Whitney's Theorem for Line Graphs of Multi-Graphs

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WHITNEY’S THEOREM FOR LINE GRAPHS OF MULTI-GRAPHS

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Abstract. Whitney’s Theorem states that every graph, different from $K_3$ or $K_{1,3}$, is uniquely determined by its line graph. A 1-line graph of a multi-graph is the graph with as vertices the edges of the multi-graph, and two edges adjacent if and only if there is a unique vertex on both edges. The $\geq 1$-line graph of a multi-graph is the graph on the edges of the multi-graph, where two edges are adjacent if and only if there is at least one vertex on both edges. We extend Whitney’s theorem to such line graphs of multi-graphs, and show that most multi-graphs are uniquely determined by their line graph. Moreover, we present an algorithm to determine for a given graph $\Gamma$, if possible, a multi-graph with $\Gamma$ as line graph.

1. Introduction

Let $\Delta = (V, E)$ be an ordinary graph, i.e. a graph without loops or multiple edges. So, we can consider each edge $e \in E$ as a set of two distinct vertices. Then its line graph $L(\Delta)$ is the graph with vertex set $E$ and two vertices $e, f \in E$ adjacent if and only if they intersect in a single vertex of $\Delta$. The graph $\Delta$ is called a root graph for its line graph.

An isomorphism $\phi$ between two graphs $\Delta = (V, E)$ and $\Delta' = (V', E')$ is a bijection $\phi : V \rightarrow V'$, such that the map on $E$ induced by $\phi$, i.e. $\phi([v, w]) = \{\phi(v), \phi(w)\}$ for all $[v, w] \in E$, is a bijection from $E$ to $E'$.

Clearly, if $\phi$ is an isomorphism between two graphs $\Delta = (V, E)$ and $\Delta' = (V', E')$, then $\phi$ induces also an isomorphism between the two line graphs $L(\Delta)$ and $L(\Delta')$.

Whitney’s Theorem is concerned with the reverse problem:

Does an isomorphism between line graphs of graphs $\Delta$ and $\Delta'$ also imply the existence of an isomorphism between $\Delta$ and $\Delta'$?

In other words, is the root graph of a line graph unique?

The answer to this question is stated in the following:

Theorem 1.1 (Whitney’s Theorem, [13]). Suppose $\Delta$ and $\Delta'$ are connected graphs and $\phi$ an isomorphism between the line graph $L(\Delta)$ and $L(\Delta')$.

Then there is a unique isomorphism $\psi$ between $\Delta$ and $\Delta'$, inducing $\phi$, unless $\Delta$ and $\Delta'$ are (up to permutation) $K_{1,3}$ and $K_3$.

An elementary and short proof of this result can be found in [7].

In this note we consider the same problem for multi-graphs. Here a multi-graph $\Delta = (V, E)$ consists of a set of vertices $V$ and a set of edges $E$ together with a relation $\sim$ between $V$ and $E$ such that for any element $e \in E$ there are exactly two distinct elements $v, w \in V$ with $v, w \sim e$. If $v \sim e$, then we will say $v$ is on $e$. Of course, an ordinary graph is a multi-graph in which the relation $\sim$ is the relation $\in$. 
There are at least two different obvious choices for the definition of the line graph of a multi-graph. The vertices of the line graph of \( \Delta = (V, E) \) are the elements from \( E \). But two elements \( e, f \in E \) can be defined to be adjacent if and only if there is a unique vertex with \( v \sim e \) and \( v \sim f \), or, if and only if there is at least one vertex \( v \) with \( v \sim e \) and \( v \sim f \). We consider both situations.

The 1-line graph \( L_1(\Delta) \) of a multi-graph \( \Delta \) is the graph with vertex set \( E \), such that two vertices \( e, f \in E \) are adjacent if and only if there is a unique vertex \( v \) of \( \Delta \) with \( v \sim e \) and \( v \sim f \). By \( L_{\geq 1}(\Delta) \) we denote the \( \geq 1 \)-line graph of \( \Delta \), the graph with vertex set \( E \) and two edges \( e, f \) adjacent if there is a at least one vertex \( v \) of \( \Delta \) with \( v \sim e \) and \( v \sim f \).

If it is clear from the context whether we are considering the 1-line graph or \( \geq 1 \)-line graph, we will just refer to these graphs as line graphs. In both cases the graph \( \Delta \) is called a root graph for its line graph.

An isomorphism between two multi-graphs \( \Delta = (V, E) \) and \( \Delta' = (V', E') \) is a pair \((\phi_V, \phi_E)\) of bijections

\[
\phi_V : V \to V', \phi_E : E \to E'
\]
such that for all \( v \in V \) and \( e \in E \) we have

\[
v \sim e \iff \phi_V(v) \sim \phi_E(e).
\]

An isomorphism between two multi-graphs \( \Delta = (V, E) \) and \( \Delta' = (V', E') \) induces an isomorphism, the isomorphism \( \phi_E \), between the graphs \( L_1(\Delta) \) and \( L_1(\Delta') \), as well as between \( L_{\geq 1}(\Delta) \) and \( L_{\geq 1}(\Delta') \).

Again we can ask the following questions:

Does an isomorphism between line graphs of multi-graphs \( \Delta \) and \( \Delta' \) also imply the existence of an isomorphism between \( \Delta \) and \( \Delta' \)?

In other words, is the root graph of a line graph of a multi-graph unique?

The answers to these questions are given in the following theorems:

**Theorem 1.2.** Suppose \( \Delta \) and \( \Delta' \) are connected multi-graphs and \( \phi \) an isomorphism between the line graph \( L_1(\Delta) \) and \( L_1(\Delta') \).

Then there is a unique isomorphism between \( \Delta \) and \( \Delta' \), inducing \( \phi \), unless \( \Delta \) or \( \Delta' \) contains four vertices.

Moreover, up to isomorphism, there is a unique connected multi-graph \( \Delta'' \), not on 4 vertices, such that \( L_1(\Delta'') \) is isomorphic to \( L_1(\Delta) \) and \( L_1(\Delta') \).

**Theorem 1.3.** Suppose \( \Delta \) and \( \Delta' \) are connected multi-graphs and \( \phi \) an isomorphism between the line graph \( L_{\geq 1}(\Delta) \) and \( L_{\geq 1}(\Delta') \).

Then there is a unique isomorphism between \( \Delta \) and \( \Delta' \), inducing \( \phi \), unless \( \Delta \) or \( \Delta' \) contains a subgraph \( \Delta_0 \)

\[
\begin{array}{c}
\text{\( x \)} \\
\text{\( y \)} \\
\text{\( z \)}
\end{array}
\]

such that the vertices \( x \) and \( y \) have no other neighbors outside \( \Delta_0 \).

Moreover, up to isomorphism, there is a unique connected multi-graph \( \Delta'' \) not containing such subgraph \( \Delta_0 \), with \( L_{\geq 1}(\Delta'') \) isomorphic to \( L_{\geq 1}(\Delta) \) and \( L_{\geq 1}(\Delta') \).
Notice that $\Delta_0$ is not necessarily an induced graph, we allow multiple edges on $x,z$ or $y,z$ or $x,y$.

After finishing the research for this note, we found that results similar to Theorem 1.3 have also been obtained by Zverovich [14].

We present a uniform proof for the above theorems in the next section. This proof can be transformed into an algorithm that on input of an ordinary graph $\Gamma = (V, E)$ decides whether $\Gamma$ is a line graph of a multi-graph and also determines in the case that $\Gamma$ is indeed a line graph a root graph $\Delta$ for $\Gamma$.

The algorithm has complexity $O(|V| + |E|)$. It is discussed in Section 3.

A class of finite $l$-line graphs of multi-graphs that received a lot of attention is the class of so-called generalized line graphs, as they are graphs with eigenvalues $\geq -2$. See for example [2] or [5], and various references in [5].

A generalized line graph $\Gamma$ is a graph which can be constructed from a line graph of an ordinary graph $\Delta = (V, E)$ and for each vertex $v$ a cocktail party graph $\Delta_v$ (a possibly empty ordinary graph in which each vertex is adjacent to precisely all but one other vertices). The graph $\Gamma$ is the union of the line graph $L(\Delta)$ and all the graphs $\Delta_v$, to which the following edges are added: a vertex $e$ of $L(\Delta)$ is adjacent to all vertices of $\Delta_v$, where $v$ is a vertex on $e$.

An equivalent definition is the following. A graph $\Gamma$ is a generalized line graph if and only if it is the $l$-line graph of a multi-graph $\Delta$ such that:

(i) Two vertices of $\Delta$ are on at most two common edges.
(ii) If $v, w$ are two vertices of $\Delta$ which are on two common edges, then one of the vertices is on no other edges.

See [4, Proposition 6.2].

Using this definition of a generalized line graph we can apply Theorem 1.2 and easily deduce the analogue of Whitney’s Theorem for generalized line graphs as in [3] and [5, Theorem 2.3.3, 2.3.4].

Moreover, our algorithm of Section 3 can be modified for generalized line graphs. See also [12] for a different algorithm to find the root graph of a generalized line graph.

2. Line graphs

In this section we prove Theorem 1.2 and Theorem 1.3.

We start with some definitions.

Let $\Gamma = (V, E)$ be an ordinary graph. On the vertex set of $\Gamma$ we define two equivalence relations $\equiv$ and $\bowtie$ by the following: two vertices $v, w$ of $\Gamma$ are in relation $\equiv$ if and only if they have the same set of neighbors (and hence are themselves not adjacent), while two vertices $v, w$ of $\Gamma$ are in relation $\bowtie$ if and only if they are adjacent and each other vertex adjacent to one of them is also adjacent to the other. For $v$ a vertex of $\Gamma$, denote by $[v]$ the $\equiv$-class of $v$ and by $\langle v \rangle$ the $\bowtie$-class of $v$. Two $\equiv$-equivalent vertices are also called false twins, while $\bowtie$-equivalent vertices are called true twins.

By $\overline{\Gamma}$ we denote the graph with vertex set the $\equiv$-classes of vertices of $\Gamma$, and two such classes $[v]$ and $[w]$ adjacent if and only if $v$ and $w$ are adjacent in $\Gamma$. The graph on the $\bowtie$-classes, where two such classes $\langle v \rangle$ and $\langle w \rangle$ are adjacent if and only if $v$ and $w$ are adjacent in $\Gamma$, is denoted by $\overline{\Gamma}$. 

If an $\equiv$- or $\equiv'$-class has size one, we often identify it with the element it contains. In this way we justify a statement like $\Gamma = \bar\Gamma = \Gamma$, or $\Gamma = \bar\Gamma$, in case all equivalence classes have size one.

**Lemma 2.1.** Suppose $\Gamma$ is a line graph of a connected ordinary graph $\Delta$. Then the following hold:

(i) $\Gamma = \bar\Gamma$ or $\Delta$ contains 4 vertices.
(ii) $\bar\Gamma = \Gamma$ or $\Delta$ contains a (not necessarily induced) subgraph $\Delta_0$

\[ \begin{array}{c}
\text{z} \\
\text{x} \\
\downarrow \\
\text{y} \\
\end{array} \]

such that the vertices $x$ and $y$ have no other neighbors outside $\Delta_0$.

**Proof.** Let $\Gamma = L(\Delta)$ for some graph $\Delta$.

Suppose $\Gamma \neq \bar\Gamma$. Then there are two edges in $\Delta$, say $e \neq f$ that do not meet each other, such that each edge meeting one of $e$ or $f$ meets the other. But then, by connectedness of $\Delta$, every edge of $\Delta$ meets both $e$ and $f$ and we find $\Delta$ to be a connected graph on 4 vertices.

Now suppose $\bar\Gamma \neq \Gamma$. Then there are two edges $e = \{x, z\}$ and $f = \{y, z\}$ of $\Delta$ on a common vertex $z$ such that each edge which meets $e$ also meets $f$. But that implies that an edge meeting $e$ or $f$ meets them in $z$, or it is the unique edge $\{x, y\}$. So, the subgraph $\Delta_0$ of $\Delta$ on the vertices $x, y$ and $z$ and the edges $e$ and $f$ is a graph as in the statement of the lemma. $\square$

For a multi-graph $\Delta$ we denote by $\Delta$ the ordinary graph obtained from $\Delta$ by replacing all multiple edges by single ones.

If two edges of $\Delta$ are on the same vertices, then in its line graphs $L_1(\Delta)$ and $L_{\geq 1}(\Delta)$ they are in relation $\equiv$ and $\equiv'$, respectively. So, if $v, w$ are two vertices of $\Delta$ on an edge $e$, then identifying the vertex $\{v, w\}$ of $L(\Delta)$ with the $\equiv$- or $\equiv'$-equivalence class of $e$ in $L_1(\Delta)$ or $L_{\geq 1}(\Delta)$, respectively, yields the following:

**Lemma 2.2.** Let $\Delta$ be a multi-graph. Then $\overline{L_1(\Delta)} = L(\Delta)$ and $\overline{L_{\geq 1}(\Delta)} = L(\Delta)$.

A multi-graph $\Delta$ that does not contain any (not necessarily induced) subgraph $\Delta_0$

\[ \begin{array}{c}
\text{z} \\
\text{x} \\
\downarrow \\
\text{y} \\
\end{array} \]

such that the vertices $x$ and $y$ have no other neighbors outside $\Delta_0$, is called $\Delta_0$-free.

**Corollary 2.3.** Let $\Delta$ be a connected multi-graph. Then we have:

(i) $\overline{L_1(\Delta)} = L(\Delta)$, or $\Delta$ has 4 vertices.
(ii) $\overline{L_{\geq 1}(\Delta)} = L(\Delta)$, or $\Delta$ is not $\Delta_0$-free.

**Proof.** This follows from Lemma 2.1 and Lemma 2.2 applied to $\Gamma = L(\Delta)$. $\square$

**Proposition 2.4.** Let $\Delta$ and $\Delta'$ be two connected multi-graphs.
Suppose \( \phi \) is an isomorphism between \( L_1(\Delta) \) and \( L_1(\Delta') \). Assume \( \Delta \) and \( \Delta' \) do not contain 4 vertices. Then there is a unique isomorphism between \( \Delta \) and \( \Delta' \) inducing \( \phi \).

(ii) Suppose \( \hat{\phi} \) is an isomorphism between \( L_{\geq 1}(\Delta) \) and \( L_{\geq 1}(\Delta') \). Assume \( \Delta \) and \( \Delta' \) are \( \Delta_0 \)-free. Then there is a unique isomorphism between \( \Delta \) and \( \Delta' \) inducing \( \hat{\phi} \).

Proof. Suppose we are in case (i) or (ii). By Corollary 2.3 and the assumptions, \( L_1(\Delta) = L(\Delta) \) and \( L_1(\Delta') = L(\Delta') \) are isomorphic in case (i), and \( L_1(\Delta) = L(\Delta) \) and \( L_1(\Delta') = L(\Delta') \) are isomorphic in case (ii).

Then \( \phi \) induces a unique isomorphism \( \overline{\phi} \) between \( L_1(\Delta) \) and \( L_1(\Delta') \) in case (i) and \( \hat{\phi} \) between \( L_1(\Delta) \) and \( L_1(\Delta') \) in case (ii). Now, by Whitney’s Theorem 1.1 there is a unique isomorphism \( \overline{\phi} \) from \( \Delta \) to \( \Delta' \), which induces \( \phi \) between \( L_1(\Delta) = L(\Delta) \) and \( L_1(\Delta') = L(\Delta') \). As \( K_{1,3} \) is not \( \Delta_0 \)-free, Whitney’s Theorem 1.1 also implies that there is a unique isomorphism \( \hat{\phi} \) from \( \Delta \) to \( \Delta' \), which induces \( \hat{\phi} \) between \( L_{\geq 1}(\Delta) = L(\Delta) \) and \( L_{\geq 1}(\Delta') = L(\Delta') \).

By the uniqueness of \( \overline{\phi} \) and \( \hat{\phi} \) we find that the pairs \( (\overline{\phi}, \phi) \) and \( (\hat{\phi}, \hat{\phi}) \) are the unique isomorphisms between \( \Delta \) and \( \Delta' \) we are looking for in case (i) and (ii), respectively.

It remains to consider the cases excluded in the above Proposition 2.4. These are handled in the following result.

Proposition 2.5. Let \( \Delta \) be a connected multi-graph.

(i) There is, up to isomorphism, a unique connected multi-graph \( \Delta' \), not on 4 vertices, with \( L_1(\Delta') \) isomorphic to \( L_1(\Delta) \).

(ii) There is, up to isomorphism, a unique \( \Delta_0 \)-free multi-graph \( \Delta' \) with \( L_{\geq 1}(\Delta') \) and \( L_{\geq 1}(\Delta) \) isomorphic.

Proof. We first consider case (i). By Proposition 2.4(i), we only have to consider the case that \( \Delta \) contains four vertices.

Then \( \Delta \) is the multi-graph

\[
\begin{array}{c}
\text{n} \\
\text{k} \\
\text{m} \\
\text{p} \\
\text{q} \\
\text{l}
\end{array}
\]

with \( k, l, m \) cardinal numbers at least 1 and \( n, p, q \) cardinals at least 0.

Let \( \Delta' \) be the following multi-graph:

\[
\begin{array}{c}
\text{n + l} \\
\text{k + m} \\
\text{q + p}
\end{array}
\]

Then \( L_1(\Delta') \) is isomorphic to \( L_1(\Delta) \) and by Proposition 2.4 it is the unique such graph which is not on 4 vertices.
Now consider (ii). We show that there does exist a \( \Delta_0 \)-free root multi-graph \( \Delta' \) with \( L_{\geq 1}(\Delta') \) and \( L_{\geq 1}(\Delta) \) isomorphic.

Suppose \( \Delta \) does contain subgraphs \( \Delta_0 \), then we can consider the graph \( \Delta' \) which we obtain from \( \Delta \) by identifying all the vertices in \([x]\) if \( x \) and \( y \) of \( \Delta_0 \) are not adjacent in \( \Delta \) or, if \( x \) and \( y \) are not adjacent in \( \Delta \), by identifying \( x \) and \( y \) (so all vertices in \( \langle x' \rangle \)) and adding a new vertex \( y' \) with for each edge between \( x \) and \( y \) an edge between \( x \) and \( y' \). See Fig. 1.

By Proposition 2.4(ii), this graph \( \Delta' \) is then the unique \( \Delta_0 \)-free root graph of \( \Gamma \). \( \square \)

Theorem 1.2 and Theorem 1.3 follow from Proposition 2.4 and Proposition 2.5.

3. FIND THE ROOT GRAPH OF A LINE GRAPH

Let \( \Gamma \) be a finite graph. In [1] it has been shown that \( \Gamma \) is the \( \geq 1 \)-line graph of a multi-graph \( \Delta \) if and only if \( \Gamma \) does not contain one of 7 graphs on at most 6 vertices as induced subgraph. A similar result for 1-line graphs has been obtained in [4], where a complete list of 33 forbidden graphs, all on 6 vertices, has been found. This implies that there is a polynomial time algorithm to determine whether \( \Gamma \) is indeed a line graph.

The results of the previous section imply that, in case \( \Gamma \) is indeed a line graph, there is a unique root graph \( \Delta \), not on four vertices if \( \Gamma \) is a 1-line graph, or \( \Delta_0 \)-free, if \( \Gamma \) is a \( \geq 1 \)-line graph.

The following algorithm has as an input of a finite graph \( \Gamma \) as output this unique root graph \( \Delta \) if \( \Gamma \) is a 1- or \( \geq 1 \)-line graph, or returns the message that the graph \( \Gamma \) is not a line graph.

Correctness of the algorithm follows easily from the results of the previous section.

(i) Find the false (or true) twin decomposition of \( \Gamma \). A decomposition into the \( \equiv \)-equivalence (or \( \preceq \)-) classes can be found by a standard partition refinement. See for example Algorithm 2 in [6].

(ii) Determine \( \Gamma \) (or \( \hat{\Gamma} \)). Take in each equivalence class a unique vertex and form the induced subgraph on these vertices.

(iii) Find a root graph \( \underline{\Delta} \) for the graph \( \Gamma \) (or \( \hat{\Gamma} \)), or return the message that the graph \( \Gamma \) is not a line graph. Several efficient algorithms exist. See for example [8, 9, 10, 11].

(iv) Determine and output \( \Delta \). Each vertex \( v \) of \( \Gamma \) determines a unique edge of \( \Gamma \) (or \( \hat{\Gamma} \)) and hence of \( \underline{\Delta} \). So, we can assign to \( v \) an edge of \( \Delta \) on the same vertices of the corresponding edge of \( \underline{\Delta} \).
The above algorithm can of course also be used to determine a root graph of a generalized line graph. Indeed, after step (i) one can check whether all -classes have size at most 2, in step (iv) whether for two vertices of Δ which are on two common edges, one is on no other edges. In [12] a different algorithm is discussed to find a root graph of a generalized line graph.

Finally we notice that each of the steps in the above algorithm requires at most \(O(|V| + |E|)\) computations. So, the complexity of the algorithm is \(O(|V| + |E|)\).

References


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