on the stationary measure π.

The dissipation potential enjoys the following properties.

Lemma 2.2.5 ([12, Lemma 4.10]). *The dissipation potential $\mathcal{R}$ is convex and sequentially lower semicontinuous with respect to the weak*-topology on $\mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$.*

Proof. By the linearity of the map $(\rho, j) \mapsto \boldsymbol{\sigma}_{\rho,j}$ combined with convexity of $\mathcal{I}$, by Lemma 2.2.3, it follows that the mapping $(\rho, j) \mapsto \mathcal{I}(\boldsymbol{\sigma}_{\rho,j}) = \mathcal{R}(\rho, j)$ is convex.

For the sequential lower semicontinuity, we claim that $(\rho, j) \mapsto \boldsymbol{\sigma}_{\rho,j} = (\vartheta_\rho^-, \vartheta_\rho^+, j)$ is weakly*-continuous on $\mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$. Namely, let $(\rho_i, j_i)_{i\in I}$ be a net in $\mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ that converges to $(\rho, j) \in \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$. For any $(\varphi^-, \varphi^+, \psi) \in C_0(\mathcal{E}; \mathbb{R}^3)$ we show that the net $(\vartheta_{\rho_i}^-(\varphi^+), \vartheta_{\rho_i}^+(\varphi^-), j_i(\psi))_{i\in I}$ in $\mathbb{R}^3$ converges to $(\vartheta_\rho^-(\varphi^+), \vartheta_\rho^+(\varphi^-), j(\psi))$. Note for the last component that $j_i(\psi) \to j(\psi)$ by assumption. For the first component, we see that

$$\vartheta_{\rho_i}^-(\varphi^-) = \int_{x\in V} \left(\int_{y\in V} \varphi(x, y)\kappa(x, dy)\right) \rho_i(dx)\,. \tag{2.1}$$

We claim that the map $x \mapsto \int_{y\in V} \varphi^-(x, y)\kappa(x, dy) \in C_0(V)$. Namely, let $\varepsilon > 0$ and $x_0 \in V$. We see that

$$\begin{aligned}\left|\int_{y\in V} \varphi^-(x, y)\kappa(x, dy) - \int_{y\in V} \varphi^-(x_0, y)\kappa(x_0, dy)\right| &\leq \int_{y\in V} |\varphi^-(x, y) - \varphi^-(x_0, y)|\kappa(x, dy) \\ &\quad + \left|\int_{y\in V} \varphi^-(x_0, y)\,[\kappa(x, dy) - \kappa(x_0, dy)]\right|.\end{aligned}$$

Since $\varphi \in C_0(\mathcal{E})$, there exist compact sets $K \subset V$ such that $|\varphi^-(x, y)| < \varepsilon/(6c_\kappa)$ for all $(x, y) \in \mathcal{E} \setminus (K \times K)$.

Recall that V is metrizable and denote by d_V a metric that induces the topology on V. Since φ^- is continuous, it follows by [14, Theorem 4.19] that φ^- is uniformly continuous with respect to d_V on $K \times K$. So there exists $\delta_1 > 0$ such that $|\varphi(x, y) - \varphi(x', y')| < \varepsilon/(6c_\kappa)$ for all $d_{V\times V}((x, y), (x', y')) := d_V(x, x') + d_V(y, y') < \delta_1$. It follows for all $x \in V$ with

$d_V(x, x_0) < \delta_1$ that

$$\begin{aligned}\int_{y\in V} |\varphi^-(x,y) - \varphi^-(x_0,y)|\kappa(x,dy) &= \int_{y\in V\setminus K} |\varphi^-(x,y) - \varphi^-(x_0,y)|\kappa(x,dy)\\ &\quad + \int_{y\in K} |\varphi^-(x,y) - \varphi^-(x_0,y)|\kappa(x,dy)\\ &\le 2\frac{\varepsilon}{6c_\kappa}c_\kappa + \frac{\varepsilon}{6c_\kappa}c_\kappa\\ &= \frac{\varepsilon}{2}\,.\end{aligned}$$

Furthermore, since $y \mapsto \varphi^-(x_0, y)$ is an element of $C_0(V)$, it follows by the Feller property that there exists $\delta_2 > 0$ such that

$$\left|\int_{y\in V} \varphi^-(x_0,y)\,[\kappa(x,dy) - \kappa(x_0,dy)]\right| < \frac{\varepsilon}{2}\,,$$

for all $x \in V$ with $d_V(x, x_0) < \delta_2$. Hence, it holds for all $x \in V$ with $d_V(x, x_0) < \min\{\delta_1, \delta_2\}$ that

$$\left|\int_{y\in V} \varphi^-(x,y)\kappa(x,dy) - \int_{y\in V} \varphi^-(x_0,y)\kappa(x_0,dy)\right| < \varepsilon\,,$$

so $x \mapsto \int_{y\in V} \varphi^-(x,y)\kappa(x,dy)$ is continuous.

Moreover, for $x \in V \setminus K$ we have that $|\varphi^-(x,y)| < \varepsilon/c_\kappa$ for every $y \in V$, and

$$\left|\int_{y\in V} \varphi^-(x,y)\kappa(x,dy)\right| \le \int_{y\in V} |\varphi^-(x,y)|\kappa(x,dy) < \varepsilon\,,$$

so $x \mapsto \int_{y\in V} \varphi^-(x,y)\kappa(x,dy) \in C_0(V)$, proving the claim.

With this and the assumption that $\rho_i \overset{*}{\rightharpoonup} \rho$, it follows from (2.1) that $\vartheta^-_{\rho_i}(\varphi^-) \to \vartheta^-_\rho(\varphi^-)$. Similarly, it follows that $\vartheta^+_{\rho_i}(\varphi^+) \to \vartheta^+_\rho(\varphi^+)$. Hence, the map $(\rho, j) \mapsto \boldsymbol{\sigma}_{\rho,j}$ is weakly*-continuous. With the sequential weak*-lower semicontinuity of $\mathcal{I}$, Lemma 2.2.3, and Lemma A.7, it follows that $(\rho, j) \mapsto \mathcal{I}(\boldsymbol{\sigma}_{\rho,j}) = \mathcal{R}(\rho, j)$ is sequentially weakly*-lower semicontinuous. This concludes the proof. □

If the potential of a pair (ρ, j) with $\rho \ll \pi$ is finite, then there is an alternative expression for $\mathcal{R}(\rho, j)$.

Lemma 2.2.6 ([12, Lemma 4.10]). *Let $(\rho, j) \in \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ such that $\mathcal{R}(\rho, j) < +\infty$ and $\rho \ll \pi$. Then $j \ll \vartheta_\pi$ and, setting $u := d\rho/d\pi$, $w := dj/d\vartheta_\pi$ and*

$$\mathcal{E}_u := \{(x,y) \in \mathcal{E} : u(x) > 0,\ u(y) > 0\}\,,$$

it holds that $w = 0$ for ϑ_π-almost all $(x,y) \in \mathcal{E} \setminus \mathcal{E}_u$, and

$$\mathcal{R}(\rho, j) = \iint_{\mathcal{E}_u} \Psi\left(\frac{w(x,y)}{\alpha(u(x),u(y))}\right) \alpha(u(x),u(y))\vartheta_\pi(dx,dy)\,.$$

2.3 The continuity equation

In this section we start by introducing the continuity equation in the setting of measure-valued mappings and define the notion of solutions of the continuity equation. Then we discuss several ways of constructing new solutions from given ones. Many of these results also contain statements about finiteness of the $\mathcal{R}$-action of the new solution. We will use these constructions frequently in Section 3.4. Furthermore, we show that solutions of the continuity equation satisfy the principle of conservation of mass.

In the well-known Euclidean setting the continuity equation is given by

$$\partial_t \rho + \operatorname{div} j = 0\,, \tag{2.2}$$

with $\rho : (0,\tau) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and flux $j : (0,\tau) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Therefore, to formulate a discrete analogue (of its weak formulation) we need discrete notions of both the gradient and the divergence.

Definition 2.3.1 (Discrete gradient and divergence)**.** The discrete gradient $\overline{\nabla} : C_0(V) \longrightarrow C_0(\mathcal{E})$ and divergence $\overline{\operatorname{div}} : \mathcal{M}(\mathcal{E}) \longrightarrow \mathcal{M}(V)$ are defined by

$$\begin{aligned} \overline{\nabla}\varphi(x,y) &:= \varphi(y) - \varphi(x)\,, \\ \overline{\operatorname{div}} j(dx) &:= \int_{y\in V} \left[j(dx,dy) - j(dy,dx) \right]\,. \end{aligned}$$

Remark. Via the canonical embeddings $\mathcal{M}(V) \hookrightarrow (C_0(V))^*$ and $\mathcal{M}(\mathcal{E}) \hookrightarrow (C_0(\mathcal{E}))^*$, it holds that $\overline{\operatorname{div}} = -\overline{\nabla}^*|_{\mathcal{M}(\mathcal{E})}$. Here $\overline{\nabla}^* : (C_0(\mathcal{E}))^* \longrightarrow (C_0(V))^*$ is the dual of $\overline{\nabla}$ given by $\overline{\nabla}^* f^* = f^* \circ \overline{\nabla}$ for any $f^* \in (C_0(\mathcal{E}))^*$. Namely, it follows for $\varphi \in C_0(V)$ and $j \in \mathcal{M}(\mathcal{E})$ that

$$\begin{aligned} (\overline{\nabla}^* j)(\varphi) &= j(\overline{\nabla}\varphi) \\ &= \iint_{\mathcal{E}} \overline{\nabla}\varphi(x,y) j(dx,dy) \\ &= \int_{y\in V} \varphi(y) \int_{x\in V} j(dx,dy) - \int_{x\in V} \varphi(x) \int_{y\in V} j(dx,dy) \\ &= \int_{x\in V} \varphi(x) \left[\int_{y\in V} j(dy,dx) - j(dx,dy) \right] \\ &= -(\overline{\operatorname{div}} j)(\varphi)\,. \end{aligned}$$

We consider the formal discrete analogue of (2.2) given by

$$\partial_t \rho_t + \overline{\operatorname{div}} j_t = 0\,,$$

where $\rho : [0,\tau] \longrightarrow \mathcal{M}^+(V)$ and $j : [0,\tau] \longrightarrow \mathcal{M}(\mathcal{E})$. Passing to a weak formulation of this formal equation inspires the following definition of solutions.

Definition 2.3.2 (Solutions of the continuity equation)**.** Let $a, b \in \mathbb{R}$ with $a < b$. We say that a curve $(\rho, j) : [a, b] \longrightarrow \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ is an element of $\mathscr{CE}([a, b])$ if and only if:

(1) the curve $t \mapsto \rho_t \in \mathcal{M}^+(V)$ is weakly*-continuous;

(2) the curve $t \mapsto j_t \in \mathcal{M}(\mathcal{E})$ is a Borel family and $\int_a^b |j_t|(\mathcal{E})dt < +\infty$;

(3) for any $\varphi \in C_c([a, b] \times V)$ with $\partial_t \varphi \in C_c([a, b] \times V)$, it holds that

$$\int_V [\varphi(b, x)\rho_b(dx) - \varphi(a, x)\rho_a(dx)] = \int_a^b \int_V \partial_t \varphi(t, x)\rho_t(dx) + \int_a^b \iint_{\mathcal{E}} \overline{\nabla}\varphi(t, x, y) j_t(dx, dy)\, dt\,.$$

Furthermore, for $\mu, \nu \in \mathcal{M}^+(V)$, we define $\mathscr{CE}([a, b]; \mu, \nu) := \{(\rho, j) \in \mathscr{CE}([a, b]) \mid \rho_a = \mu\,, \rho_b = \nu\}$. In addition, for $\tau > 0$, we set $\mathscr{CE}_\tau := \mathscr{CE}([0, \tau])$ and $\mathscr{CE}_\tau(\mu \to \nu) := \mathscr{CE}([0, \tau]; \mu, \nu)$.

Remark. Alternatively, one could define $(\rho, j) : [a, b] \longrightarrow \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ to be a solution if and only if it satisfies conditions (2) and (3) of Definition 2.3.2. By [12, Lemma 4.4] there exists weakly*-continuous curve $[0, \tau] \ni t \mapsto \widetilde{\rho}_t \in \mathcal{M}^+(V)$ such that $\rho_t = \widetilde{\rho}_t$ for almost all $t \in [0, \tau]$ and $(\widetilde{\rho}, j)$ satisfies (2) and (3). So (ρ, j) and $(\widetilde{\rho}, j)$ have the same $\mathcal{R}$-action and therefore both definitions will result in the same notion of DVT-cost, which we will define in the next section.

Any locally compact space admits an *exhaustion by compact sets*. For V we fix the exhaustion $\{K_n\}_{n\mathbb{N}}$. That is, the set K_n is compact for each n and $K_m \subset K_n^\circ$ for all $m < n$. Furthermore, $\{K_n^\circ\}_{n\in\mathbb{N}}$ is an open cover of V, i.e., $V = \bigcup_{n\in\mathbb{N}} K_n^\circ$.

We show that the restriction of a solution to a subinterval of its domain is again a solution.

Lemma 2.3.3 (Restriction of solution)**.** *Let $\tau > 0$ and $(\rho, j) \in \mathscr{CE}([0, \tau])$. Then for any $t_1, t_2 \in [0, \tau]$ with $t_1 < t_2$ it holds that the restriction of (ρ, j) to $[t_1, t_2]$ is an element of $\mathscr{CE}([t_1, t_2])$.*

Proof. Remark that the restriction $\rho|_{[t_1,t_2]}$ is weakly*-continuous, and that $j|_{[t_1,t_2]}$ is a Borel family with

$$\int_{t_1}^{t_2} |j_t|(\mathcal{E})dt \leq \int_0^\tau |j_t|(\mathcal{E})dt\,.$$

Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family in $C_c^\infty(t_1, t_2)$ such that

$$0 \leq \psi_\varepsilon \leq 1\,, \qquad \psi_\varepsilon(t) \to \mathbb{1}_{(t_1,t_2)}(t)\,, \qquad \psi'_\varepsilon \overset{*}{\rightharpoonup} \delta_{t_1} - \delta_{t_2}\,,$$

for all $t \in (t_1, t_2)$ as $\varepsilon \downarrow 0$. Let $\varphi \in C_c([0,\tau]\times V)$ with $\partial_t\varphi \in C_c([0,\tau]\times V)$. Define $\varphi_\varepsilon : [0,\tau]\times V \longrightarrow \mathbb{R}$ by $\varphi_\varepsilon(t,x) := \psi_\varepsilon(t)\varphi(t,x)$. Then it holds that $\varphi_\varepsilon \in C_c([0,\tau]\times V)$ and $\partial_t\varphi_\varepsilon \in C_c([0,\tau]\times V)$. It follows from the weak formulation with test function φ_ε that

$$\begin{aligned}
0 &= \int_V [\varphi_\varepsilon(\tau,x)\rho_\tau(dx) - \varphi_\varepsilon(0,x)\rho_0(dx)] \\
&= \int_0^\tau \int_V \partial_t\varphi_\varepsilon(t,x)\rho_t(dx)\,dt + \int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi_\varepsilon(t,x,y)j_t(dx,dy)\,dt \\
&= \int_0^\tau \psi_\varepsilon'(t)\left(\int_V \varphi(t,x)\rho_t(dx)\right)dt \\
&\quad + \int_0^\tau \psi_\varepsilon(t)\left(\int_V \partial_t\varphi(t,x)\rho_t(dx) + \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y)j_t(dx,dy)\right)dt\,.
\end{aligned} \tag{2.3}$$

We claim that the map $[0,\tau] \ni t \mapsto \int_V \varphi(t,x)\rho_t(dx)$ is continuous. Namely, let $\varepsilon' > 0$ and $t \in [0,\tau]$. We see that

$$\begin{aligned}
\left|\int_V \varphi(s,x)\rho_s(dx) - \int_V \varphi(t,x)\rho_t(dx)\right| &\le \left|\int_V (\varphi(s,x) - \varphi(t,x))\,\rho_s(dx)\right| \\
&\quad + \left|\int_V \varphi(t,x)(\rho_s(dx) - \rho_t(dx))\right|
\end{aligned}$$

Recall that V is metrizable and denote by d_V a metric that induces the topology on V. Firstly, since φ is continuous and has compact support, φ is uniformly continuous on $\operatorname{supp}\varphi$ by [14, Theorem 4.19]. That is, there exists $\delta_1 > 0$ such that for any $(s,x),(s',x') \in [0,\tau]\times V$ with $|s-s'| + d_V(x,x') < \delta_1$, it holds that $|\varphi(s,x) - \varphi(s',x)| < \varepsilon'$. Furthermore, denoting by $\pi_2 : [0,\tau]\times V \longrightarrow V$ the canonical projection onto the second component, we have that $K := \pi_2(\operatorname{supp}\varphi)$ is compact. There exists $N \in \mathbb{N}$ such that $K \subset K_N^\circ$, and, by Urysohn's Lemma [5, Theorem 5.21], a cut-off function $\chi \in C_c(V)$ with

$$0 \le \chi \le 1\,, \quad \chi \equiv 1 \text{ on } K_N\,, \quad \operatorname{supp}\chi \subset K_{N+1}^\circ\,.$$

By weak*-continuity of ρ, we have that

$$[0,\tau] \ni s \mapsto \int_V \chi(x)\rho_s(dx) \in [0,\infty)\,,$$

is bounded from above. It follows for $s \in [0,\tau]$ with $|s-t| < \delta_1$ that

$$\begin{aligned}
\left|\int_V (\varphi(s,x) - \varphi(t,x))\,\rho_s(dx)\right| &= \left|\int_K (\varphi(s,x) - \varphi(t,x))\,\rho_s(dx)\right| \\
&\le \int_K |\varphi(s,x) - \varphi(t,x)|\,\rho_s(dx) \\
&\le \varepsilon' \int_V \mathbb{1}_K(x)\rho_s(dx) \\
&\le \varepsilon' \sup_{s\in[0,\tau]} \int_V \chi(x)\rho_s(dx)\,.
\end{aligned}$$

Secondly, by the Feller property, there exists $\delta_2 > 0$ such that for all $s \in [0,\tau]$ with $|s-t| < \delta_2$, it holds that

$$\left| \int_V \varphi(t,x)(\rho_s(dx) - \rho_t(dx)) \right| < \varepsilon' .$$

Hence, it follows that

$$\left| \int_V \varphi(s,x)\rho_s(dx) - \int_V \varphi(t,x)\rho_t(dx) \right| < \left(1 + \sup_{s\in[0,\tau]} \int_V \chi(x)\rho_s(dx) \right) \varepsilon' ,$$

for all $s \in [0,\tau]$ with $|s-t| < \min\{\delta_1, \delta_2\}$, proving the claim.

Therefore, it follows that

$$\int_0^\tau \psi_\varepsilon'(t) \left(\int_V \varphi(t,x)\rho_t(dx) \right) dt \to \int_V \left[\varphi(t_1,x)\rho_{t_1}(dx) - \varphi(t_2,x)\rho_{t_2}(dx) \right] ,$$

as $\varepsilon \downarrow 0$.

By similar reasoning it follows that the map $[0,\tau] \ni t \mapsto \int_V \partial_t\varphi(t,x)\rho_t(dx)$ is continuous and therefore bounded, and

$$\begin{aligned} \left| \int_V \partial_t\varphi(t,x)\rho_t(dx) + \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y) j_t(dx,dy) \right| &\le \sup_{t\in[0,\tau]} \left| \int_V \partial_t\varphi(t,x)\rho_t(dx) \right| \\ &\quad + 2\|\varphi\|_{C([0,\tau]\times V)} |j_t|(\mathcal{E}) , \end{aligned}$$

which is integrable over $[0,\tau]$.

Hence, by letting $\varepsilon \downarrow 0$ in the weak formulation (2.3), it follows by Dominated Convergence that

$$\begin{aligned} \int_V \left[\varphi(t_2,x)\rho_{t_2}(dx) - \varphi(t_1,x)\rho_{t_1}(dx) \right] &= \int_{t_1}^{t_2} \int_V \partial_t\varphi(t,x)\rho_t(dx)\, dt \\ &\quad + \int_{t_1}^{t_2} \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y) j_t(dx,dy)\, dt , \end{aligned}$$

which proves the claim. □

The following result is an important consequence.

Corollary 2.3.4 (Conservation of mass)**.** *Let $\tau > 0$ and $(\rho, j) \in \mathscr{C}\mathscr{E}([0,\tau])$. Then for any $t_1, t_2 \in [0,\tau]$ and bounded continuous map $\varphi : V \longrightarrow \mathbb{R}$, it holds that*

$$\int_V \varphi(x) \left[\rho_{t_2}(dx) - \rho_{t_1}(dx) \right] = \int_{t_1}^{t_2} \iint_{\mathcal{E}} \overline{\nabla}\varphi(x,y) j_t(dx,dy)\, dt .$$

Consequently, the mass along ρ is conserved, i.e., $\rho_t(V) = \rho_0(V)$ for all $t \in [0,\tau]$.

Proof. Let $\varphi \in C_b(V)$ and let $\{K_n\}_{n\mathbb{N}}$ be an exhaustion of V by compact sets. That is, for the set K_n is compact for each n and $K_m \subset K_n^\circ$ for all $m < n$. Furthermore, $\{K_n^\circ\}_{n\in\mathbb{N}}$ is an open cover of V, i.e., $V = \bigcup_{n\in\mathbb{N}} K_n^\circ$. By Urysohn's Lemma we find cut-off functions $\chi_n \in C_c^\infty(V)$ with

$$0 \le \chi_n \le 1\,, \qquad \chi_n \equiv 1 \text{ on } K_n\,, \qquad \operatorname{supp}\chi_n \subset K_{n+1}^\circ\,.$$

It follows that $\lim_{n\to\infty}\chi_n(x) = 1$ for all $x \in V$. We define $\varphi_n : [0,\tau]\times V \longrightarrow \mathbb{R}$ by $\varphi_n(t,x) = \chi_n(x)\varphi(x)$. Then $\varphi_n \in C_c([0,\tau]\times V)$ and $\partial_t\varphi_n \equiv 0 \in C_c([0,\tau]\times V)$. By Lemma 2.3.3 it follows for any $t_1, t_2 \in [0,\tau]$ with $t_1 < t_2$ that

$$\begin{aligned}\int_V \chi_n(x)\varphi(x)\,[\rho_{t_2}(dx) - \rho_{t_1}(dx)] &= \int_V [\varphi_n(t_2,x)\rho_{t_2}(dx) - \varphi_n(t_1,x)\rho_{t_1}(dx)]\\ &= \int_{t_1}^{t_2}\iint_{\mathcal{E}} \overline{\nabla}\varphi_n(t,x,y) j_t(dx,dy)dt\\ &= \int_{t_1}^{t_2}\iint_{\mathcal{E}} [\chi_n(y)\varphi(y) - \chi_n(x)\varphi(x)]\, j_t(dx,dy)dt\,.\end{aligned}$$

Letting $n\to\infty$, it follows by Dominated Convergence that

$$\int_{t_1}^{t_2} \varphi(x)\,[\rho_{t_2}(dx) - \rho_{t_1}(dx)] = \int_{t_1}^{t_2}\iint_{\mathcal{E}} \overline{\nabla}\varphi(x,y) j_t(dx,dy)\,dt\,.$$

The final assertion follows by this equality with $\varphi \equiv 1$. This concludes the proof. □

The remaining of the section treats several ways of constructing new solutions from given ones. We start with a simple result, making use of the linearity of the continuity equation.

Let X be a real vector space. By a *conical combination* we mean a linear combination $\sum_{i=1}^n \lambda_i v_i$, where $n\in\mathbb{N}$, $v_1,\dots,v_n \in X$ and $\lambda_1,\dots,\lambda_n \ge 0$.

Lemma 2.3.5 (Conical combinations of solutions)**.** *Let $\tau > 0$ and $(\rho^1, j^1), (\rho^2, j^2) \in \mathscr{C}\mathscr{E}([0,\tau])$. Then any conical combination of (ρ^1, j^1) and (ρ^2, j^2) is again an element of $\mathscr{C}\mathscr{E}([0,\tau])$.*

Proof. Let $(\rho^1, j^1), (\rho^2, j^2) \in \mathscr{C}\mathscr{E}([0,\tau])$ and $\lambda_1, \lambda_2 \ge 0$. Because $\mathcal{M}^+(V)$ is a cone, it holds that $\lambda_1\rho_t^1 + \lambda_2\rho_t^2 \in \mathcal{M}^+(V)$ for all $t\in[0,\tau]$. Furthermore, $t \mapsto \lambda_1\rho_t^1 + \lambda_2\rho_t^2$ is weakly*-continuous.

In addition, $t \mapsto \lambda_1 j_t^1 + \lambda_2 j_t^2$ is a Borel family and

$$\int_0^\tau |\lambda_1 j_t^1 + \lambda_2 j_t^2|(\mathcal{E})dt \le \lambda_1\int_0^\tau |j_t^1|(\mathcal{E})dt + \lambda_2\int_0^\tau |j_t^2|(\mathcal{E})dt < +\infty\,.$$

By linearity of the continuity equation, it follows that $\lambda_1(\rho^1, j^1) + \lambda_2(\rho^2, j^2) \in \mathscr{C}\mathscr{E}([0,\tau])$. □

The next lemma deals with time rescaling of solutions.

Lemma 2.3.6 (Time rescaling)**.** *Let $\tau > 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $\Phi : [a, b] \longrightarrow [0, \tau]$ be a strictly increasing C^1-diffeomorphism. Then a curve $(\rho, j) : [0, \tau] \longrightarrow \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ with $\rho_t \ll \pi$ for a.e. $t \in [0, \tau]$ is an element of $\mathscr{CE}_\tau(\mu \to \nu)$ if and only if the curve $\mathsf{S}_\Phi(\rho, j) : [a, b] \longrightarrow \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ defined by $\mathsf{S}_\lambda(\rho_t, j_t) := (\rho_{\Phi(t)}, \Phi'(t) j_{\Phi(t)})$ is an element of $\mathscr{CE}([a, b]; \mu, \nu)$. Moreover, $\mathsf{S}_\Phi : \mathscr{CE}_\tau(\mu \to \nu) \longrightarrow \mathscr{CE}([a, b]; \mu, \nu)$ is a bijection with $(\mathsf{S}_\Phi)^{-1} = \mathsf{S}_{\Phi^{-1}}$.*

Proof. Assume that $(\rho, j) \in \mathscr{CE}([0, \tau]; \mu, \nu)$. Then $t \mapsto \rho_{\Phi(t)}$ is weakly*-continuous. Furthermore, $t \mapsto \Phi'(t) j_{\Phi(t)}$ is a Borel family and, because $\Phi'(t) > 0$ for all $t \in [0, \tau]$, it follows that

$$\int_a^b |\Phi'(t) j_{\Phi(t)}|(\mathcal{E})\, dt = \int_a^b \Phi'(t) |j_{\Phi(t)}|(\mathcal{E})\, dt = \int_0^\tau |j_t|(\mathcal{E})\, dt < +\infty\,.$$

Hence, $\mathsf{S}_\Phi(\rho, j)$ satisfies (1) and (2) of Definition 2.3.2. By application of the change of variables formula for integration, it follows that $\mathsf{S}_\Phi(\rho, j)$ satisfies (3) of Definition 2.3.2, the weak formulation of the continuity equation, and hence that $\mathsf{S}_\Phi(\rho, j) \in \mathscr{CE}([a, b]; \mu, \nu)$.

Remark that Φ^{-1} is also a strictly increasing C^1-diffeomorphism, so

$$\mathsf{S}_{\Phi^{-1}} : \mathscr{CE}([a, b]; \mu, \nu) \longrightarrow \mathscr{CE}_\tau(\mu \to \nu)\,,$$

is well-defined. It is straightforward to see that $\mathsf{S}_{\Phi^{-1}} \circ \mathsf{S}_\Phi = \mathrm{id}_{\mathscr{CE}_\tau(\mu \to \nu)}$ and $\mathsf{S}_\Phi \circ \mathsf{S}_{\Phi^{-1}} = \mathrm{id}_{\mathscr{CE}([a,b];\mu,\nu)}$. Hence, S_Φ is invertible with $(\mathsf{S}_\Phi)^{-1} = \mathsf{S}_{\Phi^{-1}}$.

Assume that $\mathsf{S}_\Phi(\rho, j) \in \mathscr{CE}([a, b]; \mu, \nu)$. Then it follows that $(\rho, j) = \mathsf{S}_{\Phi^{-1}} \circ \mathsf{S}_\Phi(\rho, j) \in \mathscr{CE}([0, \tau]; \mu, \nu)$, which concludes the proof. □

2.3.1 The $\mathcal{R}$-action

An important subset of the solutions of solutions of the continuity equation is the following.

Definition 2.3.7. We define class of curves of finite $\mathcal{R}$-action by

$$\mathscr{A}_\tau := \left\{ (\rho, j) \in \mathscr{CE}_\tau \;:\; t \mapsto \mathcal{R}(\rho_t, j_t) \text{ is measurable, } \int_0^\tau \mathcal{R}(\rho_t, j_t) dt < +\infty \right\}\,,$$

and for $\mu, \nu \in \mathcal{M}^+(V)$, the class $\mathscr{A}_\tau(\mu \to \nu) := \{(\rho, j) \in \mathscr{A}_\tau : \rho_0 = \mu\,, \rho_\tau = \nu\,\}$.

Definition 2.3.8 ($\mathcal{R}$-action)**.** Let $\tau > 0$. The $\mathcal{R}$-action $\mathcal{F} : \mathscr{CE}_\tau \longrightarrow [0, +\infty]$ is defined by

$$\mathcal{F}(\rho, j) := \begin{cases} \int_0^\tau \mathcal{R}(\rho_t, j_t) dt & \text{if } t \mapsto \mathcal{R}(\rho_t, j_t) \text{ is measurable}\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Instead of writing $\mathcal{F}(\rho, j)$ in the case that $(\rho, j) \notin \mathscr{A}_\tau$, we will always use the formal notation $\int_0^\tau \mathcal{R}(\rho_t, j_t) dt$ for the $\mathcal{R}$-action.

Define for $\widetilde{\rho} \in \mathcal{M}^+(V)$ with $\widetilde{\rho} \ll \pi$ the measure $\nu_{\widetilde{\rho}} \in \mathcal{M}(\mathcal{E})$ by

$$\nu_{\widetilde{\rho}}(dx, dy) := \alpha\left(\frac{d\widetilde{\rho}}{d\pi}(x), \frac{d\widetilde{\rho}}{d\pi}(y)\right)\vartheta_\pi(dx, dy)\,.$$

Proposition 2.3.9 (Lower bound of the $\mathcal{R}$-action)**.** *Let $\tau > 0$ and $(\rho, j) \in \mathscr{C}\mathscr{E}([0,\tau])$ with $\rho_t \ll \pi$ for almost all $t \in [0,\tau]$. Define $C_{\kappa,\rho} := \rho_0(V)c_\kappa < +\infty$. Then*

$$\int_0^\tau \mathcal{R}(\rho_t, j_t)dt \geq \tau C_{\kappa,\rho}\Psi\left(\frac{1}{\tau C_{\kappa,\rho}}\int_0^\tau \|j_t\|_{\mathrm{TV}(\mathcal{E})}dt\right)\,.$$

Proof. If $(\rho_t, j_t)_t$ does not have finite $\mathcal{R}$-action, the inequality is clearly satisfied. Assume therefore that $(\rho_t, j_t)_t$ has finite action. It follows that $\mathcal{R}(\rho_t, j_t) < +\infty$ for almost all $t \in [0,\tau]$ and therefore, by Lemma 2.2.6, that

$$\mathcal{R}(\rho_t, j_t) = \iint_{\mathcal{E}_{u_t}} \Psi\left(\frac{w_t(x,y)}{\alpha(u_t(x), u_t(y))}\right)\alpha(u_t(x), u_t(y))d\vartheta_\pi(x,y)\,,$$

where $\mathcal{E}_{u_t} = \{(x,y) \in \mathcal{E} : u_t(x) > 0,\, u_t(y) > 0\}$. The set $M := \bigcup_{t\in[0,\tau]}\{t\} \times \mathcal{E}_{u_t}$ is Borel measurable, because it is the pre-image under the continuous map $[0,\tau] \times \mathcal{E} \ni (t,x,y) \mapsto (u_t(x), u_t(y))$ of the set $(0,\infty) \times (0,\infty)$. By definition of $\mathcal{E}_{u_t}$ we have that

$$\int_0^\tau \iint_{\mathcal{E}_{u_t}} \alpha(u_t(x), u_t(y))d\vartheta_\pi(x,y)dt = \int_0^\tau \iint_{\mathcal{E}} \alpha(u_t(x), u_t(y))d\vartheta_\pi(x,y)dt =: m\,.$$

Recall that Ψ is convex. So by applying Jensen's inequality [3, Theorem 2.12.19] on the measurable space $(M, \mathcal{B}(M))$ equipped with the probability measure $\frac{\nu_{\rho_t}(dx,dy)dt}{m}$, it follows that

$$\begin{aligned}
\int_0^\tau \mathcal{R}(\rho_t, j_t)dt &= m\int_0^\tau \iint_{\mathcal{E}_{u_t}} \Psi\left(\frac{|w_t(x,y)|}{\alpha(u_t(x), u_t(y))}\right)\frac{\nu_{\rho_t}(x,y)dt}{m} \\
&\geq m\Psi\left(\int_0^\tau \iint_{\mathcal{E}_{u_t}} \frac{|w_t(x,y)|}{\alpha(u_t(x), u_t(y))}\frac{\nu_{\rho_t}(x,y)dt}{m}\right) \\
&= m\Psi\left(\frac{1}{m}\int_0^\tau \iint_{\mathcal{E}_{u_t}} |w_t(x,y)|d\vartheta_\pi(x,y)dt\right) \\
&= m\Psi\left(\frac{1}{m}\int_0^\tau \|j_t\|_{\mathrm{TV}(\mathcal{E})}dt\right)\,,
\end{aligned}$$

where we used that Ψ is even for the first inequality. Furthermore, by the inequality

$\alpha(\xi,\eta) \leq (\xi+\eta)/2$ and the **Detailed Balance Condition** (page 8) we have

$$\begin{aligned} m &\leq \frac{1}{2}\int_0^\tau \iint_{\mathcal{E}} (u_t(x)+u_t(y))d\vartheta_\pi(x,y)\,dt \\ &= \int_0^\tau \iint_{\mathcal{E}} u_t(x)d\vartheta_\pi(x,y)\,dt \\ &= \int_0^\tau \int_V u_t(x)\kappa(x,V)d\pi(x)\,dt \\ &\leq \sup_{x\in V}\kappa(x,V)\int_0^\tau \rho_t(V)dt \\ &= \tau\mu(V)c_\kappa \\ &= \tau C_{\kappa,\rho}\,. \end{aligned} \tag{2.4}$$

Here the second to last inequality follows by conservation of mass, Corollary 2.3.4. Since $\xi \mapsto \xi\Psi(1/\xi)$ is decreasing by Lemma 3.1.3, it follows that

$$\int_0^\tau \mathcal{R}(\rho_t,j_t)dt \geq \tau C_{\kappa,\rho}\Psi\left(\frac{1}{\tau C_{\kappa,\rho}}\int_0^\tau \|j_t\|_{\mathrm{TV}(\mathcal{E})}dt\right)\,.$$

□

The next two results deal with the operations of time rescaling and skew-symmetrization of solutions, and also include statements about their $\mathcal{R}$-action.

Lemma 2.3.10 (Time reversal)**.** *Let $(\rho,j)\in\mathscr{CE}_\tau(\mu\to\nu)$. Then the curve $\mathsf{R}(\rho,j):[0,\tau]\longrightarrow\mathcal{M}^+(V)\times\mathcal{M}(\mathcal{E})$ defined by $\mathsf{R}(\widehat{\rho}_t,\widehat{j}_t):=(\rho_{\tau-t},-j_{\tau-t})$ is an element of $\mathscr{CE}_\tau(\nu\to\mu)$. In addition, (ρ,j) has finite $\mathcal{R}$-action if and only if $\mathsf{R}(\rho,j)$ has finite $\mathcal{R}$-action, and*

$$\int_0^\tau \mathcal{R}(\rho_t,j_t)dt = \int_0^\tau \mathcal{R}\circ\mathsf{R}(\rho_t,j_t)dt\,.$$

Proof. Writing $(\widehat{\rho},\widehat{j})=\mathsf{R}(\rho,j)$, we first remark that $\widehat{\rho}_0=\rho_\tau=\nu$ and $\widehat{\rho}_\tau=\rho_0=\mu$, so $\widehat{\rho}$. Let $\widehat{\varphi}\in C_c([0,\tau]\times V)$ with $\partial_t\widehat{\varphi}\in C_c([0,\tau]\times V)$. Define the map $\varphi:[0,\tau]\times V\longrightarrow\mathbb{R}$ by $\varphi(t,x):=\widehat{\varphi}(\tau-t)$, and note that both φ and $\partial_t\varphi$ are elements of $C_c([0,\tau]\times V)$. We see that

$$\begin{aligned} \int_0^\tau\int_V \partial_t\widehat{\varphi}(t,x)\widehat{\rho}_t(dx)\,dt &= \int_0^\tau\int_V \partial_t\widehat{\varphi}(t,x)\rho_{\tau-t}(dx)\,dt \\ &= \int_0^\tau\int_V \partial_t\widehat{\varphi}(\tau-s,x)\rho_s(dx)\,ds \\ &= -\int_0^\tau\int_V \frac{\partial}{\partial s}\left(\widehat{\varphi}(\tau-s,x)\right)\rho_s(dx)\,ds \\ &= -\int_0^\tau\int_V \partial_s\varphi(s,x)\rho_s(dx)\,ds\,, \end{aligned}$$

and

$$\begin{aligned}\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\widehat{\varphi}(t,x,y)\widehat{j}_t(dx,dy)\,dt &= -\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\widehat{\varphi}(t,x,y)j_{\tau-t}(dx,dy)\,dt \\ &= -\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\widehat{\varphi}(\tau-s,x,y)j_s(dx,dy)\,ds \\ &= -\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi(s,x,y)j_s(dx,dy)\,ds\,.\end{aligned}$$

In addition, we have that (ρ, j) is a solution of the continuity equation, so it follows that

$$\begin{aligned}&\int_0^\tau \int_V \partial_t\widehat{\varphi}(t,x)\widehat{\rho}_t(dx)\,dt + \int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\widehat{\varphi}(t,x,y)\widehat{j}_t(dx,dy)\,dt \\ &\quad= -\int_0^\tau \int_V \partial_t\varphi(t,x)\rho_t(dx)\,dt - \int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y)j_t(dx,dy)\,dt \\ &\quad= -\int_V \left[\varphi(\tau,x)\rho_\tau(dx) - \varphi(0,x)\rho_0(dx)\right] \\ &\quad= \int_V \left[\widehat{\varphi}(\tau,x)\mu(dx) - \widehat{\varphi}(0,x)\nu(dx)\right]\,,\end{aligned}$$

where we used that $\varphi(\tau,x) = \widehat{\varphi}(0,x)$ and $\varphi(0,x) = \widehat{\varphi}(\tau,x)$ for all $x \in V$ for the last equality. Hence, we have shown that $\mathsf{R}(\rho,j) \in \mathscr{C}\mathscr{E}_\tau(\nu \to \mu)$.

For the final assertion, assume that (ρ, j) has finite $\mathcal{R}$-action, i.e., $t \mapsto \mathcal{R}(\rho_t, j_t)$ is integrable. Therefore, $\mathcal{R}(\rho_t, j_t) < +\infty$ almost everywhere, and there exists $g : [0,\tau] \longrightarrow \mathbb{R}$ such that $g(t) = \mathcal{R}(\rho_t, j_t)$ for almost all $t \in [0,\tau]$. Since $\mathcal{R}$ is symmetric in the second argument by symmetry of Υ in the third argument, it follows that

$$g(\tau - t) = \mathcal{R}(\rho_{\tau-t}, j_{\tau-t}) = \mathcal{R}(\rho_{\tau-t}, -j_{\tau-t}) = \mathcal{R}\circ\mathsf{R}(\rho_t, j_t)\,,$$

for almost all $t \in [0,\tau]$. We see that

$$\begin{aligned}\int_0^\tau \mathcal{R}(\rho_t, j_t)dt &= \int_0^\tau g(t)dt \\ &= \int_0^\tau g(\tau - t)dt \\ &= \int_0^\tau \mathcal{R}\circ\mathsf{R}(\rho_t, j_t)dt\,,\end{aligned}$$

where we used a change of variables for second equality. The converse implication follows by applying the above to $\mathsf{R}(\rho, j)$ and the identity $\mathsf{R}\circ\mathsf{R} = \mathrm{id}$. □

The following will enable us to work with skew-symmetric fluxes j, i.e., $\mathsf{s}_* j = -j$, in Chapter 4, where we try to find minimizers of the $\mathcal{R}$-action.

Lemma 2.3.11 ([12, Remark 4.10]). *Let $\tau > 0$ and $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau$. Then the skew-symmetrization $(\rho, \widetilde{j})$, where $\widetilde{j}_t =: \frac{1}{2}(j_t - \mathsf{s}_* j_t)$ for all $t \in [0, \tau]$, is again an element of $\mathscr{C}\mathscr{E}_\tau$. Moreover, if (ρ, j) has finite $\mathcal{R}$-action, then so does $(\rho, \widetilde{j})$ and*

$$\int_0^\tau \mathcal{R}(\rho_t, \widetilde{j}_t) dt \leq \int_0^\tau \mathcal{R}(\rho_t, j_t) dt\,.$$

Proof. Since $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau$, it follows for all $\varphi \in C_c([0,\tau] \times V)$ with $\partial_t \varphi \in C_c([0,\tau] \times V)$ that

$$\begin{aligned}
-\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y)(\mathsf{s}_* j_t)(dx,dy)\,dt &= -\int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,\mathsf{s}(x,y)) j_t(dx,dy)\,dt \\
&= \int_0^\tau \iint_{\mathcal{E}} \overline{\nabla}\varphi(t,x,y) j_t(dx,dy)\,dt \\
&= -\int_0^\tau \int_V \varphi(t,x)\rho_t(dx) dt \\
&\quad + \int_V \left[\varphi(\tau,x)\rho_\tau(dx) - \varphi(0,x)\rho_0(dx)\right]\,.
\end{aligned}$$

Hence, $(\rho, -\mathsf{s}_* j) \in \mathscr{C}\mathscr{E}_\tau$. By Lemma 2.3.5 it follows that $(\rho, \widetilde{j}) \in \mathscr{C}\mathscr{E}_\tau$.

For the proof of the assertion about the $\mathcal{R}$-actions, see [12, Remark 4.9]. □

2.3.2 Concatenation of solutions

It will be useful to be able to concatenate solutions of the continuity equation. First we will define the concatenation both of curves and of equivalence classes of curves in a general setting.

Let X be a non-empty set and let $a, b \in \mathbb{R}$ with $a < b$. Denote by $\mathcal{C}([a,b], X)$ the set of curves $\gamma : [a,b] \longrightarrow X$. We define the equivalence relation $\sim$ on $\mathcal{C}([a,b], X)$ by $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1(t) = \gamma_2(t)$ for Lebesgue-a.e. $t \in [a,b]$.

Definition 2.3.12 (Concatenation of (equivalence classes) curves). Let X be a non-empty set and let $a, b, c \in \mathbb{R}$ with $a < b < c$.

(a) For $\gamma_1 \in \mathcal{C}([a,b], X)$ and $\gamma_2 \in \mathcal{C}([b,c], X)$ with $\gamma_1(b) = \gamma_2(b)$, we the define the concatenation $\gamma_1 \odot \gamma_2 \in \mathcal{C}([a,c], X)$ by

$$\gamma_1 \odot \gamma_2 := \begin{cases} \gamma_1(t) & \text{if } t \in [a,b]\,, \\ \gamma_2(t) & \text{if } t \in [b,c]\,. \end{cases}$$

(b) For $\gamma_1 \in \mathcal{C}([a,b], X)$ and $\gamma_2 \in \mathcal{C}([b,c], X)$ with $\gamma_1(b) \neq \gamma_2(b)$, we the define the concatenation $\gamma_1 \odot \gamma_2 \in \mathcal{C}([a,c], X)$ by

$$\gamma_1 \odot \gamma_2 = \mathbb{1}_{[a,b]}\gamma_1 + \mathbb{1}_{[b,c]}\gamma_2\,.$$

(c) For two equivalence classes $[\gamma_1] \in \mathcal{C}([a,b],X)\backslash \sim$ and $[\gamma_2] \in \mathcal{C}([b,c],X)\backslash \sim$, we the define the concatenation $[\gamma_1] \odot [\gamma_2] \in \mathcal{C}([a,c],X)\backslash \sim$ by

$$[\gamma_1] \odot [\gamma_2] := [\gamma_1 \odot \gamma_2]\,.$$

Let $k, l \in \mathbb{N}$. For two vectors $(\gamma_1, \delta_1) \in \mathcal{C}([a,b],X)^k \times (\mathcal{C}([a,b],X)\backslash \sim)^l$ and $(\gamma_2, \delta_2) \in \mathcal{C}([b,c],X)^k \times (\mathcal{C}([b,c],X)\backslash \sim)^l$ with $\gamma_1(b) = \gamma_2(b)$, we define the concatenation $(\gamma_1, \delta_1) \odot (\gamma_2, \delta_2)$ as the concatenation of their components.

Remark. The concatenation of two equivalence classes is well-defined. Indeed, let $\gamma_1, \gamma_1' \in \mathcal{C}([a,b],X)$ and $\gamma_2, \gamma_2' \in \mathcal{C}([b,c],X)$ with $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$. Let $N_1 \subset [a,b]$ and $N_2 \subset [b,c]$ be the Lebesgue-null sets such that $\gamma_1 = \gamma_1'$ on $[a,b] \setminus N_1$ and $\gamma_2 = \gamma_2'$ on $[b,c] \setminus N_2$. Then $\gamma_1 \odot \gamma_2 = \gamma_1' \odot \gamma_2'$ on $[a,c] \setminus (N_1 \cup N_2 \cup \{b\})$. Hence, $[\gamma_1] \odot [\gamma_2] = [\gamma_1'] \odot [\gamma_2']$.

Remark. It is easy to see that the concatenation of curves defines an associative operation in the sense that

$$(\gamma_1 \odot \gamma_2) \odot \gamma_3 = \gamma_1 \odot (\gamma_2 \odot \gamma_3)\,,$$

for all $a, b, c, d \in \mathbb{R}$ with $a < b < c < d$, and $\gamma_1 \in \mathcal{C}([a,b],X)$, $\gamma_2 \in \mathcal{C}([b,c],X)$ and $\gamma_3 \in \mathcal{C}([c,d],X)$. Therefore, we can write $\gamma_1 \odot \gamma_2 \odot \gamma_3$ for the concatenation of three curves. Similarly, the concatenation of equivalence classes of curves is associative.

If X is a topological space the concatenation of two continuous curves is again continuous.

Lemma 2.3.13. *Let X be a topological space and let $a, b, c \in \mathbb{R}$ with $a < b < c$. If $\gamma_1 : [a,b] \longrightarrow X$ and $\gamma_2 : [b,c] \longrightarrow X$ are continuous and $\gamma_1(b) = \gamma_2(b)$, then the concatenation $\gamma_1 \odot \gamma_2 : [a,c] \longrightarrow X$ is continuous.*

Proof. Let $U \subset X$ be an open set. We will show that $(\gamma_1 \odot \gamma_2)^{-1}(U) = \gamma_1^{-1}(U) \cup \gamma_2^{-1}(U)$ is open in $[a,c]$, i.e., $(\gamma_1 \odot \gamma_2)^{-1}(U) = V \cap [a,c]$ for some open set $V \subset \mathbb{R}$. Since γ_1 and γ_2 are continuous, there exist open subsets $V_1, V_2 \subset \mathbb{R}$ such that $\gamma_1^{-1}(U) = V_1 \cap [a,b]$ and $\gamma_2^{-1}(U) = V_2 \cap [b,c]$. It follows that

$$\begin{aligned}(\gamma_1 \odot \gamma_2)^{-1}(U) &= (V_1 \cap [a,b]) \cup (V_2 \cap [b,c]) \\ &= (V_1 \cup V_2) \cap ([a,b] \cup V_2) \cap (V_1 \cup [b,c]) \cap [a,c]\,.\end{aligned}$$

Note that $\gamma_1(b) = \gamma_2(b)$ implies that $b \in \gamma_1^{-1}(U)$ if and only if $b \in \gamma_2^{-1}(U)$, and therefore that $b \in V_1$ if and only if $b \in V_2$. If $b \in V_1$, there exists $\varepsilon > 0$ such that $(b-\varepsilon, b+\varepsilon) \subset V_1 \cap V_2$. It follows that

$$\begin{aligned}([a,b] \cup V_2) \cap [a,c] &= ([a,b] \cup (b-\varepsilon, b+\varepsilon) \cup V_2) \cap [a,c] \\ &= ([a, b+\varepsilon) \cup V_2) \cap [a,c] \\ &= ((a-\varepsilon, b+\varepsilon) \cup V_2) \cap [a,c]\,,\end{aligned}$$

and analogously that

$$(V_1 \cup [b, c]) \cap [a, c] = (V_1 \cup (b - \varepsilon, c + \varepsilon)) \cap [a, c] \,.$$

Therefore, we see that

$$(\gamma_1 \odot \gamma_2)^{-1}(U) = (V_1 \cup V_2) \cap ((a - \varepsilon, b + \varepsilon) \cup V_2) \cap (V_1 \cup (b - \varepsilon, c + \varepsilon)) \cap [a, c] \,,$$

which is open in $[a, c]$, because $(V_1 \cup V_2) \cap ((a - \varepsilon, b + \varepsilon) \cup V_2) \cap (V_1 \cup (b - \varepsilon, c + \varepsilon))$ is open in $\mathbb{R}$.

Let $\delta > 0$. If $b \notin V_1$, it follows that $V_1 \cup [a, b] = V_1 \cap [a, b)$ and $V_2 \cap [b, c] = V_2 \cap (b, c]$, which are both open in $[a, c]$. Hence, it holds that $(\gamma_1 \odot \gamma_2)^{-1}(U) = (V_1 \cap [a, b]) \cup (V_2 \cap [b, c])$ is open in $[a, c]$, which concludes the proof. □

The next lemma will be used for the concatenation of fluxes, for which we use Definition 2.3.12 (b).

Lemma 2.3.14. *Let $f_1 : [a, b] \longrightarrow \mathbb{R}$ and $f_2 : [b, c] \longrightarrow \mathbb{R}$ be Borel measurable. Then the concatenation $f_1 \odot f_2 : [a, c] \longrightarrow \mathbb{R}$ is again Borel measurable. In particular, the concatenation of two Borel families $(j^1_t)_{t\in[a,b]}$ and $(j^2_t)_{t\in[b,c]}$ is again a Borel family.*

Proof. We show that $\mathbb{1}_{[a,b]} f_1 : [a, c] \longrightarrow \overline{\mathbb{R}}$ is Borel measurable. Let $B \in \mathscr{B}(\overline{\mathbb{R}})$. If $0 \notin B$, then

$$(\mathbb{1}_{[a,b]} f_1)^{-1}(B) = f_1^{-1}(B) \in \mathscr{B}([a, b]) \subset \mathscr{B}([a, c]) \,.$$

If $0 \in B$, then

$$(\mathbb{1}_{[a,b]} f_1)^{-1}(B) = f_1^{-1}(B) \cup (b, c] \in \mathscr{B}([a, c]) \,.$$

So $\mathbb{1}_{[a,b]} f_1$ is Borel measurable. Analogously, we have that $\mathbb{1}_{[b,c]} f_2$ is Borel measurable. Hence, it follows that $f_1 \odot f_2 = \mathbb{1}_{[a,b]} f_1 + \mathbb{1}_{[b,c]} f_2$ is Borel measurable.

For the final assertion, let $A \in \mathscr{B}(\mathcal{E})$. Denoting by $j^1(A)$ and $j^2(A)$ the Borel measurable maps $t \mapsto j^1_t(A)$ and $t \mapsto j^2_t(A)$ respectively, we see that

$$t \mapsto (j^1 \odot j^2)_t(A) = (j^1(A) \odot j^2(A))(t) \,,$$

is Borel measurable by the first assertion. □

The concatenation of two solutions of the continuity equation is again a solution.

Lemma 2.3.15 (Concatenation of solutions)**.** *Let $a, b, c \in \mathbb{R}$ such that $a < b < c$ and $\mu_1, \mu_2, \mu_3 \in \mathcal{M}^+(V)$. Let $(\rho^1, j^1) \in \mathscr{CE}([a, b]; \mu_1, \mu_2)$ and $(\rho^2, j^2) \in \mathscr{CE}([b, c]; \mu_2, \mu_3)$. Then the concatenation $(\rho, j) := (\rho^1, j^1) \odot (\rho^2, j^2) : [a, c] \longrightarrow \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ is an element of $\mathscr{CE}([a, c]; \mu_1, \mu_3)$ and*

$$\int_a^c \mathcal{R}(\rho_t, j_t) dt = \int_a^b \mathcal{R}(\rho^1_t, j^1_t) dt + \int_b^c \mathcal{R}(\rho^2_t, j^2_t) dt \,.$$

Proof. We see that $(\rho, j) = (\rho^1 \odot \rho^2, j^1 \odot j^2)$. Since ρ^1 and ρ^2 are both weakly*-continuous and $\rho^1_b = \rho^2_b$, it follows by Lemma 2.3.13 that $\rho^1 \odot \rho^2$ is weakly*-continuous. The concatenation $j = j^1 \odot j^2 = \mathbb{1}_{[a,b]} j^1 + \mathbb{1}_{[b,c]} j^2$ is a Borel family by Lemma 2.3.14, and

$$\int_a^c |j_t|(\mathcal{E}) = \int_a^c |\mathbb{1}_{[a,b]}(t) j^1_t + \mathbb{1}_{[b,c]}(t) j^2_t|(\mathcal{E}) dt = \int_a^b |j^1_t|(\mathcal{E}) dt + \int_b^c |j^2_t|(\mathcal{E}) dt < +\infty\,.$$

Because $(\rho^1, j^1) \in \mathscr{CE}([a,b]; \mu_1, \mu_2)$ and $(\rho^2, j^2) \in \mathscr{CE}([b,c]; \mu_2, \mu_3)$, it follows for $\varphi \in C_c([a,c] \times V)$ with $\partial_t \varphi \in C_c([a,c] \times V)$ that

$$\begin{aligned}
\int_V [\varphi(c,x)\mu_3(dx) - \varphi(a,x)\mu_1(dx)] &= \int_V [\varphi(c,x)\mu_3(dx) - \varphi(b,x)\mu_2(dx)] \\
&\quad + \int_V [\varphi(b,x)\mu_2(dx) - \varphi(a,x)\mu_1(dx)] \\
&= \int_b^c \left[\int_V \partial_t \varphi(t,x) \rho^2_t(dx) + \iint_{\mathcal{E}} \overline{\nabla} \varphi(t,x,y) j^2_t(dx,dy) \right] dt \\
&\quad + \int_a^b \left[\int_V \partial_t \varphi(t,x) \rho^1_t(dx) + \iint_{\mathcal{E}} \overline{\nabla} \varphi(t,x,y) j^1_t(dx,dy) \right] dt \\
&= \int_a^c \left[\int_V \partial_t \varphi(t,x) \rho_t(dx) + \iint_{\mathcal{E}} \overline{\nabla} \varphi(t,x,y) j_t(dx,dy) \right] dt\,.
\end{aligned}$$

Since $\rho_a = \rho^1_a = \mu_1$ and $\rho_c = \rho^2_c = \mu_3$, we have shown that $(\rho, j) \in \mathscr{CE}([a,c]; \mu_1, \mu_3)$. For the final assertion, it follows from the definitions of the concatenation and the $\mathcal{R}$-action that

$$\int_a^c \mathcal{R}(\rho_t, j_t) dt = \int_a^b \mathcal{R}(\rho^1_t, j^1_t) dt + \int_b^c \mathcal{R}(\rho^2_t, j^2_t) dt\,.$$

□

2.4 The DVT-cost

Using the dissipation potential and the continuity equation, we can define the Dynamical-Variational Transport (DVT) cost of two positive finite Borel measures on V, depending on an additional parameter τ.

Definition 2.4.1. Let $\tau > 0$. We define the Dynamical-Variational Transport cost $\mathcal{W}^\tau : \mathcal{M}^+(V) \times \mathcal{M}^+(V) \longrightarrow [0, +\infty]$ by

$$\mathcal{W}^\tau(\mu, \nu) := \inf \left\{ \int_0^\tau \mathcal{R}(\rho_t, j_t) dt \;:\; (\rho, j) \in \mathscr{CE}_\tau(\mu \to \nu) \right\}. \tag{2.5}$$

Remark. In general there is no guarantee that $\mathscr{A}_\tau(\mu \to \nu) \neq \emptyset$, i.e., that there exists a solution of the continuity equation connecting μ and ν. So if $\mathscr{A}_\tau(\mu \to \nu) = \emptyset$, then

$\mathcal{W}^\tau(\mu \to \nu) = +\infty$. A sufficient condition for finiteness of the cost when $\mu \ll \pi$ and $\nu \ll \pi$, is the condition that (π, ϑ_π) satisfies a so-called Poincarè inequality, see [12, Section 4.6].

In the case that V is finite and $\pi(x) > 0$ for all $x \in V$, we will show that $\mathscr{A}_\tau(\mu \to \nu) \neq \emptyset$ for any pair $\mu, \nu \in \mathcal{M}^+(V)$, see Corollary 3.4.6.

We will see that minimizers of (2.5) exist if $\mathscr{A}_\tau(\mu \to \nu) \neq \emptyset$. This is due to the following result, which is concerned with sequential compactness of sets with bounded $\mathcal{R}$-action in a suitable topology, and sequential lower semicontinuity of the $\mathcal{R}$-action with respect to this topology.

Proposition 2.4.2 ([12, Proposition 4.21]). *Let $\tau > 0$ and $\{(\rho^n, j^n)\}_n$ a sequence in $\mathscr{C}\mathscr{E}_\tau$ such that*

$$\sup_{n\in\mathbb{N}} \int_0^\tau \mathcal{R}(\rho_t^n, j_t^n)dt + \sup_{n\in\mathbb{N}} \rho_0^n(V) < +\infty\,.$$

Then there exists a subsequence (not relabelled) and a pair $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau$ such that for the measures $j^n \in \mathcal{M}([0,\tau]\times\mathcal{E})$ given by $j^n(dt, dx, dy) := j_t(dx, dy)dt$, it holds that

$$\rho_t^n \overset{*}{\rightharpoonup} \rho_t \text{ weakly* in } \mathcal{M}^+(V) \text{ for all } t \in [0,\tau]\,, \tag{2.6a}$$

$$j^n \overset{*}{\rightharpoonup} j \text{ weakly* in } \mathcal{M}([0,\tau]\times\mathcal{E})\,, \tag{2.6b}$$

where $j(dt, dx, dy) = j_t(dx, dy)dt$ for a Borel family $(j_t)_{t\in[0,\tau]} \subset \mathcal{M}(\mathcal{E})$. In addition, for any (ρ^n, j^n) converging to (ρ, j) in the sense of (2.6), we have

$$\int_0^\tau \mathcal{R}(\rho_t, j_t)dt \le \liminf_{n\to\infty} \int_0^\tau \mathcal{R}(\rho_t^n, j_t^n)dt\,.$$

Corollary 2.4.3 (Existence of minimizers). *Let $\tau > 0$ and $\mu, \nu \in \mathcal{M}^+(V)$. If $\mathscr{A}_\tau(\mu \to \nu) \neq \emptyset$, then there exists $(\rho, j) \in \mathscr{A}_\tau(\mu \to \nu)$ such that*

$$\mathcal{W}^\tau(\mu, \nu) = \int_0^\tau \mathcal{R}(\rho_t, j_t)dt\,.$$

Proof. Let $\{(\rho^n, j^n)\}_n \subset \mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$ be a minimizing sequence for $\mathcal{W}^\tau(\mu, \nu) < +\infty$. That is,

$$\lim_{n\to\infty} \int_0^\tau \mathcal{R}(\rho_t^n, j_t^n)dt = \mathcal{W}^\tau(\mu, \nu)\,.$$

Then

$$M := \sup_{n\in\mathbb{N}} \int_0^\tau \mathcal{R}(\rho_t^n, j_t^n)dt + \sup_{n\in\mathbb{N}} \rho_0^n(V) = \sup_{n\in\mathbb{N}} \int_0^\tau \mathcal{R}(\rho_t^n, j_t^n)dt + \mu(V) < +\infty\,.$$

By Proposition 2.4.2, there exists a subsequence $\{(\rho^{n_k}, j^{n_k})\}_k \in \mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$ and $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau$ such that this subsequence converges to (ρ, j) as $k \to \infty$ in the sense of (2.6). For the boundary points of ρ this means that

$$\int_V \varphi(x)\rho_0(dx) = \int_V \varphi(x)\mu(dx)\,, \qquad \int_V \varphi(x)\rho_\tau(dx) = \int_V \varphi(x)\nu(dx)\,,$$

for all $\varphi \in C_0(V)$. It follows by Riesz' Representation Theorem [4, Theorem C.18] that $\rho_0 = \mu$ and $\rho_\tau - \nu$. It holds by sequential lower semicontinuity of the $\mathcal{R}$-action with respect to (2.6) that

$$\int_0^\tau \mathcal{R}(\rho_t, j_t) dt \leq \liminf_{k \to \infty} \int_0^\tau \mathcal{R}(\rho_t^{n_k}, j_t^{n_k}) dt = \mathcal{W}^\tau(\mu, \nu) \,,$$

which proves the claim. □

Chapter 3

Short-time limit of the rescaled DVT-cost

In the introduction we mentioned the expectation that a sequence of rescaled action functionals, which we specify in the next section, will converge in the framework of Γ-convergence to the Monge–Kantorovich functional, similar to the convergence of the rescaled entropy functionals in [10]. First we will motivate this more precisely.

Let $\mu, \nu \in \mathcal{M}^+(V)$. Assuming that there exist curves of finite $\mathcal{R}$-action, or equivalently that $\mathscr{A}_\tau(\mu \to \nu) \neq \emptyset$, it follows by Corollary 2.4.3 that

$$\mathcal{W}^\tau(\mu, \nu) = \min \left\{ \int_0^\tau \mathcal{R}(\rho_t, j_t) dt \,\middle|\, (\rho, j) \in \mathscr{A}_\tau(\mu \to \nu) \right\} .$$

We remove τ from the integration domain by a change of variables
$\Phi_\tau : [0,1] \longrightarrow [0,\tau]$, $t \mapsto \tau t$ to the fixed domain $[0,1]$. By Lemmas 2.3.6 and 3.1.5 the map $\mathsf{S}_\Phi : \mathscr{A}_\tau(\mu \to \nu) \longrightarrow \mathscr{A}_1(\mu \to \nu)$ is a bijection. It follows that

$$\begin{aligned}
\mathcal{W}^\tau(\mu, \nu) &= \min \left\{ \int_0^1 \Phi_\tau'(t) \mathcal{R}(\rho_{\Phi(t)}, j_{\Phi(t)}) dt \,\middle|\, (\rho, j) \in \mathscr{A}_\tau(\mu \to \nu) \right\} \\
&= \min \left\{ \tau \int_0^1 \mathcal{R}\left((\mathsf{S}_{\Phi_\tau}\rho)_t, \frac{1}{\tau}(\mathsf{S}_{\Phi_\tau}\rho)_t \right) dt \,\middle|\, (\rho, j) \in \mathscr{A}_\tau(\mu \to \nu) \right\} \\
&= \min \left\{ \tau \int_0^1 \mathcal{R}\left(\widehat{\rho}_t, \frac{1}{\tau}\widehat{j}_t \right) dt \,\middle|\, (\widehat{\rho}, \widehat{j}) \in \mathscr{A}_1(\mu \to \nu) \right\} \\
&= \min \left\{ \tau \int_0^1 \iint_{\mathcal{E}_{\widehat{u}_t}} \widehat{\alpha}_t \Psi\left(\frac{|\widehat{w}_t|}{\tau \widehat{\alpha}_t} \right) d\vartheta_\pi dt \,\middle|\, (\widehat{\rho}, \widehat{j}) \in \mathscr{A}_1(\mu \to \nu) \right\} ,
\end{aligned}$$

and for $\tau < 1$, or equivalently $c := 1/\tau > 1$, that

$$\frac{\mathcal{W}^\tau(\mu, \nu)}{\log(\tau^{-1})} = \min \left\{ \int_0^1 \iint_{\mathcal{E}_{\widehat{u}_t}} \frac{\widehat{\alpha}_t}{c \log c} \Psi\left(\frac{c|\widehat{w}_t|}{\widehat{\alpha}_t} \right) d\vartheta_\pi dt \,\middle|\, (\widehat{\rho}, \widehat{j}) \in \mathscr{A}_1(\mu \to \nu) \right\} .$$

The relative entropy functional of [10], rescaled with precisely the same factor $\log(\tau^{-1})^{-1}$, also has an integrand of the form $\frac{\phi(cs)}{c\log(c)}$ with $\phi(s) = s\log(s) - s + 1$, the Boltzmann–Shannon entropy. Therefore, choosing

$$\Psi(\xi) = (\cosh -1)^*(\xi) = x \log\left(x + \sqrt{x^2+1}\right) - \sqrt{x^2+1} + 1\,,$$

we expect the same kind of limiting behavior, and therefore that we also have Γ-convergence to the Monge–Kantorovich in our case.

In addition, we choose $\alpha(u,v) := \sqrt{uv}$. In [12] it is shown that in their gradient flow framework with this choice of (Ψ, α), the heat equation on a graph can be reformulated as a gradient flow with driving functional the relative entropy. This choice is motivated by the Riemannian case, in which the heat equation can also be reformulated as the gradient flow with driving functional the relative entropy, and its connection to curvature via contraction estimates.

The main result of this chapter is the existence of the Γ-limit of the rescaled action functionals by Theorem 3.2.7. Together with a strong lower bound of the rescaled action functional, we will show that the heuristic limit,

$$\begin{aligned}
\lim_{\tau\downarrow 0} \frac{\mathcal{W}^\tau(\mu,\nu)}{\log(\tau^{-1})} &= \min\left\{ \lim_{c\to\infty} \int_0^1 \iint_{\mathcal{E}_{\widehat{u}_t}} \frac{\widehat{\alpha}_t}{c\log c} \Psi\left(\frac{c|\widehat{w}_t|}{\widehat{\alpha}_t}\right) d\vartheta_\pi dt \,\middle|\, (\widehat{\rho}, \widehat{\jmath}) \in \mathscr{A}_1(\mu\to\nu) \right\} \\
&= \min\left\{ \int_0^1 \iint_{\mathcal{E}_{\widehat{u}_t}} |\widehat{w}_t|\, d\vartheta_\pi dt \,\middle|\, (\widehat{\rho}, \widehat{\jmath}) \in \mathscr{A}_1(\mu\to\nu) \right\} \\
&= \min\left\{ \int_0^1 |\widehat{\jmath}_t|(\mathcal{E}) dt \,\middle|\, (\widehat{\rho}, \widehat{\jmath}) \in \mathscr{A}_1(\mu\to\nu) \right\}\,,
\end{aligned}$$

of the minimal values of the rescaled functionals to the minimal value of the Γ-limit holds true in Proposition 3.5.1. In the last section we will prove an inequality of W^1 and the limit of the rescaled cost via the continuity equation if we equip the graph V with the combinatorial graph distance in Corollary 3.6.5.

3.1 The choice of Ψ and α

As announced in the introduction of this chapter, we will choose a specific Ψ and α for the remaining of the thesis. In this section we will prove some useful properties of this specific Ψ.

As stated in the introduction we fix

$$\Psi(\xi) = (\Psi^*)^*(\xi) = \xi \log\left(\xi + \sqrt{\xi^2+1}\right) - \sqrt{\xi^2+1} + 1\,, \qquad \xi \in \mathbb{R}\,,$$

which is readily seen to satisfy **Assumption** Ψ on page 9. So Ψ is even, strictly increasing, strictly convex with $\Psi(0) = 0$, takes values in $(0,\infty)$ if $x \in \mathbb{R} \setminus \{0\}$ and is superlinear.

The flux density α was chosen $\alpha(u,v) := \sqrt{uv}$, the geometric mean, for which **Assumption** α on page 9 is readily verified.

The following lemmas make explicit use of our choice of Ψ.

Lemma 3.1.1. *For all $\xi \in \mathbb{R}$ it holds that*

$$\lim_{c\to\infty} \frac{\Psi(c\xi)}{c\log c} = |\xi|\,.$$

Proof. Since $\Psi(0) = 0$ the Lemma holds true for $\xi = 0$. Let $\xi \neq 0$. We see that

$$\begin{aligned}\Psi(c\xi) &= c\xi \log\left(c\xi + \sqrt{c^2\xi^2+1}\right) - \sqrt{c^2\xi^2+1} + 1\\ &= \xi c\log c + c\left(\xi\log\left(\xi + \sqrt{\xi^2+1/c^2}\right) - \sqrt{\xi^2+1/c^2}\right) + 1\,,\end{aligned}$$

so

$$\frac{\Psi(c\xi)}{c\log c} = \xi + \frac{\xi\log\left(\xi+\sqrt{\xi^2+1/c^2}\right) - \sqrt{\xi^2+1/c^2}}{\log c} + \frac{1}{c\log c}\,. \tag{3.1}$$

Because Ψ is even, it follows that

$$\lim_{c\to\infty}\frac{\Psi(c\xi)}{c\log c} = \lim_{c\to\infty}\frac{\Psi(c|\xi|)}{c\log c} = |\xi|\,.$$

□

Lemma 3.1.2. *The dissipation density Ψ is quadratically bounded. That is, $\Psi(\xi) \leq \xi^2$ for all $\xi \in \mathbb{R}$.*

Proof. Let $\xi, \eta \geq 0$. It follows that

$$\Psi'(\eta) = \log(\eta + \sqrt{\eta^2+1}) \leq \log(2\eta+1) \leq 2\eta\,.$$

Integrating both sides over $[0,\xi]$, it follows that

$$\Psi(\xi) = \Psi(\xi) - \Psi(0) = \int_0^\xi \Psi'(\eta)d\eta \leq \int_0^\xi 2\eta d\eta = \xi^2\,.$$

Because Ψ and $\xi \mapsto \xi^2$ are even, we also have $\Psi(\xi) \leq \xi^2$ for $\xi \leq 0$. □

Lemma 3.1.3. *The map* $(0,\infty) \ni \xi \mapsto \xi\Psi\left(\frac{1}{\xi}\right)$ *is strictly decreasing.*

Proof. We show that $\frac{d}{d\xi}\left(\xi\Psi\left(\frac{1}{\xi}\right)\right) = \Psi\left(\frac{1}{\xi}\right) - \frac{1}{\xi}\Psi'\left(\frac{1}{\xi}\right) < 0$, or equivalently that $\xi\Psi\left(\frac{1}{\xi}\right) < \Psi'\left(\frac{1}{\xi}\right)$, for all $\xi > 0$. We see that

$$\begin{aligned}
\xi\Psi\left(\frac{1}{\xi}\right) &= \xi\left(\frac{1}{\xi}\log\left(\frac{1}{\xi} + \sqrt{1+1/\xi^2}\right) - \sqrt{1+1/\xi^2} + 1\right) \\
&= \log\left(\frac{1}{\xi}(1+\sqrt{1+\xi^2})\right) - \sqrt{1+\xi^2} + \xi \\
&= \log\left(1+\sqrt{1+\xi^2}\right) - \log(\xi) - \sqrt{1+\xi^2} + \xi\,.
\end{aligned}$$

Furthermore, it follows that

$$\begin{aligned}
\Psi'(\eta) &= \log\left(\eta + \sqrt{\eta^2+1}\right) + \frac{\eta}{\eta+\sqrt{\eta^2+1}}\left(1 + \frac{\eta}{\sqrt{\eta^2+1}}\right) - \frac{\eta}{\sqrt{\eta^2+1}} \\
&= \log\left(\eta + \sqrt{\eta^2+1}\right)\,,
\end{aligned}$$

so $\Psi'\left(\frac{1}{\xi}\right) = \log\left(1+\sqrt{1+\xi^2}\right) - \log(\xi)$. Since $-\sqrt{\xi^2+1}+\xi < 0$, it follows that $\xi\Psi\left(\frac{1}{\xi}\right) < \Psi'\left(\frac{1}{\xi}\right)$ for all $\xi > 0$. □

The following lemma will be used in the proofs of Lemma 3.1.5 and Proposition 3.4.2.

Lemma 3.1.4. *Let* $\lambda > 0$ *and let* $(\widetilde{\rho}, \widetilde{j}) \in \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ *such that* $\widetilde{\rho} \ll \pi$ *and* $\widetilde{j} \ll \vartheta_\pi$*. Then, writing*

$$\widetilde{\alpha} := \alpha\left(\frac{d\widetilde{\rho}}{d\pi}(x), \frac{d\widetilde{\rho}}{d\pi}(y)\right)\,, \qquad \widetilde{w} := \frac{d\widetilde{j}}{d\vartheta_\pi}(x,y)\,,$$

it holds that

$$\widetilde{\alpha}\Psi\left(\frac{\lambda|\widetilde{w}|}{\widetilde{\alpha}}\right) \le \lambda\log\lambda|\widetilde{w}| + \lambda\widetilde{\alpha}\Psi\left(\frac{|\widetilde{w}|}{\widetilde{\alpha}}\right) + 2\max\{0, 1-\lambda\}\widetilde{\alpha}\,.$$

Proof. By the reverse triangle inequality, it holds that

$$\sqrt{|\widetilde{w}|^2 + \frac{\widetilde{\alpha}^2}{\lambda^2}} = \sqrt{|\widetilde{w}|^2 + \left(\widetilde{\alpha} + \left(\frac{1}{\lambda} - 1\right)\widetilde{\alpha}\right)^2} \ge \sqrt{|\widetilde{w}|^2 + \widetilde{\alpha}^2} - \left|1 - \frac{1}{\lambda}\right|\widetilde{\alpha}\,.$$

With this it follows that

$$\begin{aligned}
\widetilde{\alpha}\Psi\left(\lambda\frac{|\widetilde{w}|}{\widetilde{\alpha}}\right) &= \lambda\log\lambda|\widetilde{w}| + \lambda\left(|\widetilde{w}|\log\left(\frac{|\widetilde{w}|}{\widetilde{\alpha}} + \sqrt{\frac{|\widetilde{w}|^2}{\widetilde{\alpha}^2} + \frac{1}{\lambda^2}}\right) - \sqrt{|\widetilde{w}|^2 + \frac{\widetilde{\alpha}^2}{\lambda^2}}\right) + \widetilde{\alpha} \\
&\leq \lambda\log\lambda|\widetilde{w}| + \lambda\left(|\widetilde{w}|\log\left(\frac{|\widetilde{w}|}{\widetilde{\alpha}} + \sqrt{\frac{|\widetilde{w}|^2}{\widetilde{\alpha}^2} + 1}\right) - \sqrt{|\widetilde{w}|^2 + \widetilde{\alpha}^2} + \left|1 - \frac{1}{\lambda}\right|\widetilde{\alpha}\right) \\
&\quad + \widetilde{\alpha} \\
&= \lambda\log\lambda|\widetilde{w}| + \lambda\left(|\widetilde{w}|\log\left(\frac{|\widetilde{w}|}{\widetilde{\alpha}} + \sqrt{\frac{|\widetilde{w}|^2}{\widetilde{\alpha}^2} + 1}\right) - \sqrt{|\widetilde{w}|^2 + \widetilde{\alpha}^2} + \widetilde{\alpha}\right) \\
&\quad + (|1-\lambda| + 1 - \lambda)\,\widetilde{\alpha} \\
&= \lambda\log\lambda|\widetilde{w}| + \lambda\widetilde{\alpha}\Psi\left(\frac{|\widetilde{w}|}{\widetilde{\alpha}}\right) + 2\max\{0, 1-\lambda\}\widetilde{\alpha}\,.
\end{aligned}$$

□

Lemma 3.1.5. *Let $\tau > 0$ and $a, b \in \mathbb{R}$ with $a < b$. Let $\Phi : [a,b] \longrightarrow [0,\tau]$ be a strictly increasing C^1-diffeomorphism. Let $(\rho, j) \in \mathscr{CE}_\tau(\mu \to \nu)$ such that both $t \mapsto \mathcal{R}(\rho_t, j_t)$ and $t \mapsto \mathsf{S}(\rho_t, j_t)$ are measurable. Then (ρ, j) has finite $\mathcal{R}$-action if and only if $\mathsf{S}_\Phi(\rho, j) \in \mathscr{CE}([a,b];\mu,\nu)$ has finite $\mathcal{R}$-action.*

Proof. Assume that $(\rho, j) \in \mathscr{CE}_\tau(\mu \to \nu)$ has finite $\mathcal{R}$-action. Furthermore, the disjoint sets $A = (\Phi')^{-1}((0,1))$ and $B = (\Phi')^{-1}([1,\infty))$ are Borel measurable by the continuity of Φ', and $[a,b] = (\Phi')^{-1}((0,1)) \cup (\Phi')^{-1}([1,\infty)) = A \cup B$. Using Lemmas 2.2.6 and 3.1.4, and writing

$$\alpha_{\Phi(t)} = \alpha\left(\frac{d\rho_{\Phi(t)}}{d\pi}(x), \frac{d\rho_{\Phi(t)}}{d\pi}(y)\right), \qquad w_{\Phi(t)} := \frac{dj_{\Phi(t)}}{d\vartheta_\pi}(x,y)\,,$$

it follows that

$$\begin{aligned}
\int_B \mathcal{R}\circ\mathsf{S}_\Phi(\rho_t, j_t)\,dt &= \int_B \iint_{\mathcal{E}_{u_{\Phi(t)}}} \alpha_{\Phi(t)}\Psi\left(\frac{\Phi'(t)|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right)d\vartheta_\pi\,dt \\
&\leq \int_B \Phi'(t)\iint_{\mathcal{E}_{u_{\Phi(t)}}}\left[\log(\Phi'(t))|w_{\Phi(t)}| + \alpha_{\Phi(t)}\Psi\left(\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right)\right]d\vartheta_\pi\,dt \\
&\leq \int_B \Phi'(t)\iint_{\mathcal{E}_{u_{\Phi(t)}}}\left[\log(\|\Phi'\|_\infty)|w_{\Phi(t)}| + \alpha_{\Phi(t)}\Psi\left(\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right)\right]d\vartheta_\pi\,dt \\
&\leq \log(\|\Phi'\|_\infty)\int_a^b\iint_{\mathcal{E}}\Phi'(t)|w_{\Phi'(t)}|d\vartheta_\pi\,dt + \int_a^b\Phi'(t)\mathcal{R}(\rho_{\Phi(t)}, j_{\Phi(t)})dt \\
&= \log(\|\Phi'\|_\infty)\int_0^\tau |j_t|(\mathcal{E})\,dt + \int_0^\tau\mathcal{R}(\rho_t, j_t)dt \\
&< +\infty\,,
\end{aligned}$$

where we used Lemma 3.1.4 for the first inequality and change of variables for the last equality. Furthermore,

$$
\begin{aligned}
\int_A \mathcal{R}\circ \mathsf{S}_\Phi(\rho_t, j_t)\, dt &= \int_A \iint_{\mathcal{E}_{u_{\Phi(t)}}} \alpha_{\Phi(t)} \Psi\left(\frac{\Phi'(t)|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right) d\vartheta_\pi\, dt \\
&\leq \int_A \Phi'(t) \iint_{\mathcal{E}_{u_{\Phi(t)}}} \left[\log(\Phi'(t))|w_{\Phi(t)}| + \alpha_{\Phi(t)}\Psi\left(\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right)\right] d\vartheta_\pi\, dt \\
&\qquad + 2\int_A (1-\Phi'(t)) \iint_{\mathcal{E}_{u_{\Phi(t)}}} \alpha_{\Phi(t)} d\vartheta_\pi\, dt \\
&\leq \int_A \Phi'(t) \iint_{\mathcal{E}_{u_{\Phi(t)}}} \alpha_{\Phi(t)}\Psi\left(\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right) d\vartheta_\pi\, dt + 2\int_A \iint_{\mathcal{E}_{u_{\Phi(t)}}} \alpha_{\Phi(t)} d\vartheta_\pi\, dt \\
&\leq \int_0^\tau \mathcal{R}(\rho_t, j_t)dt + 2\sup_{x\in V}\kappa(x,V)\int_a^b \rho_{\Phi(t)}(V)dt \\
&= \int_0^\tau \mathcal{R}(\rho_t, j_t)dt + 2(b-a)\mu(V)\sup_{x\in V}\kappa(x,V) \\
&< +\infty\,,
\end{aligned}
$$

where we used change of variables and the estimate (2.4) for the second to last inequality. Hence,

$$
\int_a^b \mathcal{R}\circ \mathsf{S}_\Phi(\rho_t, j_t)dt = \int_A \mathcal{R}\circ \mathsf{S}_\Phi(\rho_t, j_t)dt + \int_B \mathcal{R}\circ \mathsf{S}_\Phi(\rho_t, j_t)dt < +\infty\,,
$$

and $\mathsf{S}_\Phi(\rho, j)$ has finite $\mathcal{R}$-action. For the reverse implication, assume that $\mathsf{S}_\Phi(\rho, j)$ has finite $\mathcal{R}$-action. Since Φ^{-1} is a strictly increasing C^1-diffeomorphism, it follows by what we have already shown that $(\rho, j) = \mathsf{S}_{\Phi^{-1}} \circ \mathsf{S}_\Phi(\rho, j)$ has finite $\mathcal{R}$-action. This concludes the proof. □

3.2 Main result

In this section we define the sequence of rescaled functionals as well as the candidate Γ-limit. Subsequently, we state our main result Theorem 3.2.7, which shows that our candidate limit is the actual Γ-limit.

Definition 3.2.1 (Rescaled dissipation potential)**.** For $c > 1$ and $(\rho, j) \in \mathcal{M}^+(V)\times\mathcal{M}(\mathcal{E})$ we define

$$
\mathcal{R}_c(\rho, j) = \frac{1}{c\log c}\mathcal{R}(\rho, cj)\,.
$$

An immediate consequence of Lemmas 2.2.5 and 2.2.6 is the following.

Lemma 3.2.2. *Let $c > 1$ and $(\rho, j) \in \mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$ such that $\rho \ll \pi$ and $\mathcal{R}_c(\rho, j) < +\infty$. Then $j \ll \vartheta_\pi$ and, denoting $u := d\rho/d\pi$, $w := dj/d\vartheta$, it holds that $w = 0$, ϑ_π-a.e. on $\mathcal{E} \setminus \mathcal{E}_u$. Moreover,*

$$\begin{aligned}\mathcal{R}_c(\rho, j) &= \frac{1}{c\log c}\int_{\mathcal{E}} \Upsilon(u(x), u(y), cw(x,y))\vartheta_\pi(dx, dy) \\ &= \iint_{\mathcal{E}_u} \frac{\alpha(u(x), u(y))}{c\log c}\Psi\left(\frac{cw(x,y)}{\alpha(u(x), u(y))}\right)\vartheta_\pi(dx, dy)\,,\end{aligned}$$

and $\mathcal{R}_c$ is sequentially weakly-lower semicontinuous on $\mathcal{M}^+(V) \times \mathcal{M}(\mathcal{E})$.*

This motivates the definition of the rescaled action densities.

Definition 3.2.3 (Rescaled action density function)**.** Let $c > 1$. We define the mapping $\Upsilon_c : [0, \infty) \times [0, \infty) \times \mathbb{R} \longrightarrow [0, +\infty]$ by

$$\Upsilon_c(u, v, w) = \frac{\Upsilon(u, v, cw)}{c\log c}\,.$$

Since we will often work on the interval $[0, 1]$, we define $\mathscr{C}\mathscr{E} := \mathscr{C}\mathscr{E}_1$. Analogous with the definition of the $\mathcal{R}$-action, we introduce the rescaled actions.

Definition 3.2.4 (Rescaled action)**.** Let $c > 1$ and $\mu, \nu \in \mathcal{M}^+(V)$. We define the $\mathcal{R}_c$-action $\mathcal{F}_c : \mathscr{C}\mathscr{E}(\mu \to \nu) \longrightarrow [0, +\infty]$ by

$$\mathcal{F}_c(\rho, j) := \begin{cases} \int_0^\tau \mathcal{R}_c(\rho_t, j_t)dt & \text{if } t \mapsto \mathcal{R}_c(\rho_t, j_t) \text{ is measurable}\,, \\ +\infty & \text{otherwise.}\end{cases}$$

Instead of writing $\mathcal{F}_c(\rho, j)$ in the case that $t \mapsto \mathcal{R}_c(\rho_t, j_t)$ is not measurable, we will always use the formal notation $\int_0^\tau \mathcal{R}_c(\rho_t, j_t)dt$ for the $\mathcal{R}_c$-action.

Assumption. For the remainder of the chapter, unless specified otherwise, we will assume that V is a finite set and $\pi(x) > 0$ for all $x \in V$.

Lemma 3.2.5. *Let $(\rho, j) \in \mathscr{C}\mathscr{E}$. Then the maps $t \mapsto \mathcal{R}(\rho_t, j_t)$ and $t \mapsto \mathcal{R}_c(\rho_t, j_t)$ are Borel measurable for all $c > 1$.*

Proof. From the definition of the dissipation potential we have that

$$\mathcal{R}(\rho_t, j_t) = \sum_{(x,y)\in\mathcal{E}} \Upsilon\left(\frac{d\sigma_{\rho_t, j_t}}{d|\sigma_{\rho_t, j_t}|}(x, y)\right)|\sigma_{\rho_t, j_t}|(x, y)\,.$$

We see for each $(x, y) \in \mathcal{E}$ that

$$\Upsilon\left(\frac{d\sigma_{\rho_t, j_t}}{d|\sigma_{\rho_t, j_t}|}(x, y)\right)|\sigma_{\rho_t, j_t}|(x, y) = \begin{cases}\Upsilon\left(\frac{\sigma_{\rho_t, j_t}(x,y)}{|\sigma_{\rho_t, j_t}|(x,y)}\right)|\sigma_{\rho_t, j_t}|(x, y) & \text{if } |\sigma_{\rho_t, j_t}|(x, y) \neq 0\,, \\ 0 & \text{otherwise}\,.\end{cases}$$

From the definition of σ_{ρ_t,j_t}, it is straightforward that $t \mapsto \sigma_{\rho_t,j_t}(x,y)$ and $t \mapsto |\sigma_{\rho_t,j_t}|(x,y)$ are measurable. Therefore, the set $\{t \in [0,1] : |\sigma_{\rho_t,j_t}|(x,y) \neq 0\}$ is a Borel set and it follows by basic measure theory that the map

$$\{t \in [0,1] : |\sigma_{\rho_t,j_t}|(x,y) \neq 0\} \ni t \mapsto \frac{\sigma_{\rho_t,j_t}(x,y)}{|\sigma_{\rho_t,j_t}|(x,y)},$$

is Borel measurable. Since Υ is lower semicontinuous, it is also Borel measurable, and it follows that

$$\{t \in [0,1] : |\sigma_{\rho_t,j_t}|(x,y) \neq 0\} \ni t \mapsto \Upsilon\left(\frac{\sigma_{\rho_t,j_t}(x,y)}{|\sigma_{\rho_t,j_t}|(x,y)}\right)|\sigma_{\rho_t,j_t}|(x,y),$$

is Borel measurable. By extending this map by zero on $[0,1] \setminus \{t \in [0,1] : |\sigma_{\rho_t,j_t}|(x,y) \neq 0\}$, it follows that

$$[0,1] \ni t \mapsto \Upsilon\left(\frac{d\sigma_{\rho_t,j_t}}{d|\sigma_{\rho_t,j_t}|}(x,y)\right)|\sigma_{\rho_t,j_t}|(x,y),$$

is Borel measurable. Hence, as a finite sum of measurable functions is again measurable, the measurability of $t \mapsto \mathcal{R}(\rho_t, j_t)$ follows.

The measurability of $t \mapsto \mathcal{R}_c(\rho_t, j_t)$ follows by a similar argument. □

We define our candidate for the Γ-limit of $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Definition 3.2.6 (Limiting action)**.** Let $\mu, \nu \in \mathcal{M}^+(V)$. We define the $\mathcal{R}_\infty$-action, or limiting action, $\mathcal{F}_\infty : \mathscr{CE}(\mu \to \nu) \longrightarrow [0, +\infty]$ by

$$\mathcal{F}_\infty(\rho, j) = \begin{cases} \int_0^1 |j_t|(\mathcal{E})\, dt & \text{if } j_t \ll \vartheta_\pi \text{ for a.e. } t \in [0,1], \\ +\infty & \text{otherwise}. \end{cases}$$

Remark. Intuitively speaking, if $\mathcal{F}_\infty$ is finite it only looks at the flux between neighbours, because $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$.

The topology on $\mathscr{CE}([0,\tau])$

Before giving a statement about Γ-convergence, we need to specify the topology we endow the domain of the functionals with. Since we want to use the convenient sequential characterization of Γ-convergence, see Lemma C.4, we choose a metrizable topology.

Let $\tau > 0$. For the curves $t \mapsto \rho_t \in \mathcal{M}^+(V)$ we choose the space $C([0,\tau], (\mathcal{M}^+(V), \mathrm{wk}^*))$ endowed with the (metrizable) topology of uniform converge, which we denote by $\mathcal{C}_{V,\tau}$. For the fluxes $t \mapsto j_t \in \mathcal{M}(\mathcal{E})$ we choose the normed vector space

$$L^1[0,\tau] := \mathcal{L}^1[0,\tau]/\sim := \left\{h : [0,\tau] \longrightarrow \mathcal{M}(\mathcal{E}) : h \text{ is Borel}, \int_0^\tau \|h_t\|_{\mathrm{TV}(\mathcal{E})}\, dt < +\infty\right\} \Big/ \sim,$$

with topology induces by the norm given by

$$\|[j]\|_{L^1[0,\tau]} := \|j\|_{\mathcal{L}^1[0,\tau]} = \int_0^\tau \|j_t\|_{\mathrm{TV}(\mathcal{E})} dt.$$

Let us explain these topologies in more detail. Let $(\rho, j) \in \mathscr{CE}([0,\tau])$. Then $\rho \in C([0,\tau], (\mathcal{M}^+(V), \mathrm{wk}^*))$ by (1) of Definition 2.3.2. We endow this space with topology of uniform convergence in the following way. It is straightforward that the topological space $(\mathcal{M}^+(V), \mathrm{wk}^*)$ is metrizable, because V is a finite set, and that it is induced by the ℓ^1-norm on V,

$$\|\sigma\|_{\ell^1(V)} := \sum_{x\in V} |\sigma(x)| , \qquad \sigma \in \mathcal{M}^+(V) .$$

Since $t \mapsto \|\rho_t\|_{\ell^1(V)}$ is continuous, it follows that $\sup_{t\in[0,\tau]} \|\rho_t\|_{\ell^1(V)} < +\infty$ and that

$$\|\rho\|_{\mathcal{C}_{V,\tau}} := \sup_{t\in[0,\tau]} \|\rho_t\|_{\ell^1(V)} ,$$

defines the supremum-norm, and thus induces the metric $d_{\mathcal{C}_{V,\tau}}$ of uniform convergence on $C([0,\tau], (\mathcal{M}^+(V), \mathrm{wk}^*))$. Convergence of sequences in $\mathcal{C}_{V,\tau}$ is characterized as follows. Let $\rho \in \mathcal{C}_{V,\tau}$ and $\{\rho^n\}_n$ a sequence in $\mathcal{C}_{V,\tau}$. Then $\rho^n \to \rho$ in $\mathcal{C}_{V,\tau}$ as $n \to \infty$ if and only if $\rho^n_t \overset{*}{\rightharpoonup} \rho_t$ as $n \to \infty$, uniformly in t.

By (2) of Definition 2.3.2 the flux is contained in the set

$$\mathcal{L}^1[0,\tau] := \left\{ h : [0,\tau] \longrightarrow \mathcal{M}(\mathcal{E}) : h \text{ is Borel, } \int_0^\tau \|h_t\|_{\mathrm{TV}(\mathcal{E})}\, dt < +\infty \right\} .$$

It is easily seen that

$$\|j\|_{\mathcal{L}^1[0,\tau]} := \int_0^\tau \|j_t\|_{\mathrm{TV}(\mathcal{E})}\, dt ,$$

defines a semi-norm, and therefore that $\mathcal{L}^1[0,\tau]$ is a vector space. Passing to equivalence classes, it is straightforward that $\|[j]\|_{L^1[0,\tau]} = \|j\|_{\mathcal{L}^1[0,\tau]}$ defines a norm on $L^1[0,\tau] = \mathcal{L}^1[0,\tau]/\sim$, making $(L^1[0,\tau], \|\cdot\|_{L^1}[0,\tau])$ into a normed vector space, and thus a metric space with metric $d_{L^1[0,\tau]}(j,h) := \|j-h\|_{L^1[0,\tau]}$. In the remainder of the thesis, whenever we refer to L^1, it is equipped with the topology induced by $\|\cdot\|_{L^1[0,\tau]}$.

Notationally, we will make no distinction between $j \in \mathcal{L}^1[0,\tau]$ and $[j] \in L^1[0,\tau]$; both will be denoted by j. Having introduced $\mathcal{C}_{V,\tau}$ and $L^1[0,\tau]$, we see that $\mathscr{CE}([0,\tau]) \subset \mathcal{C}_{V,\tau} \times \mathcal{L}^1[0,\tau]$ and, passing to equivalence classes, that $\mathscr{CE}([0,\tau]) \subset \mathcal{C}_{V,\tau} \times L^1[0,\tau]$. Here we do not introduce new notation for $\mathscr{CE}([0,\tau])$ after passing to equivalence classes. In the remaining of the chapter we view $\mathscr{CE}([0,\tau])$ as a subspace of the metric space $\mathcal{C}_{V,\tau} \times L^1([0,\tau])$. Furthermore, we will write $\mathcal{C}_V := \mathcal{C}_{V,1}$ and $L^1 := L^1[0,1]$.

Now we are able to state the main result of this section.

Theorem 3.2.7. *Let $\mu, \nu \in \mathcal{M}^+(V)$. The sequence $(\mathcal{F}_n)_n$ of functionals Γ-converges to $\mathcal{F}_\infty : \mathscr{CE}(\mu \to \nu) \longrightarrow [0,+\infty]$.*

The proof uses the sequential characterization of Γ-convergence of Lemma C.4, which consists of two conditions. Namely, let X be as topological space that satisfies the first axiom of countability. Then a sequence $F_n : X \longrightarrow \overline{\mathbb{R}}$ sequentially Γ-converges to $F : X \longrightarrow \overline{\mathbb{R}}$ if and only if the following conditions are satisfied:

(i) [*Liminf inequality*] for every $x \in X$ and sequence $(x_n)_n$ in X that converges to x, it holds that

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n)\,;$$

(ii) [*Limsup inequality*] for every $x \in X$ there exists a sequence $(x_n)_n$ in X that converges to x such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)\,.$$

For our result, (i) is proved in Section 3.3 and (ii) in Section 3.4.

Proof. Let $(\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$ and $\{(\rho^n, j^n)\}_n$ a sequence in $\mathscr{C}\mathscr{E}(\mu \to \nu)$ such that $(\rho^n, j^n) \to (\rho, j)$ as $n \to \infty$. By invoking Theorem 3.3.1 it follows that

$$\mathcal{F}_\infty(\rho, j) \leq \liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n)\,.$$

Hence, the liminf inequality is satisfied.

For the limsup inequality, let $(\widetilde{\rho}, \widetilde{j}) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$. By Corollary 3.4.1 there exists a sequence $(\widetilde{\rho}^n, \widetilde{j}^n) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$ such that $(\widetilde{\rho}^n, \widetilde{j}^n) \to (\widetilde{\rho}, \widetilde{j})$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \mathcal{F}_n(\widetilde{\rho}^n, \widetilde{j}^n) \leq \mathcal{F}_\infty(\widetilde{\rho}, \widetilde{j})\,.$$

This concludes the proof. □

3.3 The liminf inequality

In this section we prove the liminf inequality for the sequence $(\mathcal{F}_n)_n$ and the limiting functional $\mathcal{F}_\infty$, which is used in the proof of Theorem 3.2.7.

Theorem 3.3.1 (Liminf inequality)**.** *Let $(\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$ and $\{(\rho^n, j^n)\}_n$ a sequence in $\mathscr{C}\mathscr{E}(\mu \to \nu)$ such that $(\rho^n, j^n) \to (\rho, j)$ as $n \to \infty$. Then it holds that*

$$\mathcal{F}_\infty(\rho, j) \leq \liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n)\,.$$

The approach of the proof is based on pointwise limits of the rescaled action densities. A key ingredient of the proof of Theorem 3.3.1 is the following. In the proof of this proposition we use the notion of Γ-convergence for topological spaces as introduced in Definition C.1.

Proposition 3.3.2. *The Γ-limit of the sequence $(\Upsilon_c)_{c>1}$ as $c \to \infty$ exists and is equal to the map $\Upsilon_\infty : [0,\infty) \times [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\Upsilon_\infty(u, v, w) = |w|$.*

Proof. Let $(u_0, v_0, w_0) \in [0,\infty) \times [0,\infty) \times \mathbb{R} =: X$, where X is endowed with the subspace topology, induced by the Euclidean topology on $\mathbb{R}^3$, as usual. We calculate the lower and upper Γ-limits of (Υ_c) from their definitions

$$\left(\Gamma - \liminf_{c\to\infty} \Upsilon_c\right)(u_0, v_0, w_0) = \sup_{U\in\mathcal{B}(u_0,v_0,w_0)} \liminf_{c\to\infty} \inf_{(u,v,w)\in U} \Upsilon_c(u,v,w)\,,$$
$$\left(\Gamma - \limsup_{c\to\infty} \Upsilon_c\right)(u_0, v_0, w_0) = \sup_{U\in\mathcal{B}(u_0,v_0,w_0)} \limsup_{c\to\infty} \inf_{(u,v,w)\in U} \Upsilon_c(u,v,w)\,,$$

where $\mathcal{B}(u_0, v_0, w_0)$ is any suitable basis of open neighbourhoods of (u_0, v_0, w_0), see Definition C.1.

1. The case that $u_0v_0 > 0$ and $w_0 \neq 0$. Let $\varepsilon_0 > 0$ be such that

$$(u_0 - \varepsilon_0, u_0 + \varepsilon_0) \times (v_0 - \varepsilon_0, v_0 + \varepsilon_0) \times (w_0 - \varepsilon_0, w_0 + \varepsilon_0) \subseteq (0,\infty) \times (0,\infty) \times (\mathbb{R} \setminus \{0\})\,.$$

In this case we choose

$$\mathcal{B}(u_0, v_0, w_0) = \{U_\varepsilon \mid 0 < \varepsilon \le \varepsilon_0\}\,,$$

where $U_\varepsilon := (u_0 - \varepsilon, u_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon) \times (w_0 - \varepsilon, w_0 + \varepsilon)$. Define

$$\alpha^\varepsilon_{\min} = \inf \alpha((u_0 - \varepsilon, u_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon))$$
$$\alpha^\varepsilon_{\max} = \sup \alpha((u_0 - \varepsilon, u_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon))\,.$$

Then $\alpha((u_0 - \varepsilon, u_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon)) = (\alpha^\varepsilon_{\min}, \alpha^\varepsilon_{\max})$ and

$$\inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) = \inf\left\{\frac{\xi}{c\log c}\Psi\left(\frac{c\eta}{\xi}\right) \;:\; (\xi,\eta) \in (\alpha^\varepsilon_{\min}, \alpha^\varepsilon_{\max}) \times (w_0 - \varepsilon, w_0 + \varepsilon)\right\}\,.$$

Because Ψ is strictly increasing and even, it follows that

$$\frac{\xi}{c\log c}\Psi\left(\frac{c\eta}{\xi}\right) > \frac{\xi}{c\log c}\Psi\left(\frac{c}{\xi}\inf\left(|w_0| - \varepsilon, |w_0| + \varepsilon\right)\right) = \frac{\xi}{c\log c}\Psi\left(\frac{c(|w_0| - \varepsilon)}{\xi}\right)\,,$$

for all $(\xi,\eta) \in (\alpha^\varepsilon_{\min}, \alpha^\varepsilon_{\max}) \times (w_0 - \varepsilon, w_0 + \varepsilon)$. By Lemma 3.1.3 the right hand side above is strictly decreasing in ξ, so we obtain the inequality

$$\frac{\xi}{c\log c}\Psi\left(\frac{c\eta}{\xi}\right) > \frac{\alpha^\varepsilon_{\max}}{c\log c}\Psi\left(\frac{c(|w_0| - \varepsilon)}{\alpha^\varepsilon_{\max}}\right)\,,$$

for all $(\xi,\eta) \in (\alpha^\varepsilon_{\min}, \alpha^\varepsilon_{\max}) \times (w_0 - \varepsilon, w_0 + \varepsilon)$, so

$$\inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) \ge \frac{\alpha^\varepsilon_{\max}}{c\log c}\Psi\left(\frac{c(|w_0| - \varepsilon)}{\alpha^\varepsilon_{\max}}\right)\,.$$

By continuity of the left hand side at $(\alpha^\varepsilon_{\max}, |w_0| - \varepsilon)$, the reverse inequality also holds and

$$\inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) = \frac{\alpha^\varepsilon_{\max}}{c\log c}\Psi\left(\frac{c(|w_0| - \varepsilon)}{\alpha^\varepsilon_{\max}}\right)\,.$$

By Lemma 3.1.1 we obtain

$$\lim_{c\to\infty} \inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) = \lim_{c\to\infty} \frac{\alpha^\varepsilon_{\max}}{c\log c}\Psi\left(\frac{c(|w_0|-\varepsilon)}{\alpha^\varepsilon_{\max}}\right) = |w_0| - \varepsilon\,.$$

It follows that

$$\begin{aligned}\left(\Gamma - \liminf_{c\to\infty} \Upsilon_c\right)(u_0,v_0,w_0) &= \sup_{0<\varepsilon\le\varepsilon_0} \lim_{c\to\infty} \inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) \\ &= \left(\Gamma - \limsup_{c\to\infty} \Upsilon_c\right)(u_0,v_0,w_0) \\ &= \sup_{0<\varepsilon\le\varepsilon_0} |w_0| - \varepsilon \\ &= |w_0|\,.\end{aligned}$$

Hence, if $u_0v_0 > 0$ and $w_0 \neq 0$ the Γ-limit Υ_∞ exists and $\Upsilon_\infty(u_0,v_0,w_0) = |w_0|$.

2. The case that $u_0v_0 = 0$ and $w_0 \neq 0$. Let $\varepsilon_0 > 0$ such that $(w_0-\varepsilon_0, w_0+\varepsilon) \subset \mathbb{R}\setminus\{0\}$. We choose the neighbourhood basis $\mathcal{B}(u_0,v_0,w_0) = \{U_\varepsilon \mid 0 < \varepsilon \le \varepsilon_0\}$, where U_ε is defined by:

(i) $U_\varepsilon = [0,\varepsilon) \times (v_0-\varepsilon, v_0+\varepsilon) \times (w_0-\varepsilon, w_0+\varepsilon)$ if $u_0 = 0$ and $v_0 > 0$;

(ii) $U_\varepsilon = (u_0-\varepsilon, u_0+\varepsilon) \times [0,\varepsilon) \times (w_0-\varepsilon, w_0+\varepsilon)$ if $u_0 \neq 0$ and $v_0 = 0$;

(iii) $U_\varepsilon = [0,\varepsilon) \times [0,\varepsilon) \times (w_0-\varepsilon, w_0+\varepsilon)$ if both $u_0 = 0$ and $v_0 = 0$.

Define the projection $p_{12} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $p_{12}(u,v,w) = (u,v)$. Then $\alpha(p_{12}(U_\varepsilon)) = [0, \alpha^\varepsilon_{\max})$, where $\alpha^\varepsilon_{\max} = \sup \alpha(p_{12}(U_\varepsilon))$. Note that $\Upsilon_c(u,v,w) = +\infty$ if $\alpha(u,v) = 0$ and $w \in (w_0-\varepsilon, w_0+\varepsilon)$. Therefore, it follows that

$$\inf_{(u,v,w)\in U_\varepsilon} \Upsilon_c(u,v,w) = \inf\left\{\frac{\xi}{c\log c}\Psi\left(\frac{c\eta}{\xi}\right) \,:\, (\xi,\eta) \in (0,\alpha^\varepsilon_{\max}) \times (w_0-\varepsilon, w_0+\varepsilon)\right\}.$$

Now we can proceed as in the first case to conclude that the Γ-limit Υ_∞ at (u_0,v_0,w_0) exists and $\Upsilon_\infty(u_0,v_0,w_0) = |w|$.

3. The case that $w_0 = 0$. In this case we have for any open neighbourhood U of (u_0,v_0,w_0) and any $c > 1$ we have $\inf_{(u,v,w)\in U} \Upsilon_c(u,v,w) = 0$, because $(u',v',0) \in U$ for some $u',v' \in [0,\infty)$. Therefore, it follows that

$$\left(\Gamma - \liminf_{c\to\infty} \Upsilon_c\right)(u_0,v_0,w_0) = 0 = \left(\Gamma - \limsup_{c\to\infty} \Upsilon_c\right)(u_0,v_0,w_0)\,.$$

Hence, if $w_0 = 0$ the Γ-limit Υ_∞ exists and $\Upsilon_\infty(u_0,v_0,w_0) = |w_0|$.

□

Finally, we give the proof of Theorem 3.3.1 of which Proposition 3.3.2 and Fatou's Lemma [3, Theorem 2.8.3] are the main ingredients.

Proof of Theorem 3.3.1. First, assume that we do not have $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$. We claim that $\liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) = +\infty$. Namely, suppose that $\liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) < +\infty$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) = \liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n)\,,$$

and $\sup_{k\in\mathbb{N}} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) < +\infty$.

Furthermore,

$$\lim_{k\to\infty} \int_0^1 \|j_t^{n_k} - j_t\|_{\mathrm{TV}} dt = \lim_{k\to\infty} \|j^{n_k} - j\|_{L^1} = 0\,,$$

so $t \mapsto \|j_t^{n_k} - j_t\|_{\mathrm{TV}}$ converges to zero in $L^1[0,1]$. Therefore, there exists a further subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \|j_t^{n_{k_l}} - j_t\|_{\mathrm{TV}} = 0$ for almost all $t \in [0,1]$. Because convergence in total variation norm implies weak*-convergence, it follows that $j_t^{n_{k_l}} \overset{*}{\rightharpoonup} j_t$ as $l \to \infty$ for almost all $t \in [0,1]$.

Since $\sup_{k\in\mathbb{N}} \mathcal{F}_{n_k}((\rho_t^{n_k}, j_t^{n_k})_t) < +\infty$, it holds that $j_t^{n_k} \ll \vartheta_\pi$ for all $k \in \mathbb{N}$ for almost all $t \in [0,1]$ by Lemma 3.2.2.

The set V is finite, so $\mathbb{1}_A \in C_0(\mathcal{E})$ and consequently $j_t^{n_k}(A) \to j_t(A)$ as $l \to \infty$ for all $A \in \mathcal{E}$ and almost all $t \in [0,1]$. Let $A \in \mathcal{E}$ such that $\vartheta_\pi(A) = 0$. Then $j_t^{n_k}(A) = 0$ for all $k \in \mathbb{N}$, and consequently $j_t(A) = 0$ for almost all $t \in [0,1]$. This implies that $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$. This contradicts the assumption and hence proves the claim.

Thus, we may assume that $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$ and $\liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) < +\infty$. So there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) = \liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n)\,,$$

and $\sup_{k\in\mathbb{N}} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) < +\infty$. Since $j^n \to j$ in L^1 as $n \to \infty$, it holds that

$$\lim_{k\to\infty} \int_0^1 \|j_t^{n_k} - j_t\|_{\mathrm{TV}} dt = \lim_{k\to\infty} \|(j_t^{n_k})_t - (j_t)\|_{\mathcal{L}^1} = 0\,.$$

Therefore, there exists a further subsequence $(n_{k_l})_{l\in\mathbb{N}}$ such that $j_t^{n_{k_l}} \overset{*}{\rightharpoonup} j_t$ as $l \to \infty$ for almost all $t \in [0,1]$. Since $\sup_{k\in\mathbb{N}} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) < +\infty$, it holds that $j_t^{n_k} \ll \vartheta_\pi$ for all $k \in \mathbb{N}$ for almost all $t \in [0,1]$. Therefore, $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$.

The set V is discrete, so $\mathbb{1}_{(x,y)} \in C_0(\mathcal{E})$ and consequently $w_t^{n_{k_l}}(x,y) \to w_t(x,y)$ as $l \to \infty$ for all $(x,y) \in \mathcal{E}$ with $\vartheta_\pi(x,y) \neq 0$ and almost all $t \in [0,1]$.

Furthermore, because $\mathbb{1}_x \in C_0(V)$, $\rho_t^n \overset{*}{\rightharpoonup} \rho_t$ as $n \to \infty$ and $\pi(x) > 0$ for all $x \in V$, we have that $u_t^{n_{k_l}}(x) \to u_t(x)$ as $l \to \infty$ for all $x \in V$ and $t \in [0,1]$. By Lemma 3.3.2 and Proposition C.2, it holds that the subsequence $\Upsilon_{n_{k_l}}$ Γ-converges to Υ_∞ as $l \to \infty$, and

$$\Upsilon_\infty(u_t(x), u_t(y), w_t(x,y)) \le \liminf_{l\to\infty} \Upsilon_{n_{k_l}}(u_t^{n_{k_l}}(x), u_t^{n_{k_l}}(y), w_t^{n_{k_l}}(x,y))\,,$$

for all $(x, y) \in \mathcal{E}$ with $\vartheta_\pi(x, y) \neq 0$ and almost all $t \in [0, 1]$. It follows that

$$\begin{aligned}
\mathcal{F}_\infty((\rho_t, j_t)_t) &= \int_0^1 \iint_{\mathcal{E}} \Upsilon_\infty(u_t(x), u_t(y), w_t(x, y)) d\vartheta_\pi(x, y) dt \\
&\leq \int_0^1 \iint_{\mathcal{E}} \liminf_{l\to\infty} \Upsilon_{n_{k_l}}(u_t^{n_{k_l}}(x), u_t^{n_{k_l}}(y), w_t^{n_{k_l}}(x, y)) d\vartheta_\pi(x, y) dt \\
&\leq \liminf_{l\to\infty} \int_0^1 \iint_{\mathcal{E}} \Upsilon_{n_{k_l}}(u_t^{n_{k_l}}(x), u_t^{n_{k_l}}(y), w_t^{n_{k_l}}(x, y)) d\vartheta_\pi(x, y) dt \\
&= \lim_{l\to\infty} \mathcal{F}_{n_{k_l}}(\rho^{n_{k_l}}, j^{n_{k_l}}) \\
&= \lim_{k\to\infty} \mathcal{F}_{n_k}(\rho^{n_k}, j^{n_k}) \\
&= \liminf_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) ,
\end{aligned}$$

where we used Fatou's lemma for the second inequality. □

We have the following lower bound of the rescaled action.

Proposition 3.3.3 (Lower bound of $\mathcal{F}_n$)**.** *Let $n \geq 2$ and $(\rho, j) \in \mathscr{CE}([0, 1])$. Then*

$$\mathcal{F}_n(\rho, j) \geq \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \mathcal{F}_\infty(\rho, j)\right) \geq \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \inf \mathcal{F}_\infty\right) .$$

Proof. We may assume that (ρ, j) has finite $\mathcal{R}_n$-action. Otherwise, $\mathcal{F}_n(\rho, j) = +\infty$ and the statement holds true. So $j_t \ll \vartheta_\pi$ for almost all $t \in [0, 1]$. By Lemma 2.3.9 it follows that

$$\begin{aligned}
\mathcal{F}_n(\rho, j) &= \frac{1}{n \log n} \int_0^1 \mathcal{R}(\rho_t, n j_t) dt \\
&\geq \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \int_0^1 \|j_t\|_{\mathrm{TV}(\mathcal{E})} dt\right) \\
&= \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \mathcal{F}_\infty(\rho, j)\right) \\
&\geq \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \inf \mathcal{F}_\infty\right) ,
\end{aligned}$$

which concludes the proof. □

3.4 The limsup inequality

In this section we state and prove two main ingredients for the limsup inequality that we need for Theorem 3.2.7.

The first of these results, Corollary 3.4.3, tells us that the $\mathcal{R}_n$-action of curves of finite $\mathcal{R}$-action converges to the $\mathcal{R}_\infty$-action. The second main result, Proposition 3.4.7, shows that the set

$$\mathscr{A}_\tau(\mu \to \nu) = \left\{ (\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\mu \to \nu) : \int_0^\tau \mathcal{R}(\rho_t, j_t) dt < +\infty \right\},$$

is dense in $\mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$.

These two results are combined through the diagonal argument Lemma C.5 to prove the limsup inequality.

Corollary 3.4.1 (Limsup inequality)**.** *Let $(\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$. Then there exists a sequence $\{(\rho^n, j^n)\}_n$ in $\mathscr{C}\mathscr{E}(\mu \to \nu)$ such that*

$$\limsup_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) \leq \mathcal{F}_\infty(\rho, j).$$

Proof. Let $(\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$. If the set of $t \in [0, 1]$ for which $j_t \not\ll \vartheta_\pi$ has positive Lebesgue measure, then $\mathcal{F}_\infty(\rho, j) = +\infty$. Therefore,

$$\limsup_{n\to\infty} \mathcal{F}_n(\rho, j) \leq \mathcal{F}_\infty(\rho, j),$$

and the limsup inequality holds for the constant sequence (ρ, j).

If $j_t \ll \vartheta_\pi$ for almost all $t \in [0, 1]$, then by Proposition 3.4.7 there exists a sequence $\{(\widetilde{\rho}^n, \widetilde{j}^n)\}_n$ of curves in $\mathscr{C}\mathscr{E}(\mu \to \nu)$ with finite $\mathcal{R}$-action such that $(\widetilde{\rho}^n, \widetilde{j}^n) \to (\rho, j)$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \mathcal{F}_\infty(\widetilde{\rho}^n, \widetilde{j}^n) = \lim_{n\to\infty} \|\widetilde{j}^n\|_{\mathcal{L}^1} = \|j\|_{\mathcal{L}^1} = \mathcal{F}_\infty(\rho, j).$$

By Corollary 3.4.3 it holds that

$$\lim_{m\to\infty} \mathcal{F}_m(\widetilde{\rho}^n, \widetilde{j}^n) = \mathcal{F}_\infty(\widetilde{\rho}^n, \widetilde{j}^n),$$

for every $n \in \mathbb{N}$. Hence, by Lemma C.5, there exists a sequence $\{(\rho^n, j^n)\}_n$ in $\mathscr{C}\mathscr{E}(\mu \to \nu)$ such that $(\rho^n, j^n) \to (\rho, j)$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) \leq \mathcal{F}_\infty(\rho_t, j_t),$$

proves the limsup inequality and concludes the proof. □

3.4.1 The limiting action of curves with finite $\mathcal{R}$-action

As preparation for the proof of the first main result, we deduce an upper bound of the rescaled action.

Proposition 3.4.2 (Upper bound of $\mathcal{F}_n$)**.** *Let $n \geq 2$ and $(\rho, j) \in \mathscr{CE}([0,1])$ with $\rho_t \ll \pi$ for a.e. $t \in [0,1]$. Then*

$$\mathcal{F}_n(\rho, j) \leq \mathcal{F}_\infty(\rho, j) + \frac{1}{\log n} \int_0^1 \mathcal{R}(\rho_t, j_t) dt\,.$$

Proof. If (ρ, j) does not have finite $\mathcal{R}$-action, then the inequality is satisfied. Thus, we may assume that (ρ, j) has finite $\mathcal{R}$-action. By Lemma 2.2.6 it holds that $j_t \ll \vartheta_\pi$ and $w_t = 0$ ϑ_π-a.e. on $\mathcal{E} \setminus \mathcal{E}_{u_t}$, for a.a. $t \in [0,1]$, and it follows that

$$\begin{aligned}
\mathcal{F}_n(\rho, j) &= \int_0^1 \iint_{\mathcal{E}_{u_t}} \frac{\alpha_t}{n \log n} \Psi\left(\frac{n|w_t|}{\alpha_t}\right) d\vartheta_\pi \, dt \\
&\leq \int_0^1 \iint_{\mathcal{E}_{u_t}} \left[|w_t| + \frac{\alpha_t}{\log n} \Psi\left(\frac{|w_t|}{\alpha_t}\right)\right] d\vartheta_\pi \, dt \\
&\leq \mathcal{F}_\infty(\rho, j) + \frac{1}{\log n} \int_0^1 \mathcal{R}(\rho_t, j_t) dt\,,
\end{aligned}$$

where we used Lemma 3.1.4 for the first inequality. □

Corollary 3.4.3. *Let $(\rho, j) \in \mathscr{CE}([0,1])$ with $\rho_t \ll \pi$ for a.e. $t \in [0,1]$ and finite $\mathcal{R}$-action. Then it holds that*

$$\lim_{n \to \infty} \mathcal{F}_n(\rho, j_t) = \mathcal{F}_\infty(\rho, j)\,.$$

Proof. Since the $\mathcal{R}$-action of (ρ, j) is finite, it follows by Propositions 3.3.3 and 3.4.2 that the $\mathcal{R}_n$- and $\mathcal{R}_\infty$-actions of (ρ, j) are finite and

$$\frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \mathcal{F}_\infty(\rho, j)\right) \leq \mathcal{F}_n(\rho, j) \leq \mathcal{F}_\infty(\rho, j) + \frac{1}{\log n} \int_0^1 \mathcal{R}(\rho_t, j_t) dt\,.$$

It follows, using Lemma 3.1.1, that

$$\limsup_{n \to \infty} \mathcal{F}_n(\rho, j) \leq \mathcal{F}_\infty(\rho, j) \leq \liminf_{n \to \infty} \mathcal{F}_n(\rho, j)\,,$$

which concludes the proof. □

3.4.2 Density of curves with finite $\mathcal{R}$-action

The remainder of the section is dedicated to the second main result. The first preliminary result about quadratic interpolating curves will be important tool in the proof of Proposition 3.4.7, and is followed by an auxiliary result.

Lemma 3.4.4. *Let $\tau > 0$ and V be a finite set. Let $\mu_1, \mu_2 \in \mathcal{M}^+(V)$ with $\mu_1(V) = \mu_2(V)$ such that $\mu_2(x) > 0$ for all $x \in V$. Define the curve $\{(\sigma_t, h_t)\}_{t \in [0,\tau]}$ by*

$$\sigma_t := \frac{\mu_2 - \mu_1}{\tau^2} t^2 + \mu_1\,, \qquad h_t := \frac{2t}{\tau^2} \widetilde{h} \mathbb{1}_{\mathcal{E}_N}\,,$$

where $\widetilde{h} : \mathcal{E}_N \longrightarrow \mathbb{R}$ is a solution of the system in Lemma 3.4.5 with $c(x) = \mu_1(x) - \mu_2(x)$. This curve:

(i) is a solution of the continuity equation in the strong sense with $\sigma_0 = \mu_1$ and $\sigma_\tau = \mu_2$;

(ii) has finite $\mathcal{R}$-action.

Proof. Remark that for any $x, y \in V$ the maps $t \mapsto \sigma_t(x)$ and $t \mapsto h_t(x,y)$ are smooth. It follows that

$$\partial_t \sigma_t(x) + \overline{\operatorname{div}}\, h_t(x) = \frac{2t}{\tau^2}\left(\mu_2(x) - \mu_1(x) + \sum_{y \in V} [h(x,y) - h(y,x)]\right) = 0\,,$$

and hence that $\{(\sigma_t, h_t)\}_{t\in[0,\tau]}$ is a solution of the continuity equation in the strong sense. Because $\sigma_0 = \mu_1$ and $\sigma_\tau = \mu_2$, *(i)* follows.

For *(ii)* remark that $h_t \ll \vartheta_\pi$ for all $t \in [0,\tau]$ by definition. We define

$$s_t := \frac{d\sigma_t}{d\pi}\,, \qquad \eta_t := \frac{dh_t}{d\vartheta_\pi}\,.$$

We claim that for any $x \in V$ it holds that $s_t(x) > 0$ for all $t \in (0,\tau]$. Namely, if $\mu_2(x) - \mu_1(x) > 0$ it holds that

$$s_t(x) \geq \frac{t^2}{\tau^2}\frac{d(\mu_2 - \mu_1)}{d\pi}(x) > 0\,.$$

If $\mu_2(x) - \mu_1(x) < 0$, then

$$s_t(x) \geq \left(\frac{d\mu_2}{d\pi}(x) - \frac{d\mu_1}{d\pi}(x)\right) + \frac{d\mu_1}{d\pi}(x) = \frac{d\mu_2}{d\pi}(x) > 0\,.$$

Finally, if $\mu_1(x) = \mu_2(x)$, then $s_t(x) = \frac{d\mu_1}{d\pi}(x) = \frac{d\mu_2}{d\pi}(x) > 0$. Therefore, it follows, using the definition of Υ, that

$$\begin{aligned}\int_0^\tau \mathcal{R}(\sigma_t, h_t)dt &= \int_0^\tau \iint_{\mathcal{E}} \Upsilon(s_t(x), s_t(y), |\eta_t(x,y)|)d\vartheta_\pi(x,y)\,dt \\ &= \int_0^\tau \iint_{\mathcal{E}} \alpha(s_t(x), s_t(y))\Psi\left(\frac{|\eta_t(x,y)|}{\alpha(s_t(x), s_t(y))}\right) d\vartheta_\pi(x,y)\,dt\,,\end{aligned}$$

which we will show to be integrable. Writing $\alpha_t := \alpha(s_t(x), s_t(y))$ and $\eta_t := \eta_t(x,y)$, we have for $0 < t \leq \tau$ that

$$\begin{aligned}\alpha_t \Psi\left(\frac{|\eta_t|}{\alpha_t}\right) &\leq \frac{|\eta_t|^2}{\alpha_t} \\ &= \frac{t^2|h(x,y)|^2}{\tau^2\vartheta_\pi(x,y)^2\sqrt{((s_\tau(x) - s_0(x))t^2/\tau^2 + s_0(x))((s_\tau(y) - s_0(y))t^2/\tau^2 + s_0(y))}} \\ &= \frac{|h(x,y)|^2}{\vartheta_\pi(x,y)^2\sqrt{((s_\tau(x) - s_0(x)) + s_0(x)\tau^2/t^2)((s_\tau(y) - s_0(y)) + s_0(y)\tau^2/t^2)}} \\ &\leq \frac{|h(x,y)|^2}{\vartheta_\pi(x,y)^2\sqrt{s_\tau(x)s_\tau(y)}}\,,\end{aligned}$$

where we used that $\tau^2/t^2 \geq 1$ for the final inequality. It follows

$$\begin{aligned}\int_0^\tau \iint_{\mathcal{E}} \frac{|\eta_t|^2}{\alpha_t} d\vartheta_\pi \, dt &= \int_0^\tau \sum_{(x,y)\in\mathcal{E}_N} \frac{|\eta_t(x,y)|^2}{\alpha(s_t(x), s_t(y))} \vartheta_\pi(x,y) \, dt \\ &\leq \sum_{(x,y)\in\mathcal{E}_N} \frac{\tau |h(x,y)|^2}{\vartheta_\pi(x,y)\sqrt{s_\tau(x)s_\tau(y)}} \\ &< +\infty \,,\end{aligned}$$

which proves that $\alpha_t \Psi\left(\frac{|\eta_t|}{\alpha_t}\right)$ is integrable. Hence, the curve $\{(\sigma_t, h_t)\}_{t\in[0,\tau]}$ has finite $\mathcal{R}$-action, which concludes the proof. □

Lemma 3.4.5. *Let $|V| \geq 2$ and $c : V \longrightarrow \mathbb{R}$ be a map with $\sum_{x\in V} c(x) = 0$. Consider for the variable $h : \mathcal{E}_N \longrightarrow \mathbb{R}$ the system of linear equations given by*

$$\sum_{y\in V: y\sim x} [h(x,y) - h(y,x)] = c(x) \,, \qquad x \in V \,.$$

Then this system has at least one solution. Moreover, there exists an invertible matrix $A \in \mathbb{R}^{(|V|-1)\times(|V|-1)}$, only depending on $\mathcal{E}_N$ and independent of c, and a solution $\widetilde{h} : \mathcal{E}_N \longrightarrow \mathbb{R}$ such that $\|\widetilde{h}\|_\infty \leq \|A^{-1}\|_\infty \|c\|_\infty$.

Proof. For all $x \in V$ we define

$$E_x := \sum_{y\in V: y\sim x} [h(x,y) - h(y,x)] \,.$$

Define $b_{xy} := h(x,y) - h(y,x)$. Then $E_x = \sum_{y\in V: y\sim x} b_{xy}$ and $b_{xy} = -b_{yx}$. Since $(x,y) \in \mathcal{E}_N$ if and only if $(y,x) \in \mathcal{E}_N$, it follows that

$$\sum_{x\in V} E_x = \sum_{(x,y)\in\mathcal{E}_n} b_{xy} = \sum_{xy \text{ is edge}} (b_{xy} + b_{yx}) = 0 \,.$$

Because it also holds that $\sum_{x\in V} c_x = 0$, we can eliminate one equation. We set $m := |V|$ and enumerate the elements of V by $\{x_1, \ldots, x_m\}$. By the what we have already shown we can eliminate the equation $E_{x_m} = c(x_m)$, and we are left with the system $E_{x_i} = c(x_i)$, $1 \leq i \leq m-1$. The rows of the matrix associated to the system are given by

$$\boldsymbol{E}_{x_i} := \sum_{y\in V: y\sim x_i} \boldsymbol{h}(x_i, y) - \boldsymbol{h}(y, x_i) \,, \qquad 1 \leq i \leq m-1 \,,$$

where the $\{\boldsymbol{h}(x_i, x_j)\}_{1\leq i,j\leq m}$ a basis of $\mathbb{R}^{|V|^2}$. Define $\boldsymbol{e}_{xy} := \boldsymbol{h}(x,y) - \boldsymbol{h}(y,x)$. Then $\boldsymbol{e}_{xy} = -\boldsymbol{e}_{yx}$, $\boldsymbol{E}_{x_i} = \sum_{y\in V: y\sim x_i} \boldsymbol{e}_{x_i y}$ and the set $\{\boldsymbol{e}_{xy} : xy \text{ is an undirected edge}\}$ is linearly independent.

We claim that the set of rows $\{\boldsymbol{E}_{x_i}\}_{1\le i\le m-1}$ is linearly independent. Namely, let $(\lambda_1,\dots,\lambda_{m-1}) \in \mathbb{R}^{m-1}$ such that $\sum_{i=1}^{m-1}\lambda_i\boldsymbol{E}_{x_i} = 0$. Because the graph is connected there exists $x \in V$ such that $x \sim x_m$, say x_{m-1}. Every $\boldsymbol{e}_{xy} = -\boldsymbol{e}_{yx}$ only appears in $\boldsymbol{E}_x$ and $\boldsymbol{E}_y$, so in the sum $\sum_{i=1}^{m-1}\lambda_i\boldsymbol{E}_{x_i}$ only $\boldsymbol{E}_{x_{m-1}}$ contains the term $\boldsymbol{e}_{x_{m-1}x_m}$. Therefore,

$$\sum_{i=1}^{m-1}\lambda_i\boldsymbol{E}_{x_i} = \lambda_{m-1}\boldsymbol{e}_{x_{m-1}x_m} + \boldsymbol{R}_{m-1}\,,$$

where $\boldsymbol{R}_{m-1}$ is the residue, independent of $\boldsymbol{e}_{x_{m-1}x_m}$. By the linear independence of $\{\boldsymbol{e}_{xy} : xy \text{ is an undirected edge}\}$, it follows that $\lambda_{m-1} = 0$ and we are left with $\sum_{i=1}^{m-2}\lambda_i\boldsymbol{E}_{x_i} = 0$. By repeating this procedure for a neighbour of either x_{m-1} or x_m, which exists by connectedness of the graph, it follows that $\lambda_{m-2} = 0$ and $\sum_{i=1}^{m-3}\lambda_i\boldsymbol{E}_{x_i} = 0$. By induction it follows that $\lambda_1 = \lambda_2 = \ldots = \lambda_{m-1} = 0$ and that the set $\{\boldsymbol{E}_{x_i} : 1 \le i \le m-1\}$ is linearly independent. Therefore, the associated matrix has rank $m-1$ and its columns span $\mathbb{R}^{m-1}$. Hence, the system has at least one solution.

For the last assertion, since the set of columns spans $\mathbb{R}^{m-1}$, we can select a set of $m-1$ columns still spanning $\mathbb{R}^{m-1}$. This can be achieved by setting the appropriate $h(x,y) = 0$ and we denote by $\mathcal{E}_0$ the set of such $(x,y) \in \mathcal{E}$. Then we are left with a system of which the associated matrix, say $A \in \mathbb{R}^{(m-1)\times(m-1)}$ is invertible. So the solution $\widetilde{h} : \mathcal{E} \longrightarrow \mathbb{R}$ of $E_{x_i} = c(x_i)$, $1 \le i \le m-1$ is given by $\widetilde{h}(x,y) = 0$ if $(x,y) \in \mathcal{E}_0$ and $(\widetilde{h}(x,y))_{(x,y)\in\mathcal{E}\setminus\mathcal{E}_0} = A^{-1}(c(x_i))_{1\le i\le m-1}$. It follows that

$$\|\widetilde{h}\|_\infty = \max_{(x,y)\in\mathcal{E}\setminus\mathcal{E}_0}|\widetilde{h}(x,y)| = \|A^{-1}(c(x_i))_{1\le i\le m-1}\|_\infty \le \|A^{-1}\|_\infty\|c\|_\infty\,,$$

which concludes the proof. □

As a consequence for any pair $\mu,\nu \in \mathcal{M}^+(V)$, solutions of the continuity equation connecting μ and ν with finite $\mathcal{R}$-action exist.

Corollary 3.4.6. *Let $\tau > 0$. Then for any $\mu,\nu \in \mathcal{M}^+(V)$ with the same mass there exists $(\sigma,h) \in \mathscr{CE}_\tau(\mu\to\nu)$ with finite $\mathcal{R}$-action.*

Proof. Define $\widetilde{\pi} := \frac{\mu(V)}{\pi(V)}\pi$. Then $\widetilde{\pi}$ has the same mass as μ and ν, and $\widetilde{\pi}(x) > 0$ for all $x \in V$. By Lemma 3.4.4 with $\mu_1 = \mu$ and $\mu_2 = \widetilde{\pi}$, there exists $(\sigma^1,h^1) \in \mathscr{CE}([0,\tau/2];\mu,\widetilde{\pi})$ with finite $\mathcal{R}$-action. By the same lemma with $\mu_1 = \nu$, $\mu_2 = \widetilde{\pi}$, there exists $(\sigma^2,j^2) \in \mathscr{CE}([0,\tau/2];\nu,\widetilde{\pi})$ with finite $\mathcal{R}$-action. By time reversal, Lemma 2.3.10, we have that $\mathsf{R}(\sigma^2,h^2) \in \mathscr{CE}([0,\tau/2];\widetilde{\pi},\nu)$. Define the time shift $\Phi : [\tau/2,\tau] \longrightarrow [0,\tau/2]$, $t \mapsto t-\tau/2$. By Lemma 2.3.6, we obtain that $\mathsf{S}_\Phi \circ \mathsf{R}(\widehat{\sigma}^2,\widehat{h}^2) \in \mathscr{CE}([\tau/2,\tau];\widetilde{\pi},\nu)$ and

$$\int_{\tau/2}^{\tau}\mathcal{R}\circ\mathsf{S}_\Phi\circ\mathsf{R}(\widehat{\sigma}^2,\widehat{h}^2)dt = \int_0^{\tau/2}\mathcal{R}\circ\mathsf{R}(\widehat{\sigma}^2,\widehat{h}^2)dt = \int_0^{\tau/2}\mathcal{R}(\widehat{\sigma}^2,\widehat{h}^2)dt < +\infty\,.$$

By Lemma 2.3.15 the concatenation $(\sigma, h) := (\sigma^1, h^1) \odot (\mathsf{S}_\Phi \circ \mathsf{R}(\widehat{\sigma}^2, \widehat{h}^2))$ is an element of $\mathscr{C}\mathscr{E}([0,\tau];\mu,\nu)$ and

$$\begin{aligned}\int_0^\tau \mathcal{R}(\sigma_t, h_t)dt &= \int_0^{\tau/2} \mathcal{R}(\sigma_t^1, h_t^1)dt + \int_{\tau/2}^\tau \mathcal{R} \circ \mathsf{S}_\Phi \circ \mathsf{R}(\widehat{\sigma}^2, \widehat{h}^2)dt \\ &= \int_0^{\tau/2} \mathcal{R}(\sigma_t^1, h_t^1)dt + \int_{\tau/2}^\tau \mathcal{R}(\widehat{\sigma}_t^2, \widehat{h}_t^2)dt \\ &< +\infty\,,\end{aligned}$$

which concludes the proof. □

Now we can prove the second main result of the section.

Proposition 3.4.7. *Let $\tau > 0$ and $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$ with $j_t \ll \vartheta_\pi$ for almost all $t \in [0,\tau]$. Then there exists a sequence $(\rho^n, j^n)_n$ in $\mathscr{A}_\tau(\mu \to \nu)$ that converges to (ρ, j) in the topology of $\mathcal{C}_{V,\tau} \times L^1[0,\tau]$.*

Proof. 1. Definition of $(\rho^n, j^n)_n$. Let $(\rho_t, j_t)_{t\in[0,\tau]} \in \mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$. The elements of the sequence we define will consist of concatenations of two boundary parts, which are of the form of Lemma 3.4.4, and a middle part that arises from the limiting curve (ρ, j) by taking conical combinations and time rescaling. The boundary parts are needed, because the construction of the middle part with finite $\mathcal{R}$-action changes the endpoints of ρ. First we will construct the middle part step by step.

Step (i): bounded solutions. To ensure finite $\mathcal{R}$-action we construct from (ρ, j) a solution that is bounded in an appropriate way. For $t \in [0,\tau]$, $n \in \mathbb{N}$ and $(x,y) \in \mathcal{E}$ we define the Borel families $(j_t^{\le n})_t, (j_t^{>n})_t$ in $L^1[0,\tau]$ by

$$\begin{aligned} j_t^{\le n}(x,y) &= j_t(x,y)\mathbb{1}_{\{s\in[0,1]:|j_t(x,y)|\le n\}}\,, \\ j_t^{>n}(x,y) &= j_t(x,y)\mathbb{1}_{\{s\in[0,1]:|j_t(x,y)|> n\}}\,, \end{aligned}$$

where we use a representative of $t \mapsto j_t(x,y)$ to define to indicator functions. Clearly we have that $j_t = j_t^{\le n} + j_t^{>n}$ and, by linearity of the discrete divergence, that $\overline{\mathrm{div}} j_t = \overline{\mathrm{div}} j_t^{\le n} + \overline{\mathrm{div}} j_t^{>n}$ for all $t \in [0,\tau]$. Let $\varphi \in C^1([0,\tau]\times V)$. Then, since $(\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\mu \to \nu)$, it follows that

$$\begin{aligned} -\int_0^\tau \int_V \varphi(t,x)\,\overline{\mathrm{div}} j_t^{\le n}(dx)\,dt &= \int_V [\varphi(\tau,x)\nu(dx) - \varphi(0,x)\mu(dx)] \\ &\quad - \int_0^\tau \int_V \partial_t\varphi(t,x)\rho_t(dx)dt + \int_0^\tau \int_V \varphi(t,x)\overline{\mathrm{div}} j_t^{>n}(dx)\,dt \\ &= \int_V \left[\varphi(\tau,x)\left(\nu(dx) + \int_0^t \overline{\mathrm{div}} j_s^{>n}(dx)\,ds\right) - \varphi(0,x)\mu(dx)\right] \\ &\quad - \int_0^\tau \partial_t \int_V \varphi(t,x)\left(\rho_t(dx) + \int_0^t \overline{\mathrm{div}} j_s^{>n}(dx)\,ds\right)dt\,, \end{aligned}$$

where we used that V is finite and integration by parts for the second equality. Remark that for $x \in V$, the functions given by $t \mapsto \int_0^t \overline{\mathrm{div}} j_t^{>n}(x)\, ds = \int_0^t \sum_{y \in V} j_s^{>n}(x,y) - j_s^{>n}(y,x)\, ds$ are continuous. Thus, we see that

$$t \mapsto \rho_t + \int_0^t \overline{\mathrm{div}} j_s^{>n}\, ds\,,$$

is weakly*-continuous and

$$t \mapsto (\bar{\rho}_t^n, \bar{j}_t^n) := \left(\rho_t + \int_0^t \overline{\mathrm{div}} j_s^{>n}\, ds\,,\ j_t^{\leq n} \right) \in \mathscr{CE}([0,\tau])\,.$$

In addition, it follows by continuity that

$$M_n := \max_{x \in V} \sup_{t \in [0,1]} \left| \frac{1}{\widetilde{\pi}(x)} \int_0^t \overline{\mathrm{div}} j_s^{>n}(x)\, ds \right| < +\infty\,,$$

where $\widetilde{\pi} := \frac{\mu(V)}{\pi(V)}\pi$. Consequently, it holds that $\widetilde{\pi}(V) = \mu(V)$ and $\widetilde{\pi}(x) > 0$ for all $x \in V$.

Step (ii): convex combination. Define for $n \in \mathbb{N}$,

$$\lambda_n := \frac{1}{1 + M_n + 1/n}\,.$$

By Lemma 2.3.5 the convex combination $(\widetilde{\rho}^n, \widetilde{j}^n)_{t \in [0,\tau]}$ of this solution and the constant solution $t \mapsto (\widetilde{\pi}, 0)$ given by

$$\begin{aligned} \lambda_n(\bar{\rho}_t^n, \bar{j}_t^n) + (1-\lambda_n)(\widetilde{\pi}, 0) &= \frac{1}{1 + M_n + 1/n} \left(\rho_t + \int_0^t \overline{\mathrm{div}} j_s^{>n}\, ds\,,\ j_t^{\leq n} \right) \\ &\quad + \frac{M_n + 1/n}{1 + M_n + 1/n} (\widetilde{\pi}, 0) \\ &= \lambda_n \left(\rho_t + \int_0^t \overline{\mathrm{div}} j_s^{>n}\, ds + \left(M_n + \frac{1}{n} \right) \widetilde{\pi}\,,\ j_t^{\leq n} \right) \\ &=: (\widetilde{\rho}_t^n, \widetilde{j}_t^n)\,, \end{aligned}$$

is an element of $\mathscr{CE}([0,\tau])$. Remark for the mass of $\widetilde{\rho}_t$ that

$$\begin{aligned} \widetilde{\rho}_t(V) &= \lambda_n \left(\rho_t(V) + \int_0^t \overline{\mathrm{div}} j_s^{>n}(V)\, ds + \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(V) \right) \\ &= \lambda_n \left(\mu(V) + \left(M_n + \frac{1}{n} \right) \mu(V) \right) \\ &= \lambda_n \frac{\mu(V)}{\lambda_n}\,. \end{aligned}$$

Step (iii): rescaling time. Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $0 < \delta_n < \tau/2$ and $\lim_{n\to\infty} \delta_n = 0$. Define for $n \in \mathbb{N}$ the strictly increasing C^1-diffeomorphisms $\Phi_n : [\delta_n, \tau - \delta_n] \longrightarrow [0, \tau]$ by

$$\Phi_n(t) := \frac{t - \delta_n}{\tau - 2\delta_n}\tau \,.$$

Then, by Lemma 2.3.6, it follows that $(\widehat{\rho}^n, \widehat{j}^n) := \mathsf{S}_{\Phi_n}(\widetilde{\rho}^n, \widetilde{j}^n) \in \mathscr{CE}([\delta_n, \tau - \delta_n]; \widehat{\rho}^n_{\delta_n}, \widehat{\rho}^n_{\tau-\delta_n})$ and $\widehat{\rho}_t(V) = \mu(V)$ for all $t \in V$.

Step (iv): boundary parts. We see that $\widehat{\rho}^n_{\delta_n}(V) = \mu(V)$ and $\widehat{\rho}^n_{\delta_n} = \widetilde{\rho}^n_0 > 0$, as well as $\widehat{\rho}^n_{\tau-\delta_n}(V) = \nu(V)$ and $\widehat{\rho}^n_{\tau-\delta_n} = \widetilde{\rho}^n_\tau > 0$ by (3.3) below. Consequently, by Lemma 3.4.4 there exist $(\sigma^n, h^n) \in \mathscr{CE}([0, \delta_n]; \mu, \widehat{\rho}^n_{\delta_n})$ and $(\omega^n, p^n) \in \mathscr{CE}([0, \delta_n]; \nu, \widehat{\rho}^n_{\tau-\delta_n})$ with finite $\mathcal{R}$-action given by

$$\sigma^n_t := \frac{\widehat{\rho}^n_{\delta_n} - \mu}{\delta_n^2}t^2 + \mu \,, \qquad h^n_t := \frac{2t}{\delta_n^2}\widetilde{h}^n \mathbb{1}_{\mathcal{E}_N} \,,$$
$$\omega^n_t := \frac{\widehat{\rho}^n_{\tau-\delta_n} - \nu}{\delta_n^2}t^2 + \nu \,, \qquad p^n_t := \frac{2t}{\delta_n^2}\widetilde{p}^n \mathbb{1}_{\mathcal{E}_N} \,,$$

and a matrix A such that

$$\|\widetilde{h}^n\|_\infty \le \|A^{-1}\|_\infty \|\widehat{\rho}^n_{\delta_n} - \mu\|_\infty \,, \qquad \|\widetilde{p}^n\|_\infty \le \|A^{-1}\|_\infty \|\widehat{\rho}^n_{\tau-\delta_n} - \nu\|_\infty \,, \tag{3.2}$$

for all $n \in \mathbb{N}$. We define the time shifts $\Theta_n : [\tau - \delta_n, \tau] \longrightarrow [0, \delta_n]$, $t \mapsto t - \tau + \delta_n$. Then it follows by Lemmas 2.3.6 and 2.3.10 that

$$(\widehat{\sigma}^n, \widehat{h}^n) := \mathsf{S}_{\Theta_n} \circ \mathsf{R}(\omega^n, p^n) \in \mathscr{CE}([\tau - \delta_n, \tau]; \widehat{\rho}^n_{\tau-\delta_n}, \nu) \,,$$

and

$$\widehat{\sigma}^n_t := \frac{\widehat{\rho}^n_{\tau-\delta_n} - \nu}{\delta_n^2}(\tau - t)^2 + \nu \,, \qquad \widehat{h}^n_t := \frac{2(t - \tau)}{\delta_n^2}\widetilde{p}^n \mathbb{1}_{\mathcal{E}_N} \,.$$

In addition, $(\widehat{\sigma}^n, \widehat{h}^n)$ has finite $\mathcal{R}$-action.

Define $(\rho^n, j^n) := (\sigma^n, h^n) \odot (\widehat{\rho}^n, \widehat{j}^n) \odot (\widehat{\sigma}^n, \widehat{h}^n)$. By Lemma 2.3.15 it holds that $(\rho^n, j^n) \in \mathscr{CE}([0, \tau]; \mu, \nu)$.

2. (ρ^n, j^n) has finite $\mathcal{R}$-action. We claim that $(\widetilde{\rho}^n, \widetilde{j}^n) \in \mathscr{CE}([0, \tau])$ has finite $\mathcal{R}$-action. Then, it follows by Lemma 2.3.6 that $(\widehat{\rho}^n, \widehat{j}^n) = \mathsf{S}_{\Phi_n}(\widetilde{\rho}^n, \widetilde{j}^n)$ has finite $\mathcal{R}$-action. Hence, it follows by Lemma 2.3.15 that

$$\int_0^\tau \mathcal{R}(\rho^n_t, j^n_t)dt = \int_0^{\delta_n} \mathcal{R}(\sigma^n_t, h^n_t)dt + \int_{\delta_n}^{\tau-\delta_n} \mathcal{R}(\widehat{\rho}^n_t, \widehat{j}^n_t)dt + \int_{\tau-\delta_n}^\tau \mathcal{R}(\widehat{\sigma}^n_t, \widehat{h}^n_t)dt < +\infty \,,$$

because (σ^n, h^n) and $(\widehat{\sigma}^n, \widehat{h}^n)$ have finite $\mathcal{R}$-action by Lemma 3.4.4.

Now we prove the claim that $(\widetilde{\rho}^n, \widetilde{j}^n)$ has finite action. From the definition of M_n we see that

$$\begin{aligned}
\widetilde{\rho}_t^n(x) &= \lambda_n \left(\rho_t(x) + \int_0^t \overline{\mathrm{div}} j_s^{>n}(x)\, ds + \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(x) \right) \\
&\geq \lambda_n \left(\rho_t(x) + \int_0^t \overline{\mathrm{div}} j_s^{>n}(x)\, ds + \left| \int_0^t \overline{\mathrm{div}} j_s^{>n}(x)\, ds \right| + \frac{\widetilde{\pi}(x)}{n} \right) \\
&\geq \frac{\widetilde{\pi}(x)}{n} \\
&> 0\,,
\end{aligned} \tag{3.3}$$

for all $t \in [0, \tau]$ and $x \in V$ and therefore that

$$\widetilde{u}_t^n(x) := \frac{d\widetilde{\rho}_t}{d\pi}(x) \geq \frac{\mu(V)}{\pi(V)n} > 0\,,$$

and $\mathcal{E}_{\widehat{u}_t^n} = \mathcal{E}$. Moreover, it follows that

$$\widetilde{\alpha}_t(x, y) := \alpha_t(u_t(x), u_t(y)) \geq \frac{\mu(V)}{\pi(V)n}\,.$$

Furthermore, by definition of $(j_t^{\leq n})_t$ it follows that

$$|\widetilde{w}_t^n(x, y)| := \left| \frac{d\widetilde{j}_t^n}{d\vartheta_\pi}(x, y) \right| = \left| \frac{dj_t^{\leq n}}{d\vartheta_\pi}(x, y) \right| \leq \frac{n}{\vartheta_\pi(x, y)}\,,$$

for all $t \in [0, \tau]$ and $(x, y) \in \mathcal{E}$ such that $\vartheta_\pi(x, y) \neq 0$, and

$$0 \leq \widetilde{\alpha}_t(x, y) \Psi \left(\frac{|\widetilde{w}_t^n(x, y)|}{\widetilde{\alpha}_t(x, y)} \right) \leq \frac{|\widetilde{w}_t^n(x, y)|^2}{\widetilde{\alpha}_t(x, y)} \leq \frac{n^3 \pi(V)}{\vartheta_\pi(x, y)^2 \mu(V)}\,.$$

It follows that

$$\begin{aligned}
0 \leq \int_0^\tau \mathcal{R}(\widetilde{\rho}_t^n, \widetilde{j}_t^n) dt &= \int_0^\tau \iint_{\mathcal{E}} \Upsilon(\widetilde{u}_t^n(x), \widetilde{u}_t^n(y), |\widetilde{w}_t^n(x, y)|) d\vartheta_\pi(x, y)\, dt \\
&= \int_0^\tau \iint_{\mathcal{E}} \widetilde{\alpha}_t(x, y) \Psi \left(\frac{|\widetilde{w}_t^n(x, y)|}{\widetilde{\alpha}_t(x, y)} \right) d\vartheta_\pi(x, y)\, dt \\
&\leq \int_0^\tau \iint_{\mathcal{E}} \frac{|\widetilde{w}_t^n(x, y)|^2}{\widetilde{\alpha}_t(x, y)} d\vartheta_\pi(x, y)\, dt \\
&\leq \iint_{\mathcal{E} \cap \operatorname{supp} \vartheta_\pi} \frac{\tau n^3 \pi(V)}{\vartheta_\pi(x, y)^2 \mu(V)} d\vartheta_\pi(x, y) \\
&< +\infty\,,
\end{aligned}$$

because $\mathcal{E}$ is a finite set. This proves the claim.

3. $\{(\rho^n, j^n)\}_n$ converges to (ρ, j).
Step 1: j^n converges to j. To complete the proof, we show that $(j_t^n) \to (j_t)_t$ in L^1 as $n \to \infty$.

Remark that

$$\int_0^\tau |j_t^n - j_t|(\mathcal{E})\, dt = \sum_{\substack{(x,y)\in\mathcal{E}:\\ x\sim y}} \int_0^1 |j_t^n(x,y) - j_t(x,y)| dt\,,$$

by finiteness of V. So it suffices to show that $\int_0^\tau |j_t^n(x,y) - j_t(x,y)|dt \to 0$ as $n \to \infty$ for every $(x,y) \in \mathcal{E}$ with $x \sim y$. We see that

$$\int_0^\tau |j_t^n(x,y) - j_t(x,y)|dt \le \int_0^{\delta_n} |h_t^n(x,y)|dt + \int_0^\tau |\mathbb{1}_{[\delta_n,\tau-\delta_n]} j_t^n(x,y) - j_t(x,y)|dt + \int_{\tau-\delta_n}^\tau |\widehat{h}_t^n(x,y)|dt\,. \tag{3.4}$$

We will estimate each of the terms on the right hand side separately. First we need some limits. We see that

$$\begin{aligned} 0 \le M_n &= \max_{x\in V} \sup_{t\in[0,1]} \left| \frac{1}{\widetilde{\pi}(x)} \int_0^t \overline{\mathrm{div}} j_s^{>n}(x)\, ds \right| \\ &\le \max_{x\in V} \frac{1}{\widetilde{\pi}(x)} \int_0^\tau |j_t^{>n}|(\mathcal{E}) dt \\ &\to 0\,, \end{aligned} \tag{3.5}$$

as $n \to \infty$ by Dominated Convergence. Consequently,

$$|1 - \lambda_n| = \frac{M_n + 1/n}{1 + M_n + 1/n} \to 0\,, \tag{3.6}$$

as $n \to \infty$.

Let $\varepsilon > 0$. For the middle part it follows that

$$\begin{aligned}
\int_0^\tau |\mathbb{1}_{[\delta_n,\tau-\delta_n]} j_t^n(x,y) - j_t(x,y)| dt &= \int_0^\tau |\mathbb{1}_{[\delta_n,\tau-\delta_n]} \lambda_n \Phi_n'(t)\, j_{\Phi_n(t)}^{\leq n}(x,y) - j_t(x,y)| dt \\
&\leq \lambda_n \int_0^\tau \left| \mathbb{1}_{[\delta_n,\tau-\delta_n]} \Phi_n'(t)\, j_{\Phi_n(t)}^{\leq n}(x,y) - j_t(x,y) \right| dt \\
&\quad + |1-\lambda_n| \int_0^\tau |j_t|(\mathcal{E})\, dt \\
&\leq \int_0^\tau \mathbb{1}_{[\delta_n,1-\delta_n]} \Phi_n'(t) \left| j_{\Phi_n(t)}^{\leq n}(x,y) - j_{\Phi_n(t)}(x,y) \right| dt \\
&\quad + \int_0^\tau \left| \mathbb{1}_{[\delta_n,\tau-\delta_n]} \Phi_n'(t) j_{\Phi_n(t)}(x,y) - j_t(x,y) \right| dt \\
&\quad + |1-\lambda_n| \|(j_t)_t\|_{\mathcal{L}^1} \\
&= \int_0^\tau |j_t^{>n}(x,y)| dt + |1-\lambda_n| \|(j_t)_t\|_{\mathcal{L}^1} \\
&\quad + \int_0^\tau \left| \mathbb{1}_{[\delta_n,\tau-\delta_n]} \Phi_n'(t) j_{\Phi_n(t)}(x,y) - j_t(x,y) \right| dt\,,
\end{aligned}$$

where we applied the change of variables formula for the last equality. The first term converges to zero by Dominated Convergence, the second one by Lemma 3.4.8 below, and the third term because $M_n + 1/n \to 0$ as $n \to \infty$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\int_0^\tau |\mathbb{1}_{[\delta_n,1-\delta_n]} j_t^n(x,y) - j_t(x,y)| dt < \varepsilon/3\,,$$

for all $n \geq N$, and simultaneously $|h_1^n(x,y)| < 2\varepsilon/3$, $|h_2^n(x,y)| < 2\varepsilon/3$ by Lemma 3.4.4.

For the boundary parts we need some limits. Namely, with the limits (3.5, 3.6) it follows that

$$\begin{aligned}
\left|\widehat{\rho}_{\delta_n}^n(x) - \mu(x)\right| &= \left|\widetilde{\rho}_{\Phi_n(\delta_n)}(x) - \mu(x)\right| \\
&= \left| \lambda_n \left(\rho_0(x) + \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(x) \right) - \mu(x) \right| \\
&\leq \lambda_n \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(x) + |1-\lambda_n| \mu(x) \\
&\to 0\,,
\end{aligned}$$

and

$$\begin{aligned}
\left|\widehat{\rho}_{\tau-\delta_n}^n(x) - \nu(x)\right| &= \left|\widetilde{\rho}_{\Phi_n(\tau-\delta_n)}(x) - \nu(x)\right| \\
&= \left| \lambda_n \left(\rho_\tau(x) + \int_0^\tau \overline{\operatorname{div}} j_s^{>n}(x)\, ds + \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(x) \right) - \nu(x) \right| \\
&\leq \lambda_n \left(M_n + \frac{1}{n} \right) \widetilde{\pi}(x) + \lambda_n \int_0^\tau |j_t^{>n}|(\mathcal{E})\, dt + |1-\lambda_n| \nu(x) \\
&\to 0\,,
\end{aligned}$$

as $n \to \infty$. So by (3.2) we can assume in addition that $\|\widetilde{h}^n\|_\infty < \varepsilon/3$ and $\|\widetilde{p}^n\|_\infty < \varepsilon/3$ for all $n \geq N$. It follows that

$$\int_0^{\delta_n} |j_t^n(x,y)|dt = \int_0^{\delta_n} |\widetilde{h}^n(x,y)|\frac{2t}{\delta_n^2}dt = |\widetilde{h}^n(x,y)| < \varepsilon/3\,,$$
$$\int_{\tau-\delta_n}^{\tau} |j_t^n(x,y)|dt = \int_{\tau-\delta_n}^{\tau} |\widetilde{p}^n(x,y)|\frac{2(\tau-t)}{\delta_n^2}dt = |\widetilde{p}^n(x,y)| < \varepsilon/3\,.$$

Hence, it holds by (3.4) that

$$\int_0^{\tau} |j_t^n(x,y) - j_t(x,y)|dt < \varepsilon\,,$$

for all $n \geq N$.

Step 2: ρ^n converges to ρ. By Lemma 2.3.3 it follows for $t \in [0,\tau]$ that $(\rho, j) \in \mathscr{CE}([0,t];\mu,\rho_t)$ and $(\rho^n, j^n) \in \mathscr{CE}([0,t];\mu,\rho_t^n)$ for all $n \in \mathbb{N}$. Therefore, it holds for any $\varphi \in C^1([0,t]\times V)$ that

$$\begin{aligned}\int_V [\varphi(t,x)(\rho_t(dx) - \rho_t^n(dx))] = &\int_0^t \int_V \partial_s\varphi(s,x)(\rho_s(dx) - \rho_s^n(dx))\,ds \\ &- \int_0^t \int_V \varphi(s,x)\overline{\operatorname{div}}\,(j_s(dx) - j_s^n(dx))\,ds\,.\end{aligned}$$

In particular, for $\varphi = \mathbb{1}_x$, it follows that

$$\rho_t(x) - \rho_t^n(x) = -\int_0^t \overline{\operatorname{div}}\,(j_s - j_s^n)(x)\,ds\,,$$

and

$$|\rho_t(x) - \rho_t^n(x)| \leq \int_0^t \left|\overline{\operatorname{div}}\,(j_s - j_s^n)(x)\right|\,ds \leq \int_0^{\tau} |j_s - j_s^n|(\mathcal{E})\,ds\,.$$

Hence, by *Step 1*, $t \mapsto \rho_t^n(x)$ converges uniformly to $t \mapsto \rho_t(x)$, which concludes that proof. □

The following lemma is used in the proof of Lemma 3.4.7.

Lemma 3.4.8. *Let $f \in L^1(\mathbb{R})$ and let $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ be a sequence of C^1-diffeomorphism such that $\lim_{n\to\infty} g_n(t) = t$, $\lim_{n\to\infty} g_n'(t) = 1$ and $g_n'(t) > 0$ for all $t \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(g_n(t))g_n'(t) - f(t)|dt = 0\,.$$

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_c(\mathbb{R})$ such that $f_n \to f$ in $L^1(\mathbb{R})$ as $n \to \infty$. Let $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} |f(t) - f_M(t)|dt < \varepsilon/3\,.$$

By continuity of f_M and the properties of $(g_n)_n$, it follows that $\lim_{n\to\infty} |f_M(g_n(t))g_n'(t) - f_M(t)| = 0$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} |f_M(g_n(t))g_n'(t) - f_M(t)|dt \leq \int_{\mathbb{R}} |f_M(g_n(t))g_n'(t)|dt + \int_{\mathbb{R}} |f_M(t)|dt = 2\int_{\mathbb{R}} |f_M(t)|dt < +\infty \,.$$

To obtain the last equality we applied the change of variables formula. Therefore, it follows by dominated convergence that there exists $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}} |f_M(g_n(t))g_n'(t) - f_M(t)|dt < \varepsilon/3 \,,$$

for all $n \geq N$. It follows for $n \geq N$ that

$$\begin{aligned}\int_{\mathbb{R}} |f(g_n(t))g_n'(t) - f(t)|dt &\leq \int_{\mathbb{R}} |f(g_n(t)) - f_M(g_n(t))|g_n'(t)dt \\ &\quad + \int_{\mathbb{R}} |f_M(g_n(t))g_n'(t) - f_M(t)|dt + \int_{\mathbb{R}} |f_M(t) - f(t)|dt \\ &= 2\int_{\mathbb{R}} |f_M(t) - f(t)|dt + \int_{\mathbb{R}} |f_M(g_n(t))g_n'(t) - f_M(t)|dt \\ &< \varepsilon \,,\end{aligned}$$

which proves the lemma. □

3.5 Outlook on convergence of minimizers

Although we have not succeeded in proving convergence of minimizers of the rescaled actions to minimizers of the limiting functional, we will show the convergence of minimal values. After this we discuss some conditions in the Γ-convergence framework that may be useful for showing the former.

Proposition 3.5.1 (Convergence of minimal values)**.** *It holds that*

$$\lim_{n\to\infty} \min \mathcal{F}_n = \inf \mathcal{F}_\infty \,.$$

Proof. Let $\varepsilon > 0$ and let $(\rho^*, j^*) \in \mathscr{CE}(\mu \to \nu)$ be a ε-minimizer of $\mathcal{F}_\infty$. That is, $\mathcal{F}_\infty(\rho^*, j^*) \leq \inf \mathcal{F}_\infty + \varepsilon$. By the Limsup inequality, Corollary 3.4.1, there exists a sequence $\{(\rho^n, j^n)\}_n \in \mathscr{CE}(\mu \to \nu)$ such that $(\rho^n, j^n) \to (\rho^*, j^*)$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) \leq \mathcal{F}_\infty(\rho^*, j^*) \,.$$

It follows that

$$\limsup_{n\to\infty} \min \mathcal{F}_n \leq \limsup_{n\to\infty} \mathcal{F}_n(\rho^n, j^n) \leq \mathcal{F}_\infty(\rho^*, j^*) \leq \inf \mathcal{F}_\infty + \varepsilon \,,$$

and, letting $\varepsilon \downarrow 0$, that

$$\limsup_{n\to\infty} \min \mathcal{F}_n \le \inf \mathcal{F}_\infty .$$

By Proposition 3.3.3 we have that

$$\min \mathcal{F}_n \ge \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \inf \mathcal{F}_\infty\right) ,$$

for all $n \in \mathbb{N}$. It follows that

$$\liminf_{n\to\infty} \min \mathcal{F}_n \ge \liminf_{n\to\infty} \frac{C_{\kappa,\rho}}{n \log n} \Psi\left(\frac{n}{C_{\kappa,\rho}} \inf \mathcal{F}_\infty\right) = \inf \mathcal{F}_\infty ,$$

which concludes the proof. □

Thus, as mentioned in the introduction, we have shown that

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathcal{W}^{1/n}(\mu,\nu)}{\log(n)} &= \lim_{n\to\infty} \min \mathcal{F}_n \\ &= \inf \mathcal{F}_\infty \\ &= \inf \left\{ \int_0^1 |j_t|(\mathcal{E}) dt : (\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu),\ j_t \ll \vartheta_\pi \text{ for a.a. } t \right\} .\end{aligned}$$

All results of this chapter remain valid if we replace the sequence $(n)_{n\in\mathbb{N}}$ by an arbitrary sequence $(\tau_n^{-1})_{n\in\mathbb{N}}$ with $0 < \tau_n < 1$ and $\lim_{n\to\infty} \tau_n = 0$. It follows that

$$\lim_{n\to\infty} \frac{\mathcal{W}^{\tau_n}(\mu,\nu)}{\log(\tau_n^{-1})} = \inf \left\{ \int_0^1 |j_t|(\mathcal{E}) dt : (\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu),\ j_t \ll \vartheta_\pi \text{ for a.a. } t \right\} ,$$

and therefore that

$$\lim_{\tau\downarrow 0} \frac{\mathcal{W}^{\tau}(\mu,\nu)}{\log(\tau^{-1})} = \inf \left\{ \int_0^1 |j_t|(\mathcal{E}) dt : (\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu),\ j_t \ll \vartheta_\pi \text{ for a.a. } t \right\} .$$

To obtain convergence of minimizers of $(\mathcal{F}_n)_n$ to minimizers of $\mathcal{F}_\infty$ in the framework of Γ-convergence, we would that need $(\mathcal{F}_n)_n$ is *equi-coercive* as in Definition C.6. Then by Theorem C.7 it holds that every cluster point of a sequence $(x_n)_n$ such that $\mathcal{F}_n(x_n) = \min \mathcal{F}_n$, is a minimizer of $\mathcal{F}_\infty$.

For equi-coerciveness to be satisfied in our setting, it is necessary that the sublevel sets $\{\mathcal{F}_n \le t\}$ are relatively compact as a subspace of $\mathcal{C}_V \times L^1$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$. However, the topology of $\mathcal{C}_V \times L^1$ might be too strong for this to hold. Therefore, we would need to weaken the topology on $\mathscr{C}\mathscr{E}(\mu \to \nu)$, and prove Γ-convergence of $(\mathcal{F}_n)_n$ to $\mathcal{F}_\infty$ for this weaker topology. A good place to start would be the topology used in the compactness result Proposition 2.4.2. The limsup inequality in this topology is then readily satisfied by our result for the stronger topology we used. However, our arguments for the liminf inequality, based on pointwise convergence, cannot be applied to the weaker topology.

3.6 A connection between the 1-Wasserstein distance and the limiting action

As noted in the introduction of this chapter, there is a connection between the 1-Wasserstein distance W^1 and the limiting action $\mathcal{F}_\infty$ via the continuity equation. We equip V with a (unspecified) metric d.

First we introduce the p-Wasserstein distance between two probability measures on V, where $p \in [1, \infty)$. For $i \in \{1, 2\}$, we denote by $P^i : V \times V \longrightarrow V$ the canonical projection onto the i-th component and by $\mathcal{P}(V)$ the set of probability measures on V.

Definition 3.6.1. Let $\mu_1, \mu_2 \in \mathcal{P}(V)$. We say that $\bar{\mu} \in \mathcal{P}(V \times V)$ is a coupling between μ_1 and μ_2 if $P^1_* \bar{\mu} = \mu_1$ and $P^2_* \bar{\mu} = \mu_2$. We denote by $\Gamma(\mu_1, \mu_2)$ the set of all couplings between μ_1 and μ_2.

Remark. Intuitively, for each $x, y \in V$ the quantity $\bar{\mu}(x, y)$ describes how much mass is allocated from vertex x to vertex y.

Remark. For all $\mu_1, \mu_2 \in \mathcal{P}(V)$, the product measure $\mu_1 \otimes \mu_2$ is a coupling. Hence, $\Gamma(\mu_1, \mu_2)$ is non-empty.

Definition 3.6.2 (p-Wasserstein distance, [2, p.151, sec. 7.1])**.** Let $p \in [1, \infty)$. We define the p-Wasserstein distance W^p on $\mathcal{P}(V)$ by the minimal value of the Monge–Kantorovich functional,

$$\Gamma(\mu_1, \mu_2) \ni \gamma \mapsto \left(\int_{V \times V} d(x, y)^p d\gamma \right)^{1/p} .$$

That is,

$$W^p(\mu_1, \mu_2) := \min \left\{ \int_{V \times V} d(x, y)^p d\gamma : \gamma \in \Gamma(\mu_1, \mu_2) \right\}^{1/p} .$$

Remark. The 1-Wasserstein distance between two probability measures chooses the minimal total distance (in d) it takes to allocate the configuration of mass of one measure to the configuration of mass of the other.

We denote by $\mathrm{Lip}_V(1)$ the set of Lipschitz continuous functions on V with Lipschitz constant 1. We have the following alternative expression of the 1-Wasserstein distance.

Proposition 3.6.3 (Kantorovich–Rubenstein duality, [2, p.152, sec. 7.1])**.** *Let $\mu_1, \mu_2 \in \mathcal{P}(V)$. Then*

$$W^1(\mu_1, \mu_2) = \sup \left\{ \int_V f d\nu - \int_V f d\mu : f \in \mathrm{Lip}_V(1) \right\} .$$

Proposition 3.6.4. *Let μ, $\nu \in \mathcal{P}(V)$ and $(\rho, j) \in \mathscr{C}\mathscr{E}(\mu \to \nu)$. It holds that*

$$W^1(\mu, \nu) \le \int_0^1 \iint_{\mathcal{E}} d(x, y)\, d|j_t|(x, y) dt \, .$$

Proof. As V is finite, it holds that $C_c(V) = C(V)$ and $C_c([0,1] \times V) = C([0,1] \times V)$. Let $f \in \mathrm{Lip}_V(1)$ and define $\varphi_f \in C([0,1] \times V)$ by $\varphi_f(t,x) = f(x)$ for all $t \in [0,1]$ and $x \in V$. It follows that $\partial_t \varphi_f \equiv 0$, so $\partial_t \varphi_f \in C([0,1] \times V)$. We see by the Kantorovich–Rubinstein duality that

$$\begin{aligned}
W^1(\mu,\nu) &= \sup_{f \in \mathrm{Lip}_V(1)} \left\{ \int_V f d\nu - \int_V f d\mu \right\} \\
&= \sup_{f \in \mathrm{Lip}_V(1)} \left\{ \int_V \varphi_f(1,x) d\rho_1(x) - \int_V \varphi_f(1,x) d\rho_0(x) - \int_0^1 \int_V \partial_t \varphi_f(t,x) d\rho_t(x)\, dt \right\} \\
&= \sup_{f \in \mathrm{Lip}_V(1)} \left\{ \int_0^1 \iint_{\mathcal{E}} \overline{\nabla} \varphi_f(t,x,y) dj_t(x,y)\, dt \right\} \\
&\leq \sup_{f \in \mathrm{Lip}_V(1)} \left\{ \int_0^1 \iint_{\mathcal{E}} |\overline{\nabla} f(x,y)|\, d|j_t|(x,y)\, dt \right\} \\
&\leq \int_0^1 \iint_{\mathcal{E}} d(x,y)\, d|j_t|(x,y) dt\,,
\end{aligned}$$

which concludes the proof. □

Now we choose the metric on V to be the combinatorial graph distance.

Corollary 3.6.5. *Let V be equipped with the combinatorial graph distance and let $\mu, \nu \in \mathcal{P}(V)$. Then*

$$W^1(\mu,\nu) \leq \inf \mathcal{F}_\infty = \inf \left\{ \int_0^1 |j_t|(\mathcal{E}) dt : (\rho, j) \in \mathscr{CE}(\mu \to \nu),\ j_t \ll \vartheta_\pi \textit{ for a.a. } t \right\} .$$

Proof. We denote the combinatorial graph distance on V by d_G. Recall that $d_G(x,y) = 1$ if $x \sim y$, so $d_G \equiv 1$ on $\mathcal{E}_N$. Let $(\rho, j) \in \mathscr{CE}(\mu \to \nu)$ such that $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$. Since $\vartheta_\pi(\mathcal{E} \setminus \mathcal{E}_N) = 0$, it holds that $j_t(\mathcal{E} \setminus \mathcal{E}_N) = 0$ for almost all $t \in [0,1]$.

By Proposition 3.6.4, it follows that

$$\begin{aligned}
W^1(\mu,\nu) &\leq \int_0^1 \iint_{\mathcal{E}} d_G(x,y)\, d|j_t|(x,y) dt \\
&= \int_0^1 \iint_{\mathcal{E}_N} d_G(x,y)\, d|j_t|(x,y) dt \\
&= \int_0^1 |j_t|(\mathcal{E}_N) dt \\
&= \int_0^1 |j_t|(\mathcal{E}) dt \\
&= \mathcal{F}_\infty(\rho, j)\,.
\end{aligned}$$

The claim follows by taking the infimum over all (ρ, j) with $j_t \ll \vartheta_\pi$ for almost all $t \in [0,1]$. □

Chapter 4

Two-point configurations

In this section we consider all two-point configurations $V := \{x, y\}$ and $\pi \in \mathcal{M}^+(V)$ such that $\pi(x)\pi(y) > 0$. Here the jump kernel κ satisfies

$$\kappa(x, x) = 0\,, \qquad \kappa_1 := \kappa(x, y) > 0\,,$$
$$\kappa_2 := \kappa(y, x) > 0\,, \qquad \kappa(y, y) = 0\,.$$

In terms of graphs, this situation is that of weighted undirected, connected graph consisting of two points without self-loops.

The main goal of this chapter is to work towards a characterization of the minimizers of

$$\mathcal{W}^\tau(\delta_x, \delta_y) = \inf\left\{\int_0^\tau \mathcal{R}(\rho_t, j_t)dt : (\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\delta_x \to \delta_y)\right\}\,,$$

the DVT-cost between two Dirac measures δ_x and δ_y, for each $\tau > 0$. Remark that by combining Corollaries 2.4.3 and 3.4.6, minimizers indeed exist and that the minimum is assumed on the set $\mathscr{A}_\tau(\delta_x \to \delta_y)$ of curves with finite $\mathcal{R}$-action. Although we will not succeed in finding a characterization of minimizers, we obtain the following results:

(i) The derivation of the Euler–Lagrange equation of an equivalent minimization problem in Proposition 4.1.5 and Lemma 4.1.6. Because of the general principle that this equation characterizes minimizers of convex functionals, this seems a suitable choice of characterization, see for example [13]. We show under some addition regularity assumptions minimizers are solutions of the Euler–Lagrange equation.

(ii) A proof of existence of classical solutions to the Euler–Lagrange equation for *for some* $\tau > 0$, via an associated initial value problem in Proposition 4.2.2. Furthermore, we show that these solutions enjoy several properties in the aforementioned proposition and Lemma 4.2.1.

(iii) In Corollary 4.3.4 we show that the entropy along solutions of the initial value problem is convex in the case that τ is small enough. This will depend on both κ and π. This is important for a gradient flow formulation of curvature, because in the case of a Riemannian manifold the Ricci curvature being bounded from below by zero is equivalent to the convexity of entropy along geodesics.

4.1 The Euler–Lagrange equation

Instead of deriving the Euler–Lagrange equation of the minimization problem for $\mathcal{W}^\tau(\delta_x, \delta_y)$ given by

$$\int_0^\tau \mathcal{R}(\rho_t, j_t)dt, \qquad (\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\delta_x \to \delta_y)\,,$$

derive an equivalent minimization problem in Proposition 4.1.4, and give the Euler–Lagrange equation of the latter in Proposition 4.1.5.

Remark that $\kappa_1\pi(x) = \vartheta_\pi(x, y) = \vartheta_\pi(y, x) = \kappa_2\pi(y)$, or equivalently that

$$\mathsf{r}_{\kappa,\pi} := \pi(x)/\pi(y) = \kappa_2/\kappa_1\,,$$

by the **Detailed Balance Condition** on page 8.

Lemma 4.1.1. *Let $(\rho, j) \in \mathscr{A}_\tau(\delta_x \to \delta_y)$ such that $\rho_t = \delta_x$ or $\rho_t = \delta_y$ for some $t \in (0, \tau)$. Then there exists a curve $(\widetilde{\rho}, \widetilde{j})$ with lower $\mathcal{R}$-action such that $\widetilde{\rho}_t \neq \delta_x$, δ_y for all $t \in (0, \tau)$.*

Proof. We assume without loss of generality that there exists $t \in (0, \tau)$ such that $\rho_t = \delta_x$. Define $t_1 := \max\{t \in [0, \tau] : \rho_t = \delta_x\} > 0$, which is well-defined by weak*-continuity of ρ. Furthermore, define $t_2 := \min\{t \in [t_1, \tau] : \rho_t = \delta_y\}$. Then $\rho_t \neq \delta_x, \delta_y$ for all $t \in (t_1, t_2)$, and by Lemma 2.3.3 it follows that $(\rho, j)|_{[t_1,t_2]} \in \mathscr{C}\mathscr{E}([t_1, t_2]; \delta_x, \delta_y)$.

Define $\Phi : [0, \tau] \longrightarrow [t_1, t_2]$ by $\Phi(t) = \frac{t_2-t_1}{\tau} t + t_1$ and remark that $0 < \Phi'(t) < 1$. Then it follows by Lemma 2.3.6 that $(\widetilde{\rho}, \widetilde{j}) := \mathsf{S}_\Phi((\rho, j)|_{[t_1,t_2]}) \in \mathscr{C}\mathscr{E}_\tau(\delta_x \to \delta_y)$ has finite $\mathcal{R}$-action. Moreover,

$$\begin{aligned}
\int_0^\tau \mathcal{R}(\rho_t, j_t)dt &\geq \int_{t_1}^{t_2} \mathcal{R}(\rho_t, j_t)dt \\
&= \int_{t_1}^{t_2} \iint_{\mathcal{E}} \alpha_t \Psi\left(\frac{|w_t|}{\alpha_t}\right) d\vartheta_\pi dt \\
&= \int_0^\tau \iint_{\mathcal{E}} \Phi'(t)\alpha_{\Phi(t)} \Psi\left(\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right) d\vartheta_\pi dt \\
&\geq \int_0^\tau \iint_{\mathcal{E}} \alpha_{\Phi(t)} \Psi\left(\Phi'(t)\frac{|w_{\Phi(t)}|}{\alpha_{\Phi(t)}}\right) d\vartheta_\pi dt \\
&= \int_0^\tau \mathcal{R} \circ \mathsf{S}_\Phi((\rho, j)|_{[t_1,t_2]})dt \\
&= \int_0^\tau \mathcal{R}(\widetilde{\rho}_t, \widetilde{j}_t)dt\,.
\end{aligned}$$

Here, we used that $\xi \mapsto \xi^{-1}\Psi\left(|w_t|\Phi'(t)\xi\right)$ is increasing by Lemma 3.1.3 with

$$\frac{1}{\Phi'(t)\alpha_t} =: \xi_1 > \xi_2 := \frac{1}{\alpha_t}\,,$$

for the last inequality. This concludes the proof. □

Corollary 4.1.2. *The $\mathcal{R}$-action given by*

$$\mathscr{C}\mathscr{E}_\tau(\delta_x \to \delta_y) \ni (\rho, j) \mapsto \int_0^\tau \mathcal{R}(\rho_t, j_t)dt \in [0, +\infty]\,,$$

assumes its minimum on the set

$$\mathscr{S}_\tau := \{(\rho, j) \in \mathscr{A}_\tau(\delta_x \to \delta_y) \,:\, \mathsf{s}_* j = -j,\ \rho_t \neq \delta_x,\ \delta_y \text{ for all } t \in (0, \tau)\}\ .$$

Proof. By Lemma 4.1.1, the $\mathcal{R}$-action takes its minimum on the set

$$S := \{(\rho, j) \in \mathscr{A}_\tau(\delta_x \to \delta_y) \,:\, \rho_t \neq \delta_x,\ \delta_y \text{ for all } t \in (0, \tau)\}\ .$$

Let $(\rho, j) \in S$. Then, by Lemma 2.3.11, it holds that the skew-symmetrization of (ρ, j) has lower $\mathcal{R}$-action than (ρ, j). Hence, the $\mathcal{R}$-action assumes its minimum on the set

$$\{(\rho, j) \in \mathscr{C}\mathscr{E}_\tau(\delta_x \to \delta_y) \,:\, \mathsf{s}_* j = -j,\ \rho_t \neq \delta_x,\ \delta_y \text{ for all } t \in (0, \tau)\}\ ,$$

which concludes the proof. □

Now we derive an equivalent minimization problem.

Definition 4.1.3. We define the set of *admissible curves* $\mathscr{A}\mathscr{D}_\tau$ by

$$\mathscr{A}\mathscr{D}_\tau := \left\{ u \in C[0, \tau] \,:\, \substack{\partial_t u \text{ is integrable},\, u_0 = 1/\pi(x),\, u_\tau = 0, \\ 0 < u_t < 1/\pi(x) \text{ for all } t \in (0,\tau)} \right\}\ .$$

Proposition 4.1.4 (Equivalent minimization problem)**.** *Let $\tau > 0$. The mapping*

$$\mathscr{I} : \mathscr{S}_\tau \longrightarrow \mathscr{A}\mathscr{D}_\tau\,, \qquad (\rho, j) \mapsto \left(t \mapsto \frac{d\rho_t}{d\pi}(x) =: u_t \right)\,,$$

is bijective, and

$$\begin{aligned} \int_0^\tau \mathcal{R}(\rho_t, j_t)dt &= \mathscr{F}_\tau(\mathscr{I}(\rho, j)) \\ &= \mathscr{F}_\tau(u) \\ &:= \int_0^\tau F(u_t, \partial_t u_t)dt\,, \end{aligned}$$

where $\mathscr{F}_\tau : \mathscr{A}\mathscr{D}_\tau \longrightarrow [0, \infty)$ and

$$F(u, u') := 2\kappa_1 \sqrt{u(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u)}\,\Psi\left(\frac{u'}{2\kappa_1 \sqrt{u(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u)}} \right)\ .$$

Proof. Let $(\rho, j) \in \mathscr{S}_\tau$. Writing $u_t = \frac{d\rho_t}{d\pi}(x)$, we have by weak*-continuity of ρ that $u \in C[0,\tau]$. Furthermore, it is easy to see that $0 < \mathscr{I}(\rho,j)_t = u_t < 1/\pi(x)$ for all $t \in (0,\tau)$, $u_0 = 1/\pi(x)$ and $u_\tau = 0$.

By conservation of mass we have that $\rho_t(x) + \rho_t(y) = 1$ that

$$u_t(y) := \frac{d\rho_t}{d\pi}(y) = \frac{1}{\pi(y)}(1 - \pi(x)u_t) = \frac{1}{\pi(y)} - \mathsf{r}_{\kappa,\pi} u_t\,.$$

Furthermore, since j is anti-symmetric, i.e., $j_t(x,y) = -j_t(y,x)$ for all t, it holds that

$$-f(0) = \int_0^\tau f'(t)\rho_t(x)dt - 2\int_0^\tau f(t)j_t(x,y)dt\,,$$

for all $f \in C^1([0,\tau])$. This equation reduces to

$$\int_0^\tau f'(t)\rho_t(x)dt - 2\int_0^\tau f(t)j_t(x,y)dt = 0\,,$$

for all $f \in C_c^1(0,\tau)$, implying that $t \mapsto \rho_t(x)$ weakly differentiable with $\partial_t\rho_t(x) = -2j_t(x,y)$ and $\partial_t\rho(x)$ is integrable. It follows that u is weakly differentiable with

$$\partial_t u_t = \partial_t\rho_t(x)/\pi(x) = -2w_t(x,y)\vartheta_\pi(x,y)/\pi(x) = -2\kappa_1 w_t(x,y)\,,$$

and $\partial_t u$ is integrable. Hence, $\mathscr{I}(\rho,j) \in \mathscr{AD}_\tau$.

To see that $\mathscr{I}$ is injective, let $(\rho^1, j^1), (\rho^2, j^2) \in \mathscr{S}_\tau$ such that $u^1 = \mathscr{I}(\rho^1, j^1) = \mathscr{I}(\rho^2, j^2) = u^2$. Then $\rho^1(x) = u^1\pi(x) = u^2\pi(x) = \rho^2(x)$ and by conservation of mass that $\rho^1(y) = \rho^2(y)$. By uniqueness of weak derivatives, it follows that

$$j^1(x,y) = w^1(x,y)\vartheta_\pi(x,y) = -\frac{\pi(x)}{2}\partial_t u^1 = -\frac{\pi(x)}{2}\partial_t u^2 = w^2(x,y)\vartheta_\pi(x,y) = j^2(x,y)\,,$$

and by anti-symmetry of j^1 and j^2 that $j^1(y,x) = j^2(y,x)$. Hence, $(\rho^1, j^1) = (\rho^2, j^2)$ and $\mathscr{I}$ is injective.

To see that $\mathscr{I}$ is surjective, let $u \in \mathscr{AD}_\tau$. It is easily seen that the pair (ρ, j) defined by $\rho(x) := u\pi(x)$ and $\rho(y) := 1 - \rho(x)$, and $j^1(x,y) := -\frac{\pi(x)}{2}\partial_t u^1$ and $j(y,x) = -j(x,y)$ defines an element of $\mathscr{S}_\tau$ and $\mathscr{I}(\rho,j) = \frac{d\rho}{d\pi}(x) = u$. Hence, we have shown that $\mathscr{I}$ is a bijection.

For the second assertion, since $\frac{d\rho_t}{d\pi}(x)\frac{d\rho_t}{d\pi}(y) > 0$ for all $t \in (0,\tau)$, it follows that

$$\begin{aligned}
\int_0^\tau \mathcal{R}(\rho_t, j_t)dt &= \int_0^\tau \alpha(u_t, u_t(y))\Psi\left(\frac{w_t(x,y)}{\alpha(u_t,u_t(y))}\right)\vartheta_\pi(x,y)dt \\
&= \int_0^\tau 2\kappa_1\pi(x)\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\Psi\left(\frac{\partial_t u_t}{2\kappa_1\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}}\right)dt \\
&= \int_0^\tau F(u_t, \partial_t u_t)dt \\
&= \mathscr{F}_\tau(u)\,.
\end{aligned}$$

□

Remark. By this lemma $\mathscr{I}$ is a bijection between minimizers of the $\mathcal{R}$-action and the functional $\mathscr{F}_\tau$. Thus, we can study the minimization problem of the latter functional.

In te remaining of the section we derive the Euler–Lagrange equation of $\mathscr{F}_\tau$. The idea of this derivation is the following. Let $\phi \in C_c^\infty(0,\tau)$ and $|h| < \delta_\phi$ sufficiently small. Assume that $(u_t^*)_t$ is a minimizer of $\mathscr{F}_\tau(u) = \int_0^\tau F(u_t, u_t')dt$. Then, $h = 0$ is a minimizer of $(-\delta_\phi, \delta_\phi) \ni h \mapsto \int_0^\tau F(u_t^* + h\phi_t, (u_t^*)' + h\phi_t')dt$. Hence, if this function is differentiable at $h = 0$, then it satisfies

$$\frac{d}{dh}\left(\int_0^\tau F(u_t^* + h\phi_t, (u_t^*)' + h\phi_t')dt\right)_{h=0} = 0\,.$$

Performing some formal calculations, it follows that

$$\begin{aligned}\frac{d}{dh}\left(\int_0^\tau F(u_t^* + h\phi_t, (u_t^*)' + h\phi_t')dt\right)_{h=0} &= \int_0^\tau \frac{d}{dh}\left[F(u_t^* + h\phi_t, (u_t^*)' + h\phi_t')\right]_{h=0} dt \\ &= \int_0^\tau \left[\phi_t'\partial_{u'}F(u_t^*, (u_t^*)') + \phi_t\partial_u F(u_t^*, (u_t^*)')\right] dt\,,\end{aligned}$$

which yields the Euler–Lagrange equation

$$-\partial_t(\partial_{u'}F(u_t^*, (u_t^*)')) + \partial_u F(u_t^*, (u_t^*)') = 0\,, \quad t \in (0,\tau)\,,$$

with $u_0^* = 1/\pi(x)$ and $u_\tau^* = 0$, in the weak sense.

In the next proposition we show that minimizers of $\mathscr{F}_\tau$ with enough regularity are solutions of the Euler–Lagrange equation.

Proposition 4.1.5. *Let $u^* \in C[0,\tau] \cap C^1(0,\tau) \cap \mathscr{AD}_\tau$ be a minimizer of $\mathscr{F}_\tau$. Then u^* satisfies the Euler–Lagrange equation*

$$-\partial_t(\partial_{u'}F(u_t^*, (u_t^*)')) + \partial_u F(u_t^*, (u_t^*)') = 0\,, \qquad t \in (0,\tau)\,,$$

in the weak sense, where

$$\begin{aligned}\partial_{u'}F(u,u') &= \Psi'\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u)}}\right)\,, \\ \partial_u F(u,u') &= \kappa_1\frac{1/\pi(y) - 2\mathsf{r}_{\kappa,\pi}u}{\sqrt{u(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u)}}\left(1 - \sqrt{\frac{(u')^2}{4\kappa_1^2 u(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u)} + 1}\right)\,.\end{aligned}$$

Proof. By the chain rule we have

$$\begin{aligned}
\partial_{u'}F(u,u') &= \Psi'\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\right),\\
\partial_u F(u,u') &= \kappa_1\frac{1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u}{\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\Psi\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\right)\\
&\quad-\frac{u'(1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u)}{2u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}\Psi'\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\right)\\
&= \kappa_1\frac{1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u}{\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\left(\Psi\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\right)\right.\\
&\quad\left.-\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\Psi'\left(\frac{u'}{2\kappa_1\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\right)\right)\\
&= \kappa_1\frac{1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u}{\sqrt{u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}}\left(1-\sqrt{\frac{(u')^2}{4\kappa_1^2u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)}+1}\right),
\end{aligned}$$

where we used that $\Psi(x)=x\Psi'(x)-\sqrt{1+x^2}+1$ for the last equality. So F is continuously differentiable on any closed interval $[L_1,L_2]\subset(0,1/\pi(x))$.

Let $\psi\in C_c^\infty(0,\tau)$. Since $u^*(\operatorname{supp}\psi)\subset(0,1/\pi(x))$ by Lemma 4.1.1, there exists $\delta>0$ such that $u^*+h\psi$ is admissible for all $h\in[-\delta,\delta]$.

We claim that for each $h\in[-\delta,\delta]$ and $t\in(0,\tau)$ the map

$$G:[0,1]\ni s\mapsto F(u_t^*+sh\psi_t,(u_t^*)'+sh\psi_t'),$$

is continuously differentiable. Remark that $t\mapsto u_t^*+sh\psi_t$ is admissible for all $h\in[-\delta,\delta]$ and $s\in[0,1]$. Moreover, the map

$$\operatorname{supp}\psi\times[-\delta,\delta]\times[0,1]\ni(t,h,s)\mapsto u_t^*+sh\psi_t,$$

is continuously differentiable. Therefore, its image is compact and contained in $[L,K]\subset(0,1/\pi(x))$ for some $L,K\in(0,1/\pi(x))$. Because F is continuously differentiable on $[L,K]\times\mathbb{R}$, the map

$$\mathscr{G}:\operatorname{supp}\psi\times[-\delta,\delta]\times[0,1]\ni(t,h,s)\mapsto F(u_t^*+sh\psi_t,(u_t^*)'+sh\psi_t'),$$

is continuously differentiable by the chain rule. In particular, the claim for $t\in\operatorname{supp}\psi$ follows and

$$\begin{aligned}
\frac{d}{ds}F(u_t^*+sh\psi_t,(u_t^*)'+sh\psi_t') &= \partial_s\mathscr{G}(t,h,s)\\
&= h\psi_t'\,\partial_{u'}F(u_t^*+sh\psi_t,(u_t^*)'+sh\psi_t')\\
&\quad+h\psi_t\,\partial_uF(u_t^*+sh\psi_t,(u_t^*)'+sh\psi_t').
\end{aligned}$$

If $t \in (0, \tau) \setminus \operatorname{supp}\psi$, then the map G is constant and hence continuously differentiable, which proves the claim.

Since u^* is a minimizer, we see for h with $0 < |h| \leq \delta$ that

$$\begin{aligned}
0 \leq \frac{1}{h}\left(\mathscr{F}(u^* + h\psi) - \mathscr{F}(u^*)\right) &= \int_0^\tau \frac{1}{h}(F(u_t^* + h\psi_t, (u_t^*)' + h\psi_t') - F(u_t^*, (u_t^*)'))dt \\
&= \int_0^\tau \int_0^1 \frac{1}{h}\frac{d}{ds} F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t') ds\, dt \\
&= \int_{\operatorname{supp}\psi} \int_0^1 \psi_t'\, \partial_{u'} F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t') ds\, dt \\
&\quad + \int_{\operatorname{supp}\psi} \int_0^1 \psi_t\, \partial_u F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t') ds\, dt\,,
\end{aligned}$$

where we used the claim for the second and third equality. Since $F \in C^1([L, K] \times \mathbb{R})$ the maps

$$\begin{aligned}
&\operatorname{supp}\psi \times [-\delta, \delta] \times [0, 1] \ni (t, h, s) \mapsto \partial_u F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t')\,, \\
&\operatorname{supp}\psi \times [-\delta, \delta] \times [0, 1] \ni (t, h, s) \mapsto \partial_{u'} F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t')\,,
\end{aligned}$$

are continuous and hence bounded. Therefore, the integrand is h-uniformly bounded. By continuity we have for each $(t, s) \in \operatorname{supp}\psi \times [0, 1]$ that

$$\begin{aligned}
&\psi_t'\, \partial_{u'} F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t') + \psi_t\, \partial_u F(u_t^* + sh\psi_t, (u_t^*)' + sh\psi_t') \\
&\qquad \to \psi_t'\, \partial_{u'} F(u_t^*, (u_t^*)') + \psi_t\, \partial_u F(u_t^*, (u_t^*)')\,,
\end{aligned}$$

as $h \to 0$. Hence, it follows by the Dominated Convergence Theorem that

$$0 \leq \int_{\operatorname{supp}\psi} \psi_t'\, \partial_{u'} F(u_t^*, (u_t^*)') + \psi_t\, \partial_u F(u_t^*, (u_t^*)')\,.$$

We obtain the reverse inequality by the following the same procedure for $-\psi$ instead of ψ. This concludes the proof. □

Next, we study the strong formulation of the Euler–Lagrange equation.

Lemma 4.1.6. *The strong formulation of the Euler–Lagrange equation of $\mathscr{F}_\tau$ yields*

$$u_t'' + 2\kappa_1^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t)\left(1 - \sqrt{\frac{(u_t')^2}{4\kappa_1^2 u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)} + 1}\right)\,, \qquad t \in (0, \tau)\,, \quad (4.1)$$

with $u_0 = 1/\pi(x)$ and $u_\tau = 0$.

Proof. For $u \in C^2(0,\tau)$ we have by the chain rule that

$$\begin{aligned}\partial_t(\partial_{u'}F(u_t,u_t')) &= \frac{d}{dt}\Psi'\left(\frac{u_t'}{2\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}\right)\\ &= \Psi''\left(\frac{u_t'}{2\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}\right)\left(\frac{u_t''}{2\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}\right.\\ &\qquad\left.-\frac{(u_t')^2(1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t)}{4\kappa_1(u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t))^{3/2}}\right).\end{aligned}$$

Noting that

$$\Psi''(x) = \frac{1}{\sqrt{1+x^2}}\,,$$

the strong form yields

$$\begin{aligned}0 &= \partial_t(\partial_{u'}F(u_t,u_t')) - \partial_u F(u_t,u_t')\\ &= \Psi''\left(\frac{u_t'}{2\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}\right)\left(\frac{u_t''}{2\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}} - \frac{(u_t')^2(1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t)}{4\kappa_1(u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t))^{3/2}}\right)\\ &\quad - \kappa_1\frac{1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t}{\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}\left(1-\sqrt{\frac{(u_t')^2}{4\kappa_1^2u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}+1}\right),\end{aligned}$$

or equivalently,

$$\begin{aligned}0 &= u_t'' - (u_t')^2\frac{1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t}{2u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}\\ &\quad - 2\kappa_1^2(1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t)\left(1-\sqrt{\frac{(u_t')^2}{4\kappa_1^2u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}+1}\right)\sqrt{\frac{(u_t')^2}{4\kappa_1^2u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}+1}\\ &= u_t'' + 2\kappa_1^2(1/\pi(y)-2\mathsf{r}_{\kappa,\pi}u_t)\left(1-\sqrt{\frac{(u_t')^2}{4\kappa_1^2u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}+1}\right),\end{aligned}$$

which concludes the proof. □

In what follows, if we refer to the Boundary Value Problem (BVP), we mean the (weak or strong form of) the BVP of the Euler–Lagrange equation.

Remark. For the symmetric case given by $\kappa_1 = \kappa_2 = 1$ and $\pi(x) = \pi(y) = 1/2$, we obtain the equation

$$u_t'' + 4(1-u_t)\left(1-\sqrt{\frac{(u_t')^2}{4u_t(2-u_t)}+1}\right) = 0\,.$$

4.2 Analysis of the associated IVP

In our attempt to show existence of solutions of the BVP on $(0,\tau)$ for each $\tau > 0$, we study an associated Initial Value Problem (IVP). We show that solutions of certain IVP's are solutions of BVP's for certain values of τ. However, we do not show that every BVP has a solution that originates from an IVP.

Setting $v := u'$, we consider the IVP given by

$$\begin{cases} u_t'' + 2\kappa_1^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi}u_t)\left(1 - \sqrt{\frac{(u_t')^2}{4\kappa_1^2 u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)} + 1}\,\right) = 0\,, \\ u_0 = 1/(2\pi(x))\,,\ u_0' = c < 0\,, \end{cases}$$

which can be written as the following system of non-linear first order ODE's:

$$\begin{cases} u' = v\,, \\ v' = 2\kappa_1^2(2\mathsf{r}_{\kappa,\pi}u - 1/\pi(y))\left(1 - \sqrt{\frac{v^2}{4\kappa_1^2 u(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u)} + 1}\,\right)\,, \\ u_0 = 1/(2\pi(x))\,,\ v_0 = c\,. \end{cases} \tag{4.2}$$

The right hand side of (4.2) as a function of (u,v) with domain $(0,1/\pi(x)) \times (-\infty,0)$ is continuously differentiable, and hence locally Lipschitz continuous. By basic ODE theory the IVP has a unique maximal solution $(t_-,t_+) \ni t \mapsto (u_t,v_t)$, which is continuously differentiable on (t_-,t_+). Here, $t_- \in [-\infty,0)$ and $t_+ \in (0,+\infty]$.

In what follows, if we refer to IVP, we mean the IVP given by (4.2).

First we prove some symmetry properties of the solution the IVP.

Lemma 4.2.1. *The maximal solution $(u,v) : I \longrightarrow \mathbb{R}^2$ of the IVP (4.2) has the following properties:*

(i) the maximal interval I is symmetric, i.e., there exists $t_+ \in (0,+\infty]$ such that $I = (-t_+,t_+)$;

(ii) the map $t \mapsto u_t$ is anti-symmetric around $(t,u) = (0,1/(2\pi(x)))$, i.e., $1/(2\pi(x)) - u_t = -(1/(2\pi(x)) - u_{-t})$ for all $t \in I$;

(iii) the map $t \mapsto v_t$ is symmetric around $t = 0$, i.e., $v_t = v_{-t}$ for all $t \in I$.

Proof. Let $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \subset I$ and consider the restriction of (u,v) to this interval. Define $(\zeta,\eta) : (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}^2$ by $(\zeta_t,\eta_t) := (1/\pi(x) - u_{-t}, v_{-t})$. We show that (ζ,η) is also a solution of the IVP. Namely, it follows that

$$\zeta_t' = v_{-t}' = \eta_t'\,.$$

Furthermore, we observe that $\mathsf{r}_{\kappa,\pi}\pi(y) = \pi(x)$ implies that $\mathsf{r}_{\kappa,\pi}\zeta_t = 1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_{-t}$, or equivalently that $-(1/(2\pi(y)) - \mathsf{r}_{\kappa,\pi}\zeta_t) = 1/(2\pi(y)) - \mathsf{r}_{\kappa,\pi}u_{-t}$. With this it follows that

$$\begin{aligned}\eta_t' = -v_{-t}' &= -2\kappa_1^2(2\mathsf{r}_{\kappa,\pi}u_{-t} - 1/\pi(y))\left(1 - \sqrt{\frac{v_{-t}^2}{4\kappa_1^2 u_{-t}(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_{-t})} + 1}\right) \\ &= 2\kappa_1^2(2\mathsf{r}_{\kappa,\pi}\zeta_t - 1/\pi(y))\left(1 - \sqrt{\frac{\eta_t^2}{4\kappa_1^2\zeta_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}\zeta_t)} + 1}\right),\end{aligned}$$

which proves that (ζ, η) is also a solution of (4.2). Thus by uniqueness the maximal interval must be of the form $(-t_+, t_+)$ for some $t_+ \in (0, +\infty]$. Namely, suppose this is not the case. Then by what we have already proven, the solution can be continued on a larger interval, which contradicts the assumption that the solution is maximal. This proves *(i)*. By what we have already proven and again by uniqueness, it follows that $u_t = \zeta_t = 1/\pi(x) - u_{-t}$, or equivalently $1/(2\pi(x)) - u_t = -(1/(2\pi(x)) - u_{-t})$ for all $t \in I$, which proves *(ii)*. By invoking uniqueness once more, it follows that $v_t = \eta_t = v_{-t}$ for all $t \in I$, which proves *(iii)* and concludes the proof. □

The following result show that solutions of the IVP are solutions of the BVP if $|v_0|$ is sufficiently large. We denote by $\boldsymbol{\pi}$ the ratio of the circumference and diameter of a circle.

Proposition 4.2.2. *Let $(u, v) : I \longrightarrow \mathbb{R}^2$ be the maximal solution of the IVP (4.2) with $v_0 < -\frac{\kappa_1\sqrt{\kappa_1}}{2\pi(y)\sqrt{\kappa_2}}\boldsymbol{\pi} =: -M$. Then the following properties are satisfied:*

(i) $v_t \leq v_0 + M < 0$ for all $t \in I$. Consequently, $t \mapsto u_t$ is strictly monotone decreasing;

(ii) the maximal interval I is bounded and there exists $t_+ \in (0, \infty)$ such that $I = (-t_+, t_+)$. Furthermore, $t \mapsto u_{t+t_+} =: \widetilde{u}_t$ is a solution of the Euler–Lagrange equation (4.1) on $(0, \tau) := (0, 2t_+)$ in the classical sense, i.e., $\widetilde{u} \in C^2(0,\tau) \cap C^1[0,\tau]$. Moreover, it holds that

$$\frac{-1}{v_0\pi(x)} \leq \tau \leq \frac{-1}{(v_0 + M)\pi(x)};$$

(iii) the map $(-\infty, -M) \ni v_0 \mapsto t_+(v_0)$ is increasing.

Proof. (i): We show *(i)* for $t \in (0, t_+)$. The case that $t \in (-t_+, 0)$ follows by symmetry. Because $0 < u_t < 1/(2\pi(x))$ and $v_t < 0$ for all $t \in (0, t_+)$, we see that

$$
\begin{aligned}
v_t - v_0 &= \int_0^t v_s' ds \\
&= \int_0^t 2\kappa_1^2 (2\mathsf{r}_{\kappa,\pi} u_s - 1/\pi(y)) \left(1 - \sqrt{\frac{v_s^2}{4\kappa_1^2 u_s (1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_s)} + 1} \right) ds \\
&\le \int_0^t -\frac{\kappa_1}{\pi(y)} \frac{v_s}{\sqrt{u_s(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_s)}} ds \\
&= -\frac{\kappa_1}{\pi(y)} \int_{u_0}^{u_t} \frac{1}{\sqrt{\xi(1/\pi(y) - \mathsf{r}_{\kappa,\pi}\xi)}} d\xi \\
&\le \frac{\kappa_1}{\pi(y)} \int_0^{1/(2\pi(x))} \frac{1}{\sqrt{\xi(1/\pi(y) - \mathsf{r}_{\kappa,\pi}\xi)}} d\xi \\
&= \frac{2\pi(x)\kappa_1\sqrt{\kappa_1}}{\pi(y)\sqrt{\kappa_2}} \int_0^{1/(2\pi(x))} \frac{1}{\sqrt{1 - (2\pi(x)\xi - 1)^2}} d\xi \\
&= -\frac{\kappa_1\sqrt{\kappa_1}}{\pi(y)\sqrt{\kappa_2}} \arcsin(-1) \\
&= \frac{\kappa_1\sqrt{\kappa_1}}{2\pi(y)\sqrt{\kappa_2}} \boldsymbol{\pi},
\end{aligned}
$$

which proves *(i)*.

(ii): Since $t \mapsto u_t$ is bounded and $t \mapsto v_t$ is bounded away from zero, the maximal interval must be bounded. Hence, there exist $t_+ > 0$ such that $I = (-t_+, t_+)$. Furthermore, the image of $t \mapsto u_t$ is $(0, 1/\pi(x))$. Namely, suppose that this is not the case and assume w.l.o.g. that $\lim_{t\uparrow t_+} u_t > 0$. Then because the interval of definition is bounded and $\lim_{t\uparrow t_+} v_t \le v_0 + M < 0$, the solution can be continued, contradicting the assumption that the solution is maximal. Hence, $t \mapsto u_t$ solves the BVP (4.1) with $u_{-t_+} = 1/\pi(x)$ and $u_{t_+} = 0$. Since $t \mapsto v_t$ is increasing and by what we have already shown, we have $v_0 \le v_t \le v_0 + M$. Integrating over $(0, t_+)$ gives $v_0 t_+ \le u_{t_+} - u_0 \le (v_0 + M)t_+$. Because $u_{t_+} - u_0 = -1/(2\pi(x))$, it follows that

$$
\frac{-1}{2v_0\pi(x)} \le t_+ \le \frac{-1}{2(v_0 + M)\pi(x)},
$$

which proves *(ii)*.

(iii): Suppose that $v_0 \mapsto t_+(v_0)$ is not increasing. Then there exist $c_1, c_2 < -M$ such that $c_1 > c_2$ and $t_+(c_1) < t_+(c_2)$. We denote by (u, v) and (ζ, η) the solutions of the IVP with $v_0 = c_1$ and $\eta_0 = c_2$, respectively. By the comparison principle for first order ODEs, it holds that $v_t > \eta_t$ for all $t \in [0, t_+(c_1)]$, so

$$
\int_0^{t_+(c_1)} v_t dt \ge \int_0^{t_+(c_1)} \eta_t dt .
$$

It follows that
$$\int_{t_+(c_1)}^{t_+(c_2)} \eta_t dt = \int_0^{t_+(c_1)} v_t dt - \int_0^{t_+(c_1)} \eta_t dt \geq 0\,;$$
contradiction. Hence, $v_0 \mapsto t_+(v_0)$ is monotone increasing. □

Remark. Actually, we would like to know if *(iii)* can be improved to the statement that the map $v_0 \mapsto t_+(v_0)$ is invertible. If this is the case, the Euler–Lagrange equation (4.1) has a solution that originates from the IVP (4.2). In the present situation, we can only find a sequences $(\tau_n)_n$ with $\tau_n \to 0$ as $n \to \infty$ and IVP solution $(u_n)_n$ such that u_n solves the BVP on $(0, \tau_n)$ for each $n \in \mathbb{N}$.

4.2.1 Are solutions of the IVP minimizers?

We cannot not tell a-priori if solutions of the Euler–Lagrange equations are actually minimizers, because of the singular behavior of F. In this subsection we take a small step in this direction, proving that in the case of our classical solution u^* of the Euler–Lagrange equation, obtained through the associated IVP, the map $h \mapsto \mathscr{F}_\tau(u^* + h\psi)$ is differentiable at $h = 0$. Then we describe a potential strategy that uses this result.

Recall that we defined $\mathscr{F}_\tau(u) = \int_0^\tau F(u_t, u_t')dt$, where u continuous and weakly differentiable with integrable derivative, and with $u_0 = 1/\pi(x)$, $u_\tau = 0$ and $0 < u_t < 1/\pi(x)$ for all $t \in (0, \tau)$. We called this class of maps admissible. Let u^* be the solution of the IVP (4.2), which is a classical solution of the Euler–Lagrange equation (4.1) by Proposition 4.2.2, i.e., $u^* \in C^2(0, \tau) \cap C^1[0, \tau]$. An important property of u^* is that strictly monotone. Namely, this admits for every $\psi \in C_c^1[0, \tau] := \{\varphi \in C[0, \tau] : \operatorname{supp}\varphi \subset (0, \tau)\,, \varphi|_{\operatorname{supp}\varphi} \in C^1(\operatorname{supp}\varphi)\}$ the existence of $\delta > 0$ such that for each $h \in [-\delta, \delta]$ it holds that $u_t^* + h\psi_t \in (0, 1/\pi(x))$ for all $t \in \operatorname{supp}\psi$. Hence, $u^* + h\psi$ is admissible for all $h \in [-\delta, \delta]$. Furthermore, we have that $u^* + h\psi \in C^1[0, \tau]$.

Lemma 4.2.3. *Let u^* be the classical solution of the Euler–Lagrange equation (4.1), which is obtained by solving IVP as in Proposition 4.2.2. Let $\psi \in C_c^1[0, \tau]$ and $\delta > 0$ such that $u^* + h\psi$ is admissible for all $h \in [-\delta, \delta]$. Then the map $[-\delta, \delta] \ni h \mapsto \mathscr{F}_\tau(u^* + h\psi)$ is differentiable at $h = 0$ with*
$$\frac{d}{dh}\left(\mathscr{F}_\tau(u^* + h\psi)\right)\big|_{h=0} = \int_0^\tau \psi_t'\,\partial_{u'}F(u_t^*, (u^*)_t') + \psi_t\,\partial_u F(u_t^*, (u^*)_t')dt = 0\,.$$

Proof. By following the same reasoning as in Proposition 4.1.5, it follows that $[-\delta, \delta] \ni h \mapsto \mathscr{F}(u^* + h\psi)$ is differentiable at $h = 0$ and
$$\frac{d}{dh}\left(\mathscr{F}_\tau(u^* + h\psi)\right)\big|_{h=0} = \int_0^\tau \psi_t'\,\partial_{u'}F(u_t^*, (u^*)_t') + \psi_t\,\partial_u F(u_t^*, (u^*)_t')dt\,.$$
By using the fact that u^* is a classical solution of the Euler–Lagrange equation and integration by parts, it follows that
$$\int_0^\tau \psi_t'\,\partial_{u'}F(u_t^*, (u^*)_t') + \psi_t\,\partial_u F(u_t^*, (u^*)_t')dt = 0\,,$$

which concludes the proof. □

Remark. We give a potential strategy for proving that solutions of the Euler–Lagrange equation are actually minimizers.

Let $\psi \in C_c^1[0,\tau]$ such that $u^* \pm \psi$ is admissible. By convexity of $\mathcal{R}$, the map $[-1,1] \ni h \mapsto \mathscr{F}_\tau(u^* + h\psi) =: g(h)$ is convex, and it follows that $g(h) \geq g(0) + hg'(0)$ for all $h \in [-1,1]$. With $h = 1$ this inequality yields

$$\mathscr{F}_\tau(u^* + \psi) = g(1) \geq g(0) + g'(0) = \mathscr{F}_\tau(u^*) + \frac{d}{dh}\left(\mathscr{F}_\tau(u^* + h\psi)\right)\big|_{h=0} = \mathscr{F}_\tau(u^*)\,.$$

Let u be an admissible curve. If one can approximate $u - u^*$ by a sequence $(\psi^n)_n$ in $C_c^1[0,\tau]$ such that

$$\mathscr{F}_\tau(u) \geq \limsup_{n\to\infty} \mathscr{F}_\tau(u^* + \psi^n) \geq \mathscr{F}(u^*)\,,$$

then one has proven that u^* is indeed a minimizer.

4.3 On the convexity of entropy

In the case of the two point space, the entropy $\mathcal{S}$ of a curve $t \mapsto \rho_t \in \mathcal{M}^+(V)$ is given by

$$t \mapsto \mathcal{S}(\rho_t) = \frac{d\rho_t}{d\pi}(x)\log\left(\frac{d\rho_t}{d\pi}(x)\right)\pi(x) + \frac{d\rho_t}{d\pi}(y)\log\left(\frac{d\rho_t}{d\pi}(y)\right)\pi(y)\,.$$

In this section we show that $t \mapsto \mathcal{S}(\rho_t)$ for curves $t \mapsto \rho_t$ associated to the IVP (4.2) is convex if $|v_0|$ is sufficiently large in Corollary 4.3.4. More precisely, there exists a constant depending on $(\kappa_1, \kappa_2, \pi)$ such that the entropy of any IVP solution with $|v_0|$ greater than this constant is convex.

First we obtain two preliminary results about the convexity of entropy in the middle of the domain in Lemma 4.3.1 and close to the boundary in Lemma 4.3.3.

Recall that by definition of u_t we have $u_t(x) = \frac{d\rho_t}{d\pi}(x) = u_t$ and $u_t(y) = \frac{d\rho_t}{d\pi}(y) = 1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t$. We see that

$$\begin{aligned}
&\frac{d^2}{dt}\left(u_t(x)\log u_t(x)\right) = u_t''(1 + \log u_t) + \frac{(u_t')^2}{u_t}\,,\\
&\frac{d^2}{dt}\left(u_t(y)\log u_t(y)\right) = -\mathsf{r}_{\kappa,\pi}u_t''(1 + \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)) + \mathsf{r}_{\kappa,\pi}^2\frac{(u_t')^2}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\,.
\end{aligned}$$

Therefore, by using that $\mathsf{r}_{\kappa,\pi}\pi(y) = \pi(x)$, it follows that

$$\begin{aligned}
\frac{d^2}{dt}\mathcal{S}(\rho_t) &= \pi(x)\left(u_t''\log u_t - u_t''\log(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t) + \frac{(u_t')^2}{u_t} + \mathsf{r}_{\kappa,\pi}\frac{(u_t')^2}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right)\\
&= \pi(x)\left(u_t''\log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right) + \frac{(u_t')^2}{u_t(1 - \pi(x)u_t)}\right)\,. \qquad (4.3)
\end{aligned}$$

The following lemma shows that for any configuration (κ, π) the entropy of IVP solutions is convex in an open neighbourhood of $t = 0$.

Lemma 4.3.1. *Let $(u, u') : (-t_+, t_+) \longrightarrow \mathbb{R}^2$ be a solution of IVP (4.2). Then the entropy $t \mapsto \mathcal{S}(\rho_t)$ is convex on $(-t', t')$ for some $t' \in (0, t_+)$.*

Proof. For all $t \geq 0$ it holds that $2\mathsf{r}_{\kappa,\pi} u_t - 1/\pi(y) \leq 0$. By Lemma 4.3.2 below, with

$$\lambda = \frac{(u_t')^2}{4u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)},$$

it follows that

$$\begin{aligned} u_t'' &= 2(2\mathsf{r}_{\kappa,\pi} u_t - 1/\pi(y))\kappa_1^2 \left(1 - \sqrt{\frac{(u_t')^2}{4\kappa_1^2 u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)} + 1}\right) \\ &= 2(2\mathsf{r}_{\kappa,\pi} u_t - 1/\pi(y))\kappa_1^2 \left(1 - \sqrt{\frac{\lambda}{\kappa_1^2} + 1}\right) \\ &\leq 2(2\mathsf{r}_{\kappa,\pi} u_t - 1/\pi(y)) \lim_{\xi \to \infty} \xi^2 \left(1 - \sqrt{\frac{\lambda}{\xi^2} + 1}\right) \\ &= (1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t)\lambda \\ &= \frac{(u_t')^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t)}{4u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)}. \end{aligned}$$

Since for all $t \geq 0$ we have $u_t \leq 1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t$, it holds that $\log u_t - \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t) \leq 0$. Therefore, it follows that

$$u_t'' \log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t}\right) \geq \frac{(u_t')^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t)}{4u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)} \log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t}\right), \tag{4.4}$$

for $t \geq 0$. Similarly, for all $t \leq 0$ we have that $2\mathsf{r}_{\kappa,\pi} u_t - 1/\pi(y) \geq 0$ it follows by proceeding as above that

$$u_t'' \geq \frac{(u_t')^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t)}{4u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t)} = \frac{(u_t')^2(1 - 2\pi(x) u_t)}{4u_t(1 - \pi(x) u_t)}.$$

Since $u_t \geq 1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t$ and therefore $\log u_t - \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t) \geq 0$ for all $t \geq 0$, the inequality (4.4) also holds for all $t \leq 0$. With this it follows from (4.3) that

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{S}(\rho_t) &= \pi(x) \left(u_t'' \log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t}\right) + \frac{(u_t')^2}{u_t(1 - \pi(x) u_t)}\right) \\ &\geq \pi(x) \left(\frac{(u_t')^2(1 - 2\pi(x) u_t)}{4u_t(1 - \pi(x) u_t)} \log\left(\frac{u_t}{1/\pi(y) - 2\mathsf{r}_{\kappa,\pi} u_t}\right) + \frac{(u_t')^2}{u_t(1 - \pi(x) u_t)}\right) \\ &= \frac{\pi(x)(u_t')^2}{4u_t(1 - \pi(x) u_t)} \left((1 - 2\pi(x) u_t) \log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t}\right) + 4\right) \\ &= \frac{\pi(x)(u_t')^2}{4u_t(1 - \pi(x) u_t)} \Xi(u_t), \end{aligned}$$

for all $t \in (-t_+, t_+)$, where the map $\Xi : (0, 1/\pi(x)) \longrightarrow \mathbb{R}$ is given by

$$\Xi(\xi) = (1 - 2\pi(x)\xi) \log\left(\frac{\xi}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}\xi}\right) + 4\,.$$

Because $\Xi(1/(2\pi(x))) = 4$, there exists by continuity $\xi_0 \in (0, 1/(2\pi(x)))$ such that $\Xi(\xi) > 0$ for all $\xi \in (-\xi_0 + 1/(2\pi(x)), 1/(2\pi(x)) + \xi_0)$.

Since $u_0 = 1/(2\pi(x))$, there exists by continuity $t' \in (0, t_+)$ such that $u((-t', t')) \subset (-\xi_0 + 1/(2\pi(x)), 1/(2\pi(x)) + \xi_0)$. It follows that $\Xi(u_t) \geq 0$ for all $t \in (-t', t')$ and hence that

$$\frac{d^2}{dt^2}\mathcal{S}(\rho_t) \geq 0\,,$$

which proves the lemma. □

Remark. Since $\Xi(\xi) \to -\infty$ as $\xi \downarrow 0$ and $\xi \uparrow 1/\pi(x)$, the proof of Lemma 4.3.1 cannot be extended case that t is near the boundary of $(-t_+, t_+)$.

The following result is used in the proof of the previous lemma.

Lemma 4.3.2. *Let $\lambda > 0$. The map $(0, \infty) \ni \xi \mapsto \xi^2\left(1 - \sqrt{1 + \frac{\lambda}{\xi^2}}\right)$ is negative and strictly decreasing, and*

$$\lim_{\xi\to\infty} \xi^2\left(1 - \sqrt{1 + \frac{\lambda}{\xi^2}}\right) = -\frac{\lambda}{2}\,.$$

Proof. We set $H_\lambda(\xi) := \xi^2\left(1 - \sqrt{1 + \frac{\lambda}{\xi^2}}\right)$. Then H_λ is negative, because $1 < \sqrt{1 + \frac{\lambda}{\xi^2}}$. It follows that

$$\begin{aligned}
H_1'(\xi) &= 2\xi\left(1 - \sqrt{1 + \frac{1}{\xi^2}}\right) + \frac{1}{\xi\sqrt{1 + 1/\xi^2}} \\
&= \frac{-1}{\sqrt{1+\xi^2}}\left(2\xi^2 - 2\xi + 1\right) \\
&= \frac{-1}{\sqrt{1+\xi^2}}\left(2\left(\xi - \frac{1}{2}\right)^2 + \frac{1}{2}\right) \\
&\leq \frac{-1}{2\sqrt{1+\xi^2}} \\
&< 0\,.
\end{aligned}$$

Hence, H_1 is strictly decreasing. We see that

$$\begin{aligned} H_\lambda(\xi) &= \xi^2\left(1-\sqrt{1+\frac{\lambda}{\xi^2}}\right) \\ &= \lambda\left(\frac{\xi}{\sqrt{\lambda}}\right)^2\left(1-\sqrt{1+\frac{1}{(\xi/\sqrt{\lambda})^2}}\right) \\ &= \lambda H_1\left(\frac{\xi}{\sqrt{\lambda}}\right). \end{aligned}$$

Let $\xi_1, \xi_2 \in (0,\infty)$ with $\xi_1 < \xi_2$. It follows that

$$H_\lambda(\xi_1) = \lambda H_1\left(\frac{\xi_1}{\sqrt{\lambda}}\right) > \lambda H_1\left(\frac{\xi_2}{\sqrt{\lambda}}\right) = H_\lambda(\xi_2)\,,$$

proving that H_λ is strictly decreasing. Finally, with L'Hospital's rule it follows that

$$\begin{aligned} \lim_{\xi\to\infty}\xi^2\left(1-\sqrt{1+\frac{\lambda}{\xi^2}}\right) &= \lim_{\xi\to\infty} -\frac{1}{2\sqrt{1+\lambda/\xi^2}}\frac{-2\lambda\xi^{-3}}{-2\xi^{-3}} \\ &= -\frac{\lambda}{2}. \end{aligned}$$

□

We show that the entropy is convex near the boundary for any configuration (κ,π).

Lemma 4.3.3. *Let $(u,u') : (-t_+, t_+) \longrightarrow \mathbb{R}^2$ be a solution of IVP (4.2) with $\inf_{t\in(-t_+,t_+)}|u'_t| > 0$. Then the entropy $t \mapsto \mathcal{S}(\rho_t)$ is convex on $(-t_+, -t_+ + \delta) \cup (t_+ - \delta, t_+)$ for some $\delta > 0$.*

Proof. For $t \geq 0$ we have that

$$\begin{aligned} u''_t &= 2\kappa_1^2(1/\pi(y) - 2\mathsf{r}_{\kappa,\pi}u_t)\left(-1+\sqrt{\frac{(u'_t)^2}{4\kappa_1^2 u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}+1}\right) \\ &\leq \frac{\kappa_1}{\pi(y)}\frac{|u'_t|}{\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}. \end{aligned}$$

By symmetry of (u,u') around $t=0$, see Lemma 4.2.1, it follows that

$$u''_t \geq -\frac{\kappa_1}{\pi(y)}\frac{|u'_t|}{\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}}.$$

Since $\log u_t - \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi} u_t) \le 0$ if $t \ge 0$, it follows that

$$\begin{aligned}\frac{d^2}{dt}\mathcal{S}(\rho_t) &= \pi(x)\left(u_t'' \log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right) + \frac{(u_t')^2}{u_t(1-\pi(x)u_t)}\right)\\ &\ge \pi(x)\left(\frac{\kappa_1}{\pi(y)}\frac{|u_t'|}{\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}}\log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right) + \frac{(u_t')^2}{u_t(1-\pi(x)u_t)}\right)\\ &= \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\left(\kappa_1\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\,\log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right) + |u_t'|\right).\end{aligned} \tag{4.5}$$

Because $\lim_{t\uparrow t_+} u_t = 0$, $\lim_{\xi\downarrow 0}\sqrt{\xi}\log(\xi) \to 0$, $\lim_{t\uparrow t_+}|u_t'| = \inf_{t\in(-t_+,t_+)}|u_t'| > 0$, it follows that (4.5) tends to $+\infty$ as $t \uparrow t_+$. Hence, there exists $\delta_1 > 0$ such that $\frac{d^2}{dt}\mathcal{S}(\rho_t) \ge 0$ for all $t \in (t_+ - \delta_1, t_+)$.

Analogously, since $\log u_t - \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t) \ge 0$ for all $t \le 0$, it follows that

$$\frac{d^2}{dt}\mathcal{S}(\rho_t) \ge \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\left(\kappa_1\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\,\log\left(\frac{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}{u_t}\right) + |u_t'|\right), \tag{4.6}$$

Because $\lim_{t\downarrow -t_+} u_t = 1/\pi(x)$, $\lim_{\xi\downarrow 0}\sqrt{\xi}\log(\xi) \to 0$, $\lim_{t\downarrow -t_+}|u_t'| = \inf_{t\in(-t_+,t_+)}|u_t'| > 0$, it follows that (4.6) tends to $+\infty$ as $t \uparrow t_+$. Hence, there exists $\delta_2 > 0$ such that $\frac{d^2}{dt}\mathcal{S}(\rho_t) \ge 0$ for all $t \in (-t_+, -t_+ + \delta_2)$, which concludes the proof. □

We show that for solutions of the IVP with $|v_0|$ sufficiently large, the inequalities (4.5) and (4.6) in the proof of the previous lemma imply the convexity of entropy on the entire domain.

Corollary 4.3.4. *The entropy $t \mapsto \mathcal{S}(\rho_t)$ associated to a solution of the IVP (4.2) with $v_0 \le -\frac{2}{\pi(x)e}\max\{\kappa_1,\kappa_2\} - M$ is convex. Here, $M = \frac{\kappa_1\sqrt{\kappa_1}}{2\pi(y)\sqrt{\kappa_2}}\boldsymbol{\pi}$ as defined in Proposition 4.2.2.*

Proof. It holds by Proposition 4.2.2 that $\inf_{t\in(-t_+,t_+)}|u_t'| \ge -v_0 - M \ge \frac{2}{\pi(x)e}\max\{\kappa_1,\kappa_2\}$. Recall that $\log u_t - \log(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t) \le 0$ for all $t \ge 0$. From inequality (4.5) in the proof of Lemma 4.3.3 we know that

$$\begin{aligned}\frac{d^2}{dt}\mathcal{S}(\rho_t) &\ge \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\left(\kappa_1\sqrt{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\,\log\left(\frac{u_t}{1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t}\right) + |u_t'|\right)\\ &\ge \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y) - \mathsf{r}_{\kappa,\pi}u_t)}\left(\kappa_1\sqrt{u_t/\pi(y)}\,\log(\pi(y)u_t) + \frac{2\kappa_2}{\pi(x)e}\right)\\ &\ge 0,\end{aligned}$$

for all $t \ge 0$ by Lemma 4.3.5 below.

Similarly, it follows from (4.6) that

$$\begin{aligned}\frac{d^2}{dt}\mathcal{S}(\rho_t) &\geq \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}\left(\kappa_1\sqrt{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}\,\log\left(\frac{1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t}{u_t}\right)+|u_t'|\right)\\ &\geq \frac{\mathsf{r}_{\kappa,\pi}|u_t'|}{u_t(1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t)}\left(\frac{\kappa_1}{\sqrt{\pi(x)}}\sqrt{1/\pi(y)-\mathsf{r}_{\kappa,\pi}u_t}\,\log\left(\mathsf{r}_{\kappa,\pi}-\mathsf{r}_{\kappa,\pi}\pi(x)u_t\right)+\frac{2\kappa_1}{\pi(x)e}\right)\\ &\geq 0\,,\end{aligned}$$

for all $t \leq 0$ by Lemma 4.3.5 below, which concludes the proof. □

The following Lemma is used in the proof of Corollary 4.3.4.

Lemma 4.3.5. *The maps*

$$\begin{aligned}G_1 :&(0,\infty)\longrightarrow\mathbb{R}\,,\quad \xi\mapsto\kappa_1\sqrt{\xi/\pi(y)}\,\log\left(\pi(y)\xi\right)+\frac{2\kappa_2}{\pi(x)e}\,,\\ G_2 :&(-\infty,1/\pi(x))\longrightarrow\mathbb{R}\,,\quad \xi\mapsto\frac{\kappa_1}{\sqrt{\pi(x)}}\sqrt{1/\pi(y)-\mathsf{r}_{\kappa,\pi}\xi}\,\log\left(\mathsf{r}_{\kappa,\pi}-\mathsf{r}_{\kappa,\pi}\pi(x)\xi\right)+\frac{2\kappa_1}{\pi(x)e}\,,\end{aligned}$$

are non-negative.

Proof. For each $\xi > 0$, there exists $\eta > 0$ such that $\xi = \eta^2/\pi(y)$. It follows that

$$\begin{aligned}G_1(\xi) = G_1\left(\frac{\eta^2}{\pi(y)}\right) &= \frac{2\kappa_1}{\pi(y)}\eta\log(\eta)+\frac{2\kappa_2}{\pi(x)e}\\ &= \frac{2\kappa_1}{\pi(y)}\left(\eta\log\eta+\frac{1}{e}\right)\\ &\geq 0\,,\end{aligned}$$

since $\eta\mapsto\eta\log\eta$ has global minimum equal to $-e^{-1}$.

Similarly, for every $\xi < 1/\pi(x)$, it holds that $\mathsf{r}_{\kappa,\pi}-\mathsf{r}_{\kappa,\pi}\pi(x)\xi > 0$ and there exists $\eta > 0$ such that $\mathsf{r}_{\kappa,\pi}-\mathsf{r}_{\kappa,\pi}\pi(x)\xi = \eta^2$. It follows that

$$G_2(\xi) = \frac{2\kappa_1}{\pi(x)}\left(\eta\log\eta+\frac{1}{e}\right) \geq 0\,,$$

which concludes the proof. □

Chapter 5

Conclusion

In Chapter 1, the introduction, we set out to obtain a characterization of lower bounds of Ollivier curvature in terms of the heat flow on the graph in the framework of gradient flows introduced by Peletier et al. [12], in the spirit of the result by Von Renesse and Sturm [15] for Riemannian manifolds. Such a characterization would provide a new point of view for Ollivier curvature, which opens up the way to a better understanding of this notion of curvature on graphs. The steps we have taken in this thesis are the following.

In Chapter 3 we establish a connection between the DVT-cost $\mathcal{W}^\tau$ and the 1-Wasserstein distance W^1, which comprises several steps. Firstly, given a finite graph and two positive measures μ and ν on the vertices of the graph, we define a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ on the set of solutions of the continuity equation on the graph connecting μ and ν in Definition 3.2.4. Equipping the domain of the functionals with the topology defined in Section 3.2 on page 36, we prove that $(\mathcal{F}_n)$ converges in the framework of Γ-convergence to the limiting functional $\mathcal{F}_\infty$, as defined in Definition 3.2.6, in Theorem 3.2.7.

Secondly, we find a lower bound of $(\mathcal{F}_n)$ in terms of $\mathcal{F}_\infty$ in Proposition 3.3.3. We combine this with the Γ-convergence to obtain convergence of minimal values, i.e., that $\log(n)^{-1}\mathcal{W}^{1/n}(\mu,\nu) = \min \mathcal{F}_n \to \inf \mathcal{F}_\infty$ as $n \to \infty$, in Proposition 3.5.1. However, the framework of Γ-convergence provides tools for showing in addition the convergence of minimizers of functionals. Consequently, the question whether this can be achieved for our particular functionals, arises. This is still an open problem.

Finally, by combining the convergence of minimal values and the Kantorovich–Rubinstein duality formula for the 1-Wasserstein distance, we obtain via the continuity equation that $W^1(\mu,\nu) \leq \lim_{\tau\downarrow 0} \log(\tau^{-1})^{-1}\mathcal{W}^\tau(\mu,\nu)$ in Corollary 3.6.5. Here we equip the graph with the combinatorial graph distance. The problem whether this inequality can be improved to an equality, is still open.

Since a lower bound of the Ollivier curvature yields a contraction estimate in W^1 of a 'truncation' of the heat kernel, this connection between W^1 and $\mathcal{W}^\tau$ can be helpful for finding a relationship between lower bounds of Ollivier curvature and the heat flow on the graph as viewed as gradient flow with respect to entropy in the framework of Peletier et al. that satisfies a contraction estimate in the DVT-cost. Therefore, further research in this direction is recommended.

Again in the spirit of Von Renesse and Sturm, in Chapter 4 we set out to study the convexity of entropy along minimizers of the DVT-cost between two Dirac measures on graphs consisting of two points. The steps we took, consist of several parts that we have not been able to connect in this thesis.

The first step is to characterize the minimizers of the DVT-cost. Since the DVT-cost is the minimal value of a convex functional, we attempt to do this via its Euler–Lagrange equation. In Proposition 4.1.5 we derive this equation and prove, assuming that minimizers enjoy some extra regularity, that minimizers are solutions of the Euler–Lagrange equation. However, we have not shown that minimizers actually enjoy this additional regularity, and this problem is still open.

Next we try to show that there exists a solution of the Euler–Lagrange equation that is a minimizer of the DVT-cost. To achieve this, we study a family of associated initial value problems (IVPs), and show that some of its solutions solve the Euler–Lagrange equations in Proposition 4.2.2. Furthermore, these solutions are highly regular and enjoy symmetry properties as shown in Lemma 4.2.1. In Section 4.2.1 we outline a strategy for showing that the IVP solutions are minimizers of the DVT-cost, but this problem remains open.

Although we have not succeeded in proving that the Euler–Lagrange equations characterize the minimizers of the DVT-cost, we study the convexity of entropy along the candidate minimizers that are provided by the family of IVPs. In Corollary 4.3.4 we prove that the entropy along solutions of the IVPs, where the absolute value of the initial velocity is sufficiently large, is indeed convex.

Recommendations for further research include the study of regularity of solutions of the Euler–Lagrange equations to get closer to the characterization of minimizers. This regularity could also be of useful an attempt to establish a correspondence between solutions of the Euler–Lagrange equations and its associated IVPs. Furthermore, what can we say about the entropy along minimizers of the DVT-cost on graphs consisting of more than two vertices?

In conclusion, the work in this thesis is indeed only a first step towards a characterization of lower bounds of Ollivier's notion of curvature on graphs via gradient flows.

Appendix A

Analysis

In this appendix we introduce the concept of lower semicontinuity and state some results that are used throughout the thesis. Furthermore, we introduce the notion of weak*-topology and narrow topology on the set Borel measures with finite total variation on a topological space.

Assume that X is a topological space, unless mentioned otherwise. We denote the extended real line by $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $[0, +\infty] := [0, \infty) \cup \{+\infty\}$.

Definition A.1 ([6, Definition 1.1])**.** We say that a functional $F : X \longrightarrow \overline{\mathbb{R}}$ is *lower semicontinuous at a point* $x \in X$ if for every $t \in \mathbb{R}$ such that $t < F(x)$, there exists a neighbourhood U of x such that $t < F(y)$ for every $y \in U$. We say that F is *lower semicontinuous (on* X*)* if F is lower semiconintous at each point $x \in X$.

The definition of *upper semicontinuity* is obtained by replacing $<$ by $>$.

Remark ([6, Proposition 1.7])**.** From this definiton it is easy to see that the following are equivalent:

(1) $F : X \longrightarrow \overline{\mathbb{R}}$ is lower semicontinuous on X;

(2) for every $t \in \mathbb{R}$ the set $F^{-1}((t, +\infty])$ is open;

(3) for every $t \in \mathbb{R}$ the set $F^{-1}([-\infty, t])$ is closed.

Definition A.2. We say that a functional $F : X \longrightarrow \overline{\mathbb{R}}$ is *sequentially lower semicontinuous at a point* $x \in X$ if for every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to x as $n \to \infty$, it holds that

$$F(x) \leq \liminf_{n\to\infty} F(x_n) \,.$$

We say that that F is *sequentially lower semicontinuous (on* X*)* if F is sequentially lower semicontinuous at every point $x \in X$.

The definition of *sequential upper semicontinuity* is obtained by replacing $\leq$ by $\geq$ and inf by sup.

In the case that X satisfies the first axiom of countability the previous definitions are equivalent.

Proposition A.3 ([6, Proposition 1.3])**.** *Let X satisfy the first axiom of countability. Then the functional $F : X \longrightarrow \overline{\mathbb{R}}$ is lower semicontinuous at a point $x \in X$ if and only if F is sequentially lower semicontinuous at $x \in X$.*

An easy consequence of the definition of lower semicontinuity is the following.

Lemma A.4. *Let $F : X \longrightarrow \overline{\mathbb{R}}$ and $x \in X$. Then F is lower semicontinuous at x if and only if $-F$ is upper semicontinuous at x.*

The next two lemmas show that taking sums and suprema preserve lower semicontinuity.

Lemma A.5 ([6, Proposition 1.9])**.** *If $F : X \longrightarrow [0, +\infty]$ and $G : X \longrightarrow [0, +\infty]$ are lower semicontinuous, then $F + G$ is again lower semicontinuous.*

Lemma A.6 ([6, Proposition 1.8])**.** *Let I be a non-empty set and let $(F_i)_{i\in I}$ be a family of lower semicontinuous functionals $F_i : X \longrightarrow \overline{\mathbb{R}}$. Then the functional $F : X \longrightarrow \overline{\mathbb{R}}$ defined by*

$$F(x) := \sup_{i\in I} F_i(x) \,,$$

is lower semicontinuous.

For compositions of a functional and a continuous mapping we have the following.

Lemma A.7. *Let X and Y be topological spaces and let $g : X \longrightarrow Y$ be a continuous map. If $f : Y \longrightarrow \overline{\mathbb{R}}$ is a sequentially lower semicontinuous map, then $f \circ g$ is again sequentially lower semicontinuous.*

Proof. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence that converges to x as $n \to \infty$. Since g is continuous, it follows that $g(x_n) \to g(x)$ as $n \to \infty$. Therefore, it follows with the sequential lower semicontinuity of f that

$$(f \circ g)(x) = f(g(x)) \leq \liminf_{n\to\infty} f(g(x_n)) = \liminf_{n\to\infty} (f \circ g)(x_n) \,,$$

which concludes the proof. □

We need the following lemma for several results related to the $\mathcal{R}$-action.

Lemma A.8. *Let $F : X \longrightarrow [0, +\infty]$ be lower semicontinuous. Then F is Borel measurable.*

Proof. Let $t \in \mathbb{R}$. Then $F^{-1}((-\infty, t]) = F^{-1}([-\infty, t])$ is closed. Hence, it is contained in $\mathscr{B}(X)$ and F is Borel measurable. □

A.1 The weak*-topology

In this thesis we frequently talk about the weak*-topology on (sub)sets of the space Borel measures with finite total variation. We will introduce this topology in a general setting, which can be found in any standard textbook on Functional Analysis such as [4].

Let $(X, \|\cdot\|)$ be a normed vector space and denote by $(X^*, \|\cdot\|_*)$ its dual. It is well known that $X^* \ni x^* \mapsto x^*(x)$ defines a bounded linear functional on X^* for any $x \in X$. Define the semi-norms $p_x : X^* \longrightarrow [0, \infty)$ by $p_x(x^*) := |x^*(x)|$.

Definition A.9 ([4, Definition V.1.1])**.** The weak*-topology on X^* is the smallest topology containing all sets of the form

$$\bigcap_{i=1}^{n} \{x^* \in X^* : p_{x_i}(x^* - x_0^*) < \varepsilon_i\}\,, \qquad x_0^* \in X^*,\ n \in \mathbb{N},\ x_1, \ldots, x_n \in X,\ \varepsilon_1, \ldots, \varepsilon_n > 0\,.$$

The collection

$$\mathcal{B}(x_0^*) = \left\{ \bigcap_{i=1}^{n} \{x^* \in X^* : p_{x_i}(x^* - x_0^*) < \varepsilon_i\} : n \in \mathbb{N},\ x_1, \ldots, x_n \in X,\ \varepsilon_1, \ldots, \varepsilon_n > 0 \right\},$$

is a local neighbourhood base of x_0^*, and convergence in the weak*-topology is characterized by the following.

Lemma A.10. *Let $x^* \in X^*$ and let be $(x_n^*)_n$ a sequence in X^*. Then $(x_n^*)_n$ converges to x^* in the weak*-topology if and only if $x_n^*(x) \to x^*(x)$ as $n \to \infty$ for every $x \in X$.*

Let Y be a locally compact topological space. For the purposes of this thesis, we choose $X = C_0(Y)$, the vector space of continuous functions vanishing at infinity, equipped with the supremum norm. Its dual is isometrically isomorphic to $\mathcal{M}(Y)$ by Riesz' Representation Theorem [4, Theorem C.18] via

$$I : \mathcal{M}(Y) \ni \mu \mapsto \left(f \mapsto \int_Y f d\mu \right) =: F_\mu \in C_0(Y)^*\,.$$

Definition A.11. The weak*-topology on $\mathcal{M}(Y)$ is defined as the topology induced by weak*-topology on $C_0(Y)^*$ via I. More precisely, the weak*-topology is given by

$$\{\, I^{-1}(U) : U \subset C_0(Y)^* \text{ weakly*-open} \,\}\,.$$

An immediate consequence of this definition is the following.

Lemma A.12. *Let $\mu \in \mathcal{M}(Y)$ and let $(\mu_n)_n$ be a sequence in $\mathcal{M}(Y)$. Then $(\mu_n)_n$ converges weakly* to μ if and only if*

$$\int_Y f d\mu_n \to \int_Y f d\mu\,,$$

as $n \to \infty$ for all $f \in C_0(Y)$.

The narrow topology
We denote by $(C_b(Y), \|\cdot\|_\infty)$ the Banach space of continuous bounded functions on Y equipped with the supremum norm. Considering the canonical embedding $I_b : \mathcal{M}(Y) \hookrightarrow C_b(Y)^*$ by $\mu \mapsto F_\mu$, we define the narrow topology as follows.

Definition A.13. The narrow topology on $\mathcal{M}(Y)$ is the subspace topology induced by the weak*-topology on $(C_b(Y))^*$ via the canonical embedding I_b. More precisely, the narrow topology is given by $\{ I^{-1}(U) : U \subset I_b(\mathcal{M}(Y)) \text{ weakly*-open} \}$.

Analogous to Lemma A.7 we have the following.

Lemma A.14. *Let $\mu \in \mathcal{M}(Y)$ and let $(\mu_n)_n$ be a sequence in $\mathcal{M}(Y)$. Then $(\mu_n)_n$ converges narrowly to μ if and only if*

$$\int_Y f d\mu_n \to \int_Y f d\mu \,,$$

as $n \to \infty$ for all $f \in C_b(Y)$.

Appendix B

Measure theory

In this appendix we discuss various measure-theoretic topics that are generally not taught in an introductory course on measure theory.

Let (X, Σ) be a measurable space and $m \in \mathbb{N}$. We denote by $\mathcal{M}(X, \Sigma; \mathbb{R}^m)$ the set of (signed) measures $\mu : \Sigma \longrightarrow \mathbb{R}^n$ with finite total variation

$$\|\mu\|_{\mathrm{TV}(X)} := |\mu|(X) := \sup \left\{ \sum_{i=0}^{\infty} |\mu(X_i)| \,:\, X_i \in \Sigma, \quad X_i \text{ pairwise disjoint}, \quad \bigcup_{i=0}^{\infty} X_i = X \right\} .$$

The function $|\mu| : \Sigma \longrightarrow [0, \infty)$ defines a finite positive measure on X by [1, Theorem 1.6].

If X is a topological space, then we denote by $\mathscr{B}(X)$ the Borel σ-algebra on X. In this case we define $\mathcal{M}(X; \mathbb{R}^m) := \mathcal{M}(X, \mathscr{B}(X); \mathbb{R}^m)$, and $\mathcal{M}(X) = \mathcal{M}(X; \mathbb{R})$. We denote by $\mathcal{M}^+(X)$ the subset of $\mathcal{M}$ consisting of positive finite Borel measures.

Let in addition X be locally compact. Identifying $\mu \in \mathcal{M}(X; \mathbb{R}^m)$ with a vector $(\mu_1, \ldots, \mu_m) \in \bigotimes_{i=1}^m \mathcal{M}(X)$, it follows by Riesz' Representation Theorem [4, Theorem C.18] that the map

$$\mathcal{M}(X; \mathbb{R}^m) \ni \mu \mapsto \left((f_1, \ldots, f_m) \mapsto \sum_{i=1}^{m} \int_X f_i d\mu_i \right) =: F_\mu \in C_0(Y; \mathbb{R}^m)^* ,$$

is an isometric isomorphism. Analogous to the one-dimensional case, weak*-convergence on $\mathcal{M}(X; \mathbb{R}^m)$ is characterized by the following.

Lemma B.1. *A sequence $(\mu^n)_n$ in $\mathcal{M}(X; \mathbb{R}^m)$ converges in the weak*-topology to $\mu \in \mathcal{M}(X; \mathbb{R}^m)$ if and only if*

$$\sum_{i=1}^{m} \int_X f_i d\mu_i^n \to \sum_{i=1}^{m} \int_X f_i d\mu_i ,$$

as $n \to \infty$ for all $(f_1, \ldots, f_m) \in C_0(X; \mathbb{R}^m)$.

Push-forward of measures

Let $(Y, \mathscr{A})$ and $(Z, \mathscr{B})$ be measurable spaces, and let $T : (Y, \mathscr{A}) \longrightarrow (Z, \mathscr{B})$ be a measurable mapping.

Definition B.2. We define the push-forward mapping $T_* : \mathcal{M}(Y, \mathscr{A}) \longrightarrow \mathcal{M}(Z, \mathscr{B})$ by $(T_*\mu)(B) = \mu\left(T^{-1}(B)\right)$ for all $B \in \mathscr{B}$.

Lemma B.3 ([3, Theorem 3.6.1])**.** *Let $f : Z \longrightarrow \mathbb{R}$ be measurable and let $\mu \in \mathcal{M}(Y, \mathscr{A})$. If $f \circ T$ is μ-integrable, then f is $T_*\mu$-integrable and*

$$\int_Z f d(T_*\mu) = \int_Y f \circ T d\mu \,.$$

Borel families of measures

Definition B.4. Let Y and Z be separable metric spaces. We say that a measure-valued map $Y \ni y \mapsto \mu_y \in \mathcal{M}(Z)$ is a Borel family if the map $y \mapsto \mu_y(A)$ is Borel measurable for every $A \in \mathscr{B}(Z)$.

Lemma B.5 ([2, p. 121, section 5.3])**.** *Let Y and Z be separable metric spaces, $Y \ni y \mapsto \mu_y \in \mathcal{M}^+(Z)$ a Borel family and $\nu \in \mathcal{M}^+(Y)$. Then the map*

$$y \mapsto \int_Z f(y,z)\mu_y(dz)\,,$$

is Borel measurable for any Borel measurable function $f : Y \times Z \longrightarrow \mathbb{R}$. Moreover, the expression

$$\boldsymbol{\mu}(f) = \int_Y \left(\int_Y f(y,z)\mu_y(dz)\right)\nu(dy)\,,$$

defines a measure in $\mathcal{M}^+(X \times Y)$.

Corollary B.6. *Let Y and Z be separable metric spaces, $Y \ni y \mapsto \mu_y \in \mathcal{M}(Z)$ a Borel family. Then the map*

$$y \mapsto \int_Z f(y,z)\mu_y(dz)\,,$$

is Borel measurable for any bounded Borel measurable function $f : Y \times Z \longrightarrow \mathbb{R}$.

Proof. The statement follows by applying the previous lemma to the Hahn–Jordan decomposition of $y \mapsto \mu_y$. □

Lemma B.7. *Let Y and Z be locally compact separable metric spaces and let $\bar{\mu} : Y \ni y \mapsto \mu_y \in \mathcal{M}(Z)$ a Borel family. Then $\bar{\mu} : (Y, \mathscr{B}(Y)) \longrightarrow (\mathcal{M}(Z), \mathscr{B}(\mathrm{wk}^*))$ is measurable.*

Proof. It is sufficient to show that $\bar{\mu}^{-1}(U) \in \mathscr{B}(Y)$ for each element U of a basis of the weak*-topology. We have seen that the elements U of the form

$$\bigcap_{i=1}^{n} \{\mu \in \mathcal{M}(Z) : p_{f_i}(\mu - \mu_0) < \varepsilon_i\}\,,$$

where $\mu_0 \in \mathcal{M}(Z)$, $n \in \mathbb{N}$, $f_1, \dots, f_n \in C_0(Z)$, $\varepsilon_1, \dots, \varepsilon_n > 0$, constitute a basis of this topology, where $p_f(\mu) = \left|\int_Z f d\mu\right|$. We see that

$$\bar{\mu}^{-1}(\{\mu \in \mathcal{M}(Z) : p_{f_i}(\mu - \mu_0) < \varepsilon_i\}) = \left\{ y \in Y : \left| \int_Z f \, [d\mu_y - d\mu_0] \right| < \varepsilon_i \right\} ,$$

which is the pre-image of the set $[0, \varepsilon_i)$ under the Borel measurable map $y \mapsto \left|\int_Z f \, [d\mu_y - d\mu_0]\right|$ (by the previous corollary), and is therefore, in $\mathscr{B}(Y)$. Hence, $\bar{\mu}^{-1}(U) \in \mathscr{B}(Y)$ and $\bar{\mu}$ is measurable. □

An integral construction

Let Y be a locally compact topological space. We use the following construction for the definition of the dissipation potential $\mathcal{R}$. Let $m \in \mathbb{N}$ and let $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ be positively 1-homogeneous, convex and lower semicontinuous. We define the functional

$$\mathcal{F}_F : \mathcal{M}(Y; \mathbb{R}^m) \longrightarrow (-\infty, +\infty] , \qquad \mathcal{F}_F(\mu) = \int_Y F\left(\frac{d\mu}{d|\mu|}\right) d|\mu| .$$

Proposition B.8 ([12, Lemma 2.3])**.** *The functional $\mathcal{F}_F$ is convex, positively 1-homogeneous and sequentially weakly*-lower semicontinuous.*

Appendix C

Γ-convergence

We introduce the concept of Γ-convergence of functionals and give some results that we use throughout the thesis as treated in Dal Maso's book [6].

Unless specified otherwise, we consider a topological space X and a sequence of functionals $F_n : X \longrightarrow \overline{\mathbb{R}}$, $n \in \mathbb{N}$.

Definition C.1 ([6, Definition 4.1])**.** For $x \in X$, we define the lower Γ-limit at x, $(\Gamma - \liminf_{n\to\infty} F_n)(x)$, and the upper Γ-limit at x, $(\Gamma - \limsup_{n\to\infty} F_n)(x)$, respectively, by

$$\left(\Gamma - \liminf_{n\to\infty} F_n\right)(x) := \sup_{U\in\mathcal{B}(x)} \liminf_{n\to\infty} \inf_{y\in U} F_h(y)\,,$$

$$\left(\Gamma - \limsup_{n\to\infty} F_n\right)(x) := \sup_{U\in\mathcal{B}(x)} \limsup_{n\to\infty} \inf_{y\in U} F_h(y)\,,$$

where $\mathcal{B}(x)$ is a basis of neighbourhoods of x. If the lower and upper Γ-limits at $x \in X$ are equal, then we say that the Γ-limit of $(F_n)_n$ at x exists. If the Γ-limit exists for all $x \in X$, then we say that $(F_n)_n$ Γ-converges to $F := \Gamma - \liminf_{n\to\infty} F_n$ as $n \to \infty$.

The following property is an immediate consequence of the definition of the lower and upper Γ-limit.

Proposition C.2 ([6, Proposition 6.1])**.** *Let $(F_{n_k})_k$ be a subsequence of $(F_n)_n$. Then,*

$$\Gamma - \liminf_{n\to\infty} F_n \le \Gamma - \liminf_{k\to\infty} F_{n_k}\,,$$

$$\Gamma - \limsup_{n\to\infty} F_n \ge \Gamma - \limsup_{k\to\infty} F_{n_k}\,.$$

Moreover, if $(F_n)_n$ Γ-converges to F, then $(F_{n_k})_k$ also Γ-converges to F.

Proposition C.3 ([6, Proposition 6.8])**.** *The functions $\Gamma - \liminf_{n\to\infty} F_n$ and $\Gamma - \limsup_{n\to\infty} F_n$ are lower semicontinuous on X. In particular, if $(F_n)_n$ Γ-converges to F, then F is lower semicontinuous.*

When X satisfies the first axiom of countability, i.e., every point has a countable basis of neighbourhoods, there is a useful characterization of Γ-convergence.

Proposition C.4 ([6, Proposition 8.1])**.** *Let X satisfy the first axiom of countability. Then the sequence $(F_n)_n$ Γ-converges to $F : X \longrightarrow \overline{\mathbb{R}}$ if and only if the following conditions are satisfied:*

(i) for every $x \in X$ and sequence $(x_n)_n$ in X that converges to x it holds that

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n)\,;$$

(ii) for every $x \in X$ there exists a sequence $(x_n)_n$ in X that converges to x such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)\,.$$

A useful approximation result is the following.

Lemma C.5 ([7, Lemma 6.1.1])**.** *Let X satisfy the first axiom of countability. Let $x \in X$ and suppose that $(x_m)_m$ is a sequence in X such that*

$$\limsup_{m\to\infty} F(x_m) \leq F(x)\,.$$

Suppose in addition that for each $m \in \mathbb{N}$ there exists a sequence $(x_{m,n})_n$ in X such that

$$\limsup_{n\to\infty} F_n(x_{m,n}) \leq F(x_m)\,.$$

Then there exists a sequence $(x_n)_n$ in X converging to x such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)\,.$$

A very convenient feature of the notion of Γ-convergence is that it enables us to prove convergence minimal values and minimizers of functionals under some assumptions on these functionals. For example, we have the following.

Definition C.6 ([6, Definitions 1.12 and 7.6])**.** Let X be a first countable space. We say that a functional $F : X \longrightarrow \overline{\mathbb{R}}$ is *coercive* if for each $t \in \mathbb{R}$ the closure of the set $\{F \leq t\} := \{x \in X : F(x) \leq t\}$ is compact.

We say that the sequence $(F_n)_n$ is *equi-coercive* if for each $t \in \mathbb{R}$ there exists a compact set $K_t \subseteq X$ such that

$$\{F_n \leq t\} \subseteq K_t\,,$$

for all $n \in \mathbb{N}$.

Theorem C.7 ([6, Theorem 7.8 and Corollary 7.20])**.** *Let X be a first countable space. Suppose that $(F_n)_n$ is equi-coercive and Γ-converges to F. Then F is coercive and*

$$\lim_{n\to\infty} \min F_n = \min F\,.$$

Moreover, if $(x_n)_n$ is a sequence in X with $F_n(x_n) = \min F_n$. If x is a cluster point of $(x_n)_n$, then x is a minimizer of F and

$$F(x) = \limsup_{n\to\infty} F_n(x_n)\,.$$

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