Line graphs of Multi-Graphs and the forbidden graph E_6

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LINE GRAPHS OF MULTI-GRAPHS AND THE FORBIDDEN GRAPH E_6

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Abstract. The *line graph* Γ of a multi-graph Δ is the graph whose vertices are the edges of Δ, where two such edges are adjacent if and only if they meet in a *single* vertex of Δ.

We provide several characterizations of such line graphs and in particular show that a graph is a line graph if and only if it does not contain one of 33 graphs, all of which correspond to bases of anisotropic vectors of a 6-dimensional orthogonal geometry of $-$-type over a field with two elements, or, equivalently, to sets of 6 generating reflections in the Weyl group of type E_6.

1. Introduction

In this paper we consider (ordinary) graphs, i.e. graphs without loops and multiple edges, as well as multi-graphs, in which we allow multiple edges but no loops. By a graph we will usually mean an ordinary graph.

Let Δ be a multi-graph without loops but with possibly multiple edges. Then we define the *line graph* of Δ to be the graph whose vertices are the edges of Δ, where two edges are adjacent if and only if they meet in a single vertex. We denote this line graph by $L(\Delta)$. Notice that $L(\Delta)$ is an ordinary graph, a graph without loops or multiple edges. (In [6], the line graph of Δ is also called the 1-line graph, in contrast to the $\geqslant 1$-line graph of Δ, which is the graph whose vertices are the edges of Δ, and two edges are adjacent if and only if there is at least one vertex on both of them.)

We prove the following:

Theorem 1.1. *Let Γ be a connected ordinary graph. Then Γ is a line graph of a multi-graph if and only if it does not contain an induced subgraph of the set $\mathcal{E}_6$ of 33 graphs given in Fig. 1.*

The graphs in $\mathcal{E}_6$ can be described as follows. Consider the orthogonal space (V, Q) of dimension 6 over $\mathbb{F}_2$ and Q a quadratic form of $-$-type (i.e. with Arf invariant $+1$). For any basis of this space consisting of anisotropic vectors (vectors v with $Q(v) = 1$) we consider the graph whose vertices are the vectors in the basis and two vertices $v \neq w$ being adjacent if and only if $Q(v + w) = 1$. The set $\mathcal{E}_6$ consists of all such connected graphs and equals the set of 33 graphs given in Fig. 1. The most prominent graph in this set is of course the graph E_6. Alternatively, one can describe the graphs in $\mathcal{E}_6$ as the graphs on 6 generating reflections in the Weyl group of type E_6, where two reflections are adjacent if and only if they do not commute, or as the graph on 6 roots spanning the root lattice E_6, where two such roots are adjacent if and only if they are not perpendicular.

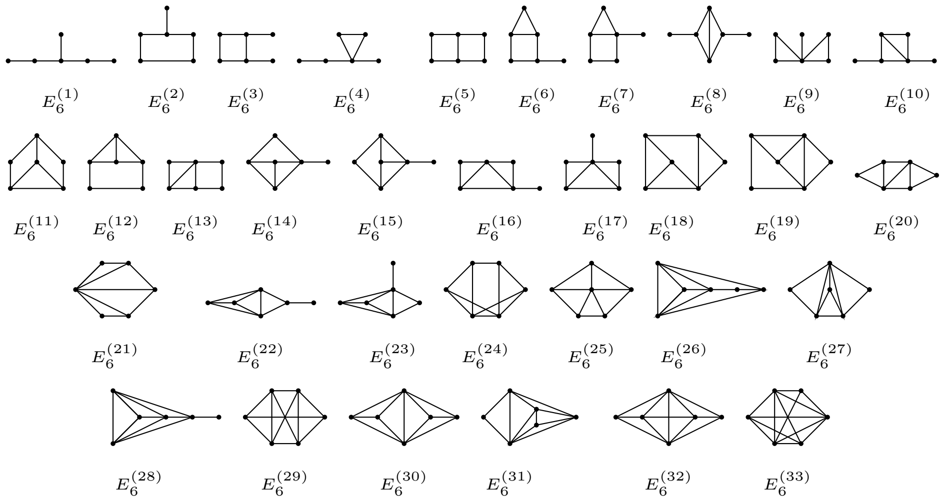
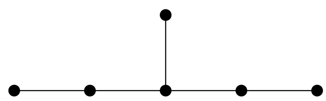


FIGURE 1. The graphs equivalent to E_6.



E_6

We call a graph $\mathcal{E}_6$-*free*, if it does not contain an induced subgraph from $\mathcal{E}_6$.

Theorem 1.1 is closely related to the following characterizations of line graphs of ordinary graphs by nine forbidden induced subgraphs due to Beineke [1] and of generalized line graphs by 31 forbidden induced subgraphs by Cvetković, Doob and Simić [7] for finite graphs and Vijayakumar [15] for infinite graphs.

Theorem 1.2 (Beineke, [1]). *Let Γ be a connected graph. Then Γ is a line graph of an ordinary graph if and only if it does not contain one of 9 graphs H_1, H_2, H_3 as in Fig. 2, or of the graphs $E_6^{(8)}, E_6^{(12)}, E_6^{(20)}, E_6^{(22)}, E_6^{(25)}$ or $E_6^{(30)}$ from Fig. 1.*

Theorem 1.3 (Cvetković et al., [7], Vijayakumar, [15]). *Let Γ be a connected graph. Then Γ is a generalized line graph if and only if Γ does not contain one of 11 graphs $G_1, \ldots, G_{11}$ from Fig. 3 or one of the 20 graphs $E_6^{(i)}$, where $i \in \{1, 2, 4, 8, 9, 10, 12, 16, 17, 20, 21, 23, 24, 25, 27, 28, 30, 31, 32, 33\}$, from Fig. 1.*

Actually, both these results follow easily from Theorem 1.1.

Theorem 1.1 not only generalizes the above two results, but also provides a common explanation why the minimal forbidden graphs in each of the three theorems above have at most 6 vertices by relating them to the Weyl group and root lattice of type E_6.

There where the main tool for studying generalized line graphs, or more generally graphs with least eigenvalue $\geqslant -2$, is embedding them into a root lattice, see [4], our main tool in proving the above theorems is to consider the graphs as embedded into an orthogonal space over $\mathbb{F}_2$, the field of 2 elements. We notice that the E_6-root lattice modulo 2 corresponds to the 6-dimensional orthogonal space of $-$-type, whose bases provide us with the class $\mathcal{E}_6$.

As our approach relates to the theory of Coxeter and Weyl groups, root lattices and related geometries and groups, see [2, 3, 4, 9, 12], we adopt the names of Dynkin diagrams A_n, D_n and E_n for the corresponding graphs.

The remainder of this paper is organised as follows.

For any ordinary graph Γ we can view its vertices as certain anisotropic vectors in an orthogonal space (V, Q), where two vertices are adjacent if and only if $Q(v+w) = 1$. We can of course assume that the vertices of Γ do linearly span V, but that does not imply that they generate the partial linear space of anisotropic vectors and elliptic lines of the orthogonal space. The case that the vertices of Γ do not generate this partial linear space can be shown to imply that Γ is the line graph of a multi-graph. The embedding of a graph Γ into an orthogonal space over $\mathbb{F}_2$ is the topic of Section 2.

To check whether the points of a graph Γ embedded in an orthogonal $\mathbb{F}_2$-space (V, Q) generate the full partial linear space of anisotropic vectors and elliptic lines, we can change the graph Γ into a new graph Δ generating the same subspace. This provides us with an equivalence relation between graphs, which is studied in Section 3 and Section 4 and leads to a proof of Theorem 1.1.

In these sections we make use of techniques and results from Brown and Humphreys [2, 3] and Seven [12]. We show that each equivalence class of graphs contains a tree. In particular, we reprove a remarkable result by Seven (Seven's Lemma 4.2) that any graph Γ equivalent to a tree with an E_6 subgraph contains itself a subgraph from the equivalence class $\mathcal{E}_6$ of the graph E_6. This lemma is the key to prove Theorem 1.1.

For completeness we provide (partly new and shorter) proofs of the results of [2, 3, 12] needed in this paper, based on a result presented by Arjeh Cohen at his PhD defence and part of the Appendix of [5].

In the two sections 5 and 6 we prove the two theorems Theorem 1.2 and Theorem 1.3. We end the paper with some final remarks in Section 7.

Acknowledgement. The author thanks Jonathan I. Hall for bringing Seven's lemma to his attention, and Arjeh Cohen for sharing the results of the Appendix of [5].

2. The orthogonal embedding of a graph

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Then consider V_Γ, the vector space of finite subsets of $\mathcal{V}$, where for two finite subsets u, w of $\mathcal{V}$ the sum $u + w$ is defined to be the symmetric difference of u and w. We identify a vertex v with the subset $\{v\} \in V_\Gamma$.

Put a total ordering $<$ on the vertex set of Γ. Let u and w be two finite subsets of $\mathcal{V}$ and let $g_\Gamma(u, w)$ denote the number of ordered pairs $(x, y) \in u \times w$, where $x < y$ and $\{x, y\}$ is an edge, or $x = y$, modulo 2. Then $g_\Gamma(u, v+w) = g_\Gamma(u, v) + g_\Gamma(u, w)$ for any finite subsets u, v, w of $\mathcal{V}$, as the ordered edges (x, z) with $x < z$ and $z \in v \cap w$ are counted twice at the right hand site of the equation, just as vertices in the intersection of u and $v \cap w$.

Similarly we find $g_\Gamma(v + w, u) = g_\Gamma(v, u) + g_\Gamma(w, u)$. So, $g_\Gamma : V_\Gamma \times V_\Gamma \to \mathbb{F}_2$ is bilinear. The map $Q_\Gamma : V \to \mathbb{F}_2$ given by $Q_\Gamma(v) = g_\Gamma(v, v)$ for all $v \in V_\Gamma$ is a quadratic form with associated symmetric (and also alternating) form f_Γ given by $f_\Gamma(u, w) = g_\Gamma(u, w) + g_\Gamma(w, u)$.

Notice that if we take a different total ordering $<'$, and let g'_Γ be the corresponding bilinear map, then $g_\Gamma(u,w) + g_\Gamma(w,u) = g'_\Gamma(u,w) + g'_\Gamma(w,u)$ and $g_\Gamma(v,v) = g'_\Gamma(v,v)$. So, Q_Γ and f_Γ are independent of the chosen ordening.

An *orthogonal embedding* of a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is a map from $\mathcal{V}$ into the set of anisotropic vectors of an orthogonal $\mathbb{F}_2$-space (V, Q), such that the images of the vertices span V and two vertices v and w of Γ are adjacent if and only if the sum of their images is also anisotropic. The embedding of Γ into (V_Γ, Q_Γ), mapping a vertex v to $\{v\}$, is an orthogonal embedding. It is called the *universal orthogonal embedding* of Γ.

Now suppose $\mathcal{V}$ is a subset of anisotropic points of an orthogonal $\mathbb{F}_2$-space (V, Q), then the graph with vertex set $\mathcal{V}$ in which two vertices v, w are adjacent if and only if $Q(v+w) = 1$ is called the graph *induced* on $\mathcal{V}$ by Q. So, a graph Γ is (isomorphic to) the induced graph of its universal orthogonal embedding into (V_Γ, Q_Γ).

Let Γ be a graph and R a subspace of the isotropic radical $\{v \in V_\Gamma \mid Q_\Gamma(v) = 0 = f_\Gamma(v,w) \text{ for all } w \in V_\Gamma\}$ of (V_Γ, Q_Γ), then we can take the quotient modulo R and find an orthogonal embedding of Γ in the quotient space. If we take for R the full isotropic radical, then we call the embedding of Γ into V_Γ/R the *minimal (orthogonal) embedding* of Γ. We denote the graph induced on the images of the elements of $\mathcal{V}$ in the minimal embedding of Γ by $\overline{\Gamma} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$.

Obviously we have:

Proposition 2.1. *Let Γ be a connected graph.*

(i) *Every orthogonal embedding of Γ can be obtained as a quotient of the universal embedding by some subspace contained in the isotropic radical of (V_Γ, Q_Γ).*
(ii) *In the minimal orthogonal embedding of Γ two vertices are mapped to the same vector if and only if they are non-adjacent and have the same set of neighbors.*
(iii) *Γ is isomorphic to $\overline{\Gamma}$ if and only if any two vertices $v, w \in \mathcal{V}$ do not have the same set of neighbors.*

Let Γ be a connected graph, then two adjacent vertices v, w are mapped to two vectors in V_Γ, also denoted by v and w, such that

$$Q_\Gamma(v) = Q_\Gamma(w) = Q_\Gamma(v+w) = 1.$$

This implies that the images of the vertices of Γ generate a connected subspace of the *orthogonal cotriangular space* of (V_Γ, Q_Γ), i.e. the partial linear space of anisotropic points outside the radical and elliptic lines of (V_Γ, Q_Γ). Here an *elliptic line* is considered to be a set of three anisotropic vectors in a 2-dimensional subspace on which Q_Γ takes the value 1 for each non-zero vector. We denote this subspace by Π_Γ. This space maps onto a subspace the cotriangular space in the quotient $(\overline{V}_\Gamma, \overline{Q}_\Gamma)$, which we then denote by $\overline{\Pi}_\Gamma$.

Lemma 2.2. $\overline{\Pi}_\Gamma = \Pi_{\overline{\Gamma}}$.

Proof. The vertices of $\overline{\Gamma}$ are contained in $\overline{\Pi}_\Gamma$, so $\Pi_{\overline{\Gamma}}$ is a subspace of $\overline{\Pi}_\Gamma$.

Now suppose $\overline{v}$ is a point of $\overline{\Pi}_\Gamma$. Then there are vertices $v_1, \ldots, v_k$ of Γ with $v_1 + \cdots + v_i$ adjacent to v_{i+1} such that $v = v_1 + \cdots + v_k$ is a point of Π_Γ mapping to $\overline{v}$. But then also $\overline{v} = \overline{v}_1 + \cdots + \overline{v}_k$ with $\overline{v_1 + \cdots + v_i} = \overline{v_1} + \cdots + \overline{v_i}$ adjacent to $\overline{v_{i+1}}$ in $\overline{\Gamma}$, and we find $\overline{v}$ to be a point of $\Pi_{\overline{\Gamma}}$. □

Example 2.3. Let Ω be a set and Γ the line graph of the complete graph Δ on Ω with vertices $v_{ij} := \{i, j\}$, where $i \neq j \in \Omega$.

Then the isotropic radical of (V_Γ, Q_Γ) contains $v_{ij} + v_{jk} + v_{kl}$, where $\{i, j, k\}$ is a subset of size 3 of Ω.

The partial linear space $\overline{\Pi}_\Gamma$ is generated by the subset of images modulo the isotropic radical of vertices v_{ij} of any line graph of a connected graph Δ_0 on the vertex set Ω.

Indeed, if $\overline{v}_{ij}$ is a point of $\overline{\Pi}_\Gamma$, with v_{ij} not an edge of Δ_0, then there is a finite path in Δ_0 from i to j. Let $v_{ik_1}, v_{k_1k_2}, \ldots, v_{k_nj}$ be the edges involved in this path. Then $\overline{v}_{ij} = \overline{v}_{ik_1} + \overline{v}_{k_1k_2} + \cdots + \overline{v}_{k_nj}$ is clearly in $\overline{\Pi}_\Gamma$.

The space $\overline{\Pi}_\Gamma$ is isomorphic to the *cotriangular space of the set* Ω, i.e. the partial linear space with as points the pairs from Ω and as lines the triples of pairs in subsets of size 3, unless $|\Omega| = 4$, in which case $\overline{\Pi}_\Gamma$ is isomorphic to the cotriangular space of a set of size 3.

If $\Omega = \{1, \ldots, 6\}$, then $(\overline{V}_\Gamma, \overline{Q}_\Gamma)$ is a 5-dimensional orthogonal space with a trivial isotropic radical and Π_Γ coincides with the orthogonal cotriangular space on 15 points. If $\Omega = \{1, \ldots, 8\}$, then the vector $v_{12} + v_{34} + v_{56} + v_{78}$ is in the isotropic radical of Q_Γ, and $(\overline{V}_\Gamma, \overline{Q}_\Gamma)$ is a 6-dimensional nondegenerate orthogonal space of $+$-type. Again Π_Γ is the full orthogonal cotriangular space on all 28 anisotropic vectors and elliptic lines of $\overline{V}_\Gamma$.

Example 2.4. If Γ is the graph E_6, then (V_Γ, Q_Γ) is a nondegenerate 6-dimensional space of $-$-type. The partial linear space Π_Γ contains all 36 anisotropic vectors. We find Π_Γ not to be isomorphic to the cotriangular space of some set. A way of seeing this, which we will use later, is that Π_Γ contains a subspace S isomorphic to Π_{D_4} and consisting of 12 points. The subspace S generates a 4-dimensional subspace of V_Γ with a 2-dimensional radical. Indeed, the subspace generated by the 4 vertices of the D_4 subgraph of Γ provides us with such a subspace. Such subspaces do not exist in any cotriangular space of a set.

We can phrase the above also in terms of subgroups of the orthogonal group of (V_Γ, Q_Γ).

Each anisotropic vector v of V_Γ is the center of a transvection defined by

$$w \in V_\Gamma \mapsto w + f_\Gamma(v, w)v.$$

The 6 transvections associated to the vertices of Γ generate the orthogonal group $O(V_\Gamma, Q_\Gamma)$ which is isomorphic to the Weyl group of type E_6.

Inside this group the 4 transvections associated to the subgraph of type D_4 of Γ_0 generate a subgroup isomorphic to $W(D_4)$, the Weyl group type D_4.

If we consider the minimal orthogonal embedding of the line graph Γ' of a complete graph on $n \geqslant 5$ vertices, then the transvections corresponding to the vertices of Γ' generate a group isomorphic to the symmetric group on n letters, i.e., a Weyl group of type A_{n-1}. Inside this group, there are no 4 transvections generating a group isomorphic to $W(D_4)$.

Theorem 2.5. *Let Γ be a connected ordinary graph. Then the following statements are equivalent:*

(i) Γ *is the line graph of some multi-graph.*
(ii) $\overline{\Gamma}$ *is the line graph of some ordinary graph.*
(iii) $\overline{\Pi}_\Gamma$ *is isomorphic to a cotriangular space of some set* Ω.
(iv) $\overline{\Pi}_\Gamma$ *contains no subspace isomorphic to* Π_{D_4}.

Proof. We first show that (i) implies (ii). Assume Γ is the line graph of a connected multi-graph Δ. Then two vertices v and w corresponding to edges on the same vertices in Δ have the same set of neighbors in Γ. This implies that they are identified in $\overline{\Gamma}$. So, if $\underline{\Delta}$ is the graph obtained from Δ by replacing all multiple edges with single edges, we find that $\overline{L(\underline{\Delta})} = \overline{\Gamma}$.

If $\overline{L(\underline{\Delta})} = L(\underline{\Delta})$, we are done. So, assume that $L(\underline{\Delta})$ contains two vertices v and w that are nonadjacent but have the same set of neighbors. That implies that $\underline{\Delta}$ is a connected graph on 4 vertices, and $\overline{\Gamma}$ a graph on at most 3 vertices, which is clearly a line graph of an ordinary graph. So (i) implies (ii).

Now assume $\overline{\Gamma}$ is a line graph of an ordinary graph $\underline{\Delta}$. Then Γ can be seen to be the line graph of the graph Δ with the same vertex set as $\underline{\Delta}$, but in which the edges $e_{\overline{v}}$ of $\underline{\Delta}$ corresponding to vertices $\overline{v}$ of $\overline{\Gamma}$ are replaced by multiple edges e_v on the same vertices in Δ, where v runs over the vertices in Γ that are mapped to $\overline{v}$. So (ii) implies (i).

Let Γ be the line graph of a multi-graph. Then, by the above we find $\overline{\Gamma}$ to be a line graph of an ordinary graph. But then Example 2.3 shows that $\overline{\Pi}_\Gamma = \Pi_{\overline{\Gamma}}$ is a cotriangular space of some set Ω. So (ii) implies (iii).

Now suppose $\overline{\Pi}_\Gamma = \Pi_{\overline{\Gamma}}$ equals a cotriangular space Π of a set Ω.

Then all the vertices of $\overline{\Gamma}$ can be identified with pairs from Ω. In particular, $\overline{\Gamma}$ is a line graph. This shows that (iii) implies (ii).

Moreover, inside $\Pi_{\overline{\Gamma}}$ we do not find subspaces isomorphic to Π_{D_4}. So (iii) also implies (iv).

Now assume $\overline{\Pi}_\Gamma$ contains no subspace isomorphic to Π_{D_4}. Then let v, w be two collinear points and $u = v + w$ the third point on the line on v and w. Any point x is collinear with 0 or two points of $\{u, v, w\}$. Moreover, if x and y are two points collinear with both v, w, but not with u, then x and y are collinear, for otherwise we find a subgraph D_4 on v, u, x, y. So, $\omega_{v,w}$, defined as the set of points x of $\overline{\Pi}_\Gamma$ that are collinear to v, w but not u, is a clique.

If x is a point in $\omega_{v,w}$ different from v, w, then any point $y \in \omega_{v,w}$ different from x, v and w, is collinear with x, v and w, and then different from $v + x$. So, $\omega_{v,w} \subseteq \omega_{x,v}$ and by symmetry of the argument $\omega_{v,w} \subseteq \omega_{x,v}$. But the we find $\omega_{v,w} = \omega_{x,y}$ for any two points in $\omega_{v,w}$.

Let Ω be the set of all cliques $\omega_{x,y}$, where x, y are collinear points of $\overline{\Pi}_\Gamma$.

The point v is contained in both $\omega_{v,w}$ and in $\omega_{v,u}$. Let ω be an arbitrary element from Ω containing v. So, $\omega = \omega_{v,x}$ for some point x collinear to x. If x is different from w, u, then it is collinear with one of w or u and we find w or u is in ω from which we deduce that ω equals $\omega_{v,w}$ or $\omega_{v,u}$. So v is in exactly two elements of Ω and two elements from Ω intersect on at most one point. But then we can identify the points of $\overline{\Gamma}$ with pairs of elements from Ω and find $\overline{\Gamma}$ to be a line graph. So (iv) implies (ii). □

Remark 2.6. The above proof is based upon the ideas of Krausz [11] characterizing line graphs of ordinary graphs, and of Hall [10, Section 6], characterizing the (finitary) symmetric groups as 3-transposition groups.

3. An equivalence between Graphs

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Without loss of generality we can assume Γ to be the induced subgraph on the set $\mathcal{V}$ of anisotropic vectors in an orthogonal $\mathbb{F}_2$-space (V, Q). Denote by f the symmetric bilinear form associated to Q.

For vertices $v, w \in \mathcal{V}$ we define $\Gamma^{(v,w)}$ to be the graph induced on $\mathcal{V}\backslash\{w\} \cup \{w + f(v,w)v\}$. So, we replace the vertex w by $\tau_v(w) := w + f(v,w)v$, all other vertices remain the same.

Such transformation is called an elementary transformation. We say that Γ is *equivalent* to Δ if it can be obtained from Δ by a series of elementary transformations.

Clearly, equivalence is an equivalence relation. This relation has been studied by Brown and Humphries [2, 3] as well as Seven [12].

In this section we analyse the equivalence classes of graphs. We use the approach of the Appendix in [5].

Lemma 3.1. *Let v, w be two vertices of an ordinary graph Γ.*

(i) *If Γ is connected, then so is $\Gamma^{(v,w)}$.*
(ii) $\Pi_\Gamma = \Pi_{\Gamma^{(v,w)}}$.

Proof. Straightforward. □

If v, w are two vertices with $v + w$ in the isotropic radical of Q, then we write $v \equiv w$. The relation $\equiv$ is an equivalence relation.

By $d_\Gamma(v)$ we denote the degree of a vertex v in the graph Γ.

Lemma 3.2. *Let Γ be a finite connected graph. Let v be a vertex of Γ and $C : v = v_1, v_2, \ldots, v_n$ a minimal cycle on the distinct vertices $v_1, \ldots, v_n$ in Γ. Let $w_1, \ldots, w_l$ be the vertices outside C.*

Then

$\Delta = \Gamma^{(v_2,v)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(v))(v_2,w_1)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(w_1))\cdots(v_2,w_l)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(w_l))}$

satisfies

$$d_\Delta(\tau_{v_{n-2}}\cdots\tau_{v_2}(v)) = d_\Gamma(v) - 1.$$

Proof. Consider

$\Delta = \Gamma^{(v_2,v)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(v))(v_2,w_1)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(w_1))\cdots(v_2,w_l)\cdots(v_{n-1},\tau_{v_{n-2}}\cdots\tau_{v_2}(w_l))}$

Under this transformation the vertex v is changed to $v' = v_1+v_2+\ldots+v_{n-1}$. All the other vertices of the cycle remain the same. Now $f(v', v_k) = 0$ for $2 < k < n-1$ and $f(v', v_n) = 0$. In particular, we find a tree of type D_n.

A vertex $w \notin C$ is replaced by

$$w' = w + f(w, v_1)v_2 + f(w, v_1 + v_2)v_2 + \ldots + f(w, v_1 + \ldots v_{n-1})v_{n-1}.$$

So,

$$\begin{aligned} f(v_i, w') &= f(v_i, w + f(w, v_1)v_1 + f(w, v_1 + v_2)v_2 + \ldots + f(w, v_1 + \ldots v_{n-1})v_{n-1}) \\ &= f(v_i, w) + f(w, v_1 + \ldots + v_{i-1}) + f(w, v_1 + \ldots + v_{i+1}) \\ &= f(v_{i+1}, w), \end{aligned}$$

for $2 \leqslant i < n-1$, while

$$f(v_{n-1}, w') = f(v_1 + \ldots + v_{n-1}, w).$$

We find

$$\begin{aligned} f(v', w') &= f(v_1 + v_2 + \ldots + v_{n-1}, w') \\ &= f(v_1, w') + \ldots + f(v_{n-1}, w') \\ &= f(v_2, w) + f(v_3, w) + \cdots + f(v_{n-1}, w) + f(v_1 + \ldots + v_{n-1}) \\ &= f(v_1, w) \\ &= f(v, w). \end{aligned}$$

So, the degree of v' in Δ is the same as the one for v in Γ except that the degree inside the transforms of the cycle C has dropped from 2 to 1. □

Theorem 3.3. *Every finite connected graph Γ is equivalent to a tree containing an E_6 induced subgraph, or the quotient $\overline{\Gamma}$ is equivalent to A_n.*

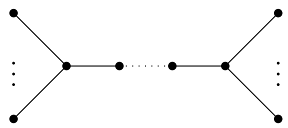
Proof. Fix a maximal subset T of $\mathcal{V}$ inducing a tree not containing two vertices of any cycle. Notice that, as a vertex is a tree not containing two vertices of a cycle, such T exist. Let v be an endpoint of T having a neighbor outside T. (If such v does not exist, Γ itself is a tree.)

Suppose v lies on a cycle. Then, by the above, the cycle lies entirely in $(\mathcal{V}\backslash T) \cup \{v\}$. Transformations involving vertices from $(\mathcal{V}\backslash T) \cup \{v\}$ have no effect on the graph structure on T. So, we can apply Lemma 3.2 to the induced graph on $(\mathcal{V}\backslash T) \cup \{v\}$ to find a graph Δ equivalent with Γ by a transformation τ with $d_\Delta(\tau(v)) < d_\Gamma(v)$.

We replace Γ by Δ and v by $\tau(v)$. We can repeat this process only a finite number of times and end up in a situation where v is not in a cycle. We pick a vertex w of v outside T and find the induced graph on $T \cup \{w\}$ again to be a tree not containing two vertices of a cycle.

Repeat the above procedure, beginning with the choice of a vertex of T separating the rest of T from a nonempty remainder in Γ, until there are no such end nodes. As at each step, the induced subtree becomes bigger, this will terminate, and we have found a tree T equivalent to Γ.

If the tree does not contain a subgraph E_6, then T is of the form



and, in $\overline{T}$, all end nodes (at the left or right) are mapped to a single vertex resulting into a graph of type A_n. □

Theorem 3.4. *A finite connected graph Γ is a line graph of a multi-graph if and only if $\overline{\Gamma}$ is equivalent to A_n for some integer n.*

Proof. Suppose $\overline{\Gamma}$ is equivalent to A_n. Then $\overline{\Pi}_\Gamma = \Pi_{\overline{\Gamma}}$ is isomorphic to Π_{A_n} and thus, by Theorem 2.5, we find Γ to be a line graph.

If Γ is equivalent to a tree containing and E_6 subdiagram, then $\overline{\Pi}_\Gamma$ contains a subspace isomorphic to Π_{E_6}, and hence it is not the cotriangular space of a set. See Example 2.4. But then, again using Theorem 2.5, we find $\overline{\Gamma}$ and also Γ not to be line graphs.

By Theorem 3.3, this proves the theorem. □

4. Line graphs and E_6

In this section we provide a proof for Theorem 1.1. We continue with the notation of previous sections.

Lemma 4.1. *Let Γ be a connected graph on 7 points with (V_Γ, Q_Γ) nondegenerate. Then Γ contains a subgraph equivalent to E_6, or Γ is equivalent to A_7.*

Proof. If Γ contains an induced subgraph Δ on 4 points which is equivalent to D_4, then the subspace V_Δ of V_Γ contains 12 anisotropic points and has a 2-dimensional isotropic radical R_Δ. As the isotropic radical of V_Γ is trivial, there are two vertices $v, w \in \mathcal{V}$ such that $\langle v, w\rangle^\perp \cap R = \{0\}$. (Here $\perp$ denotes orthogonality.) The induced subgraph Γ_0 on Δ and the vertices v, w generates a nondegenerate hyperplane V_{Γ_0} in which we find the subspace V_Δ. This implies that $(V_{\Gamma_0}, Q_{\Gamma_0})$ is 6-dimensional and of $-$-type. See Example 2.4.

So, we can now assume that Γ contains no subgraph equivalent to D_4. Let C be a maximal clique in Γ. A clique C is equivalent to $A_{|C|}$. So, we can assume $|C| \leqslant 6$. A vertex v outside C is adjacent to at most one element in C. For, otherwise, we find a subgraph $\lhd\!\rhd$ equivalent to D_4. Moreover, a point can not be in three maximal cliques, as this would mean that Γ contains a subgraph D_4.

Suppose $C_1, \ldots, C_k$ are the maximal cliques of size at least 2. As each vertex is in at most two such cliques, we find $k \leqslant 7$. Assign to a vertex v the pair $\{i, j\}$, if v is in the intersection $C_i \cap C_j$, where $i \neq j$. If v is in a unique maximal clique C_i, assign to it the pair $\{i, l\}$ for some $k < l \leqslant 8$ such that different pairs are assigned to different vertices. This is possible by the above. Two vertices are adjacent if and only if the assigned pairs meet nontrivially. But then it is easy to check that Γ is equivalent to A_7. □

Lemma 4.2 (Seven's Lemma, [12])**.** *If Γ is a finite connected graph equivalent to a tree containing E_6, then Γ contains a subgraph on 6 vertices equivalent to E_6.*

Proof. Suppose Γ contains a subgraph Δ on 6 vertices equivalent to E_6. To prove the lemma it suffices to show that for any two vertices v, w we will find $\Gamma^{(v,w)}$ also to contain a subgraph on 6 vertices equivalent to E_6.

If v, w are nonadjacent, or vertices both in Δ or both outside Δ, this is clear. So, consider the case where w is a vertex of Δ, adjacent to v, which is not in Δ. Then consider the graph $\widehat{\Delta}$ on the vertices of $\Delta^{(v,w)}$ and w.

If $(V_{\widehat{\Delta}}, Q_{\widehat{\Delta}})$ a nondegenerate 7-dimensional space, then, by Lemma 4.1, we find that the subgraph $\widehat{\Delta}$ of $\Gamma^{(v,w)}$ contains a subgraph equivalent to E_6. If $(V_{\widehat{\Delta}}, Q_{\widehat{\Delta}})$ is a degenerate 7-dimensional space, it contains a 1-dimensional radical, and modulo this radical we find a 6-dimensional orthogonal space of $-$ type and we can take any 6 vertices of $\widehat{\Delta}$ spanning a complement to the radical. □

Theorem 4.3. *Let Γ be a connected graph. Then Γ is a line graph of a multi-graph, if and only if it does not contain an induced subgraph in $\mathcal{E}_6$.*

Proof. If Γ contains a subgraph in $\mathcal{E}_6$, it can not be a line graph.

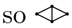
Now assume that Γ contains no induced subgraph from $\mathcal{E}_6$. Then $\overline{\Gamma}$ also has no induced subgraph from $\mathcal{E}_6$. If $\overline{\Gamma}$ is a finite graph, then we can apply Theorem 3.3 and Lemma 4.2, and find that $\overline{\Gamma}$ is equivalent to A_n for some n and hence, by Theorem 3.4, that Γ is a line graph.

The case that Γ is infinite can be dealt with in the following way.

Consider $\overline{\Gamma}$ which also has no induced subgraph from $\mathcal{E}_6$. By Theorem 2.5 we only have to show that $\Pi_{\overline{\Gamma}}$ is (isomorphic to) a cotriangular space of a set Ω. But by Theorem 2.5 this can be checked inside finite subspaces of $\Pi_{\overline{\Gamma}}$ and thus inside finite subgraphs of $\overline{\Gamma}$. □

Theorem 1.1 follows from Theorem 4.3, as the graphs in Fig. 1 are the graphs in the set $\mathcal{E}_6$.

Remark 4.4. To compute the list of graphs in $\mathcal{E}_6$ we have used the following observations.

If Γ is an element of $\mathcal{E}_6$, then it does not contain any two nonadjacent vertices having the same neighbors, for otherwise, the sum of these two vertices is a nonzero vector in the radical of Q_Γ. Moreover, if C is a minimal cycle in Γ, then there has to be a vertex that has an odd number of neighbors in the cycle, for otherwise the sum of the vertices in the cycle is in the radical of Q_Γ. Finally, as follows from the second part of the proof of Lemma 4.1, the graph Γ has an induced subgraph equivalent to D_4, so  or .

Starting with one of these two graphs on four vertices, one can add two extra vertices to obtain a graph Γ that satisfies the above requirements. We have checked our list against the list of all connected graphs on 6 vertices as given in [8].

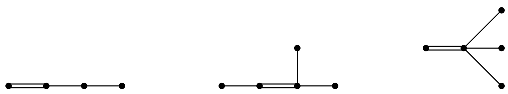
5. Line graphs of ordinary graphs

In this section we determine the minimal graphs which are not line graphs of ordinary graphs and reprove Beineke's result Theorem 5.1. So, let Γ be such a graph. Assume that Γ is $\mathcal{E}_6$-free, and hence by Theorem 4.3 a line graph of a multi-graph Δ.

As Γ is not the line graph of an ordinary graph, we can assume that Δ contains two vertices, v, w say, on which there are two or more edges.

If Δ contains a 3-fold edge, it contains (by connectedness) a subgraph (not necessarily induced) and we find Γ to contain an induced subgraph H_1. So, we can assume that on each pair of points there are at most two edges.

If Δ contains more than 4 vertices, then we find at least one of the following three subgraph:



But that implies that Γ contains an induced subgraph H_1, H_2 or H_3.

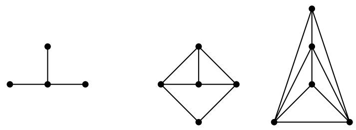
If Δ contains 3 or 4 points it is straightforward to check that Γ is not a line graph if and only if it does not contain induced subgraph H_1, H_2 or H_3. This leads to the following theorem.

Theorem 5.1 (Beineke, [1])**.** *A graph Γ is an ordinary line graph if and only if it does not contain any of the graphs H_1, H_2, H_3 as in Fig. 2, or of the graphs $E_6^{(8)}, E_6^{(12)}, E_6^{(20)}, E_6^{(22)}, E_6^{(25)}$ or $E_6^{(30)}$ from Fig. 1.*

Proof. If a graph Γ is the line graph of an ordinary graph, then clearly it does not contain any of the graphs H_1, H_2, H_3 as in Fig. 2, nor one of the graphs $E_6^{(8)}, E_6^{(12)}, E_6^{(20)}, E_6^{(22)}, E_6^{(25)}$ or $E_6^{(30)}$ from Fig. 1.

The above shows that a graph Γ which is not a line graph of an ordinary graph and does not contain one of the graphs from $\mathcal{E}_6$, will contain H_1, H_2, or H_3.

As the graphs $E_6^{(8)}, E_6^{(12)}, E_6^{(20)}, E_6^{(22)}, E_6^{(25)}$ or $E_6^{(30)}$ are precisely the graphs of $\mathcal{E}_6$ not containing an induced subgraph H_1, H_2 or H_3, we have proved the theorem. □



$H_1 = D_4$ H_2 H_3

FIGURE 2. Some forbidden subgraphs of a line graph.

6. Generalized line graphs

In this section we consider generalized line graphs and determine the minimal graphs which are not generalized line graphs.

Definition 6.1. A *cocktail party graph* is a graph in which every vertex is adjacent to precisely all, or all but one other vertices.

Suppose $\Delta = (\mathcal{V}, \mathcal{E})$ is a graph and for every vertex v of Δ let Δ_v be a (possibly empty) cocktail party graph.

Then the *generalized line graph* obtained from Δ and the various Δ_v for $v \in \mathcal{V}$ is the graph which is the union of the line graph $L(\Delta)$ and all the graphs Δ_v, to which the following edges are added: a vertex e of $L(\Delta)$ is adjacent to all vertices of Δ_v, where v is a vertex on e.

This definition of generalized line graph does not directly fit our purposes.

We use the following proposition, which shows that generalized line graphs are a special type of line graph of a multi-graph.

Proposition 6.2. *A graph Γ is a generalized line graph if and only if it is the line graph of a multi-graph Δ such that*

(i) *two vertices of Δ are on at most two common edges.*
(ii) *If v, w are two vertices of Δ which are on two common edges, then one of the vertices is on no other edges.*

Proof. Suppose we are given a generalized line graph Γ based on a graph $\Delta = (\mathcal{V}, \mathcal{E})$ and a collection Δ_v, where $v \in \mathcal{V}$, of cocktail party graphs.

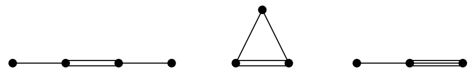
Then consider the multi-graph $\hat{\Delta}$ which consists of Δ to which we add new vertices $\delta_{v,\{w,w'\}}$ for each $v \in \mathcal{V}$ and $\{w, w'\}$ a pair of nonadjacent vertices of Δ_v, and two edges on v and $\delta_{v,\{w,w'\}}$. The graph Δ' satisfies (i) and (ii) and Γ is the line graph of Δ'.

On the other hand if Γ is the line graph of a graph $\hat{\Delta}$ satisfying (i) and (ii), then let $\mathcal{V}_0$ be the vertices that are on a multiple edge but not on another edge, and let Δ be the graph obtained from $\hat{\Delta}$ by removing all vertices of $\mathcal{V}_0$ and the edges on them. For each $v_0 \in \mathcal{V}_0$, let v_1 be the unique neighbor. Then for each such v_1 define Δ_{v_1} to be the cocktail party graph which is the line graph of the subgraph of $\hat{\Delta}$ induced on v_1 and all the $v_0 \in \mathcal{V}_0$ adjacent to v_1. Then Γ is the generalized line graph obtained from Δ and all these cocktail party graphs. □

Now assume that Γ is a connected graph which is not a generalized line graph, but $\mathcal{E}_6$-free.

We will prove that Γ contains an induced subgraph isomorphic to $G_1, \ldots, G_{10}$ or G_{11}.

By Theorem 4.3 we find that Γ is the line graph of a connected multi-graph Δ. As we assume Γ not to be a generalized line graph, Δ will have to contain one of the following subgraphs:

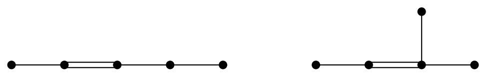


Δ_1 Δ_2 Δ_3

Suppose Δ contains a 5-fold edge. Then it contains a subgraph and Γ contains a subgraph G_5.

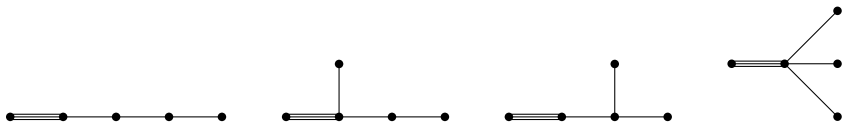
So, we can assume that Δ contains only k-fold edges, where $k \leqslant 4$.

Now assume that Δ contains at least 5 vertices. If Δ contains Δ_1, we can extend Δ_1 to one of the following graphs:



But then Γ contains G_1 or G_3.

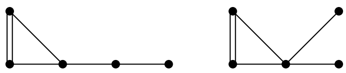
So, from now on we can and do assume that Δ contains no Δ_1. If Δ contains Δ_3, then by adding some edges, we can extend it to one of the following subgraphs:



But then we find in Γ an induced subgraph G_4, G_8, G_6, or G_{11}, respectively.

So, we can also assume that Δ does not contain Δ_3. It remains to consider the case that Δ contains Δ_2.

In this case we can extend Δ_2 to one of the following subgraphs



and find subgraphs G_7 or G_{10} inside Γ.

If Δ contains $\leqslant 4$ vertices and no k-fold edges for $k > 4$, then we can easily check that Γ is a generalized line graph if and only if it does not contain any induced subgraph isomorphic to $G_1, \ldots, G_{10}$ or G_{11}. For some examples, see Fig. 3.

We are now in a position to prove the following:

Theorem 6.3 ([7]). *A connected graph Γ is a generalized line graph if and only if it does not contain one of the* 31 *graphs* $G_1, \ldots, G_{11}$ *or* $E_6^{(i)}$ *of Fig. 1, where i is an element of* $\{1, 2, 4, 8, 9, 10, 12, 16, 17, 20, 21, 23, 24, 25, 27, 28, 30, 31, 32, 33\}$.

Proof. By the above we find that Γ is a generalized line graph if and only if it does not contain one of the graphs from $\mathcal{E}_6$ or $G_1, \ldots, G_{11}$. Removing from $\mathcal{E}_6$ graphs that contain one of $G_1, \ldots, G_{11}$ leaves us with the additional 20 graphs $E_6^{(i)}$ of Fig. 1, where $i \in \{1, 2, 4, 8, 9, 10, 12, 16, 17, 20, 21, 23, 24, 25, 27, 28, 30, 31, 32, 33\}$. □

7. Concluding remarks

We end this paper with some remarks.

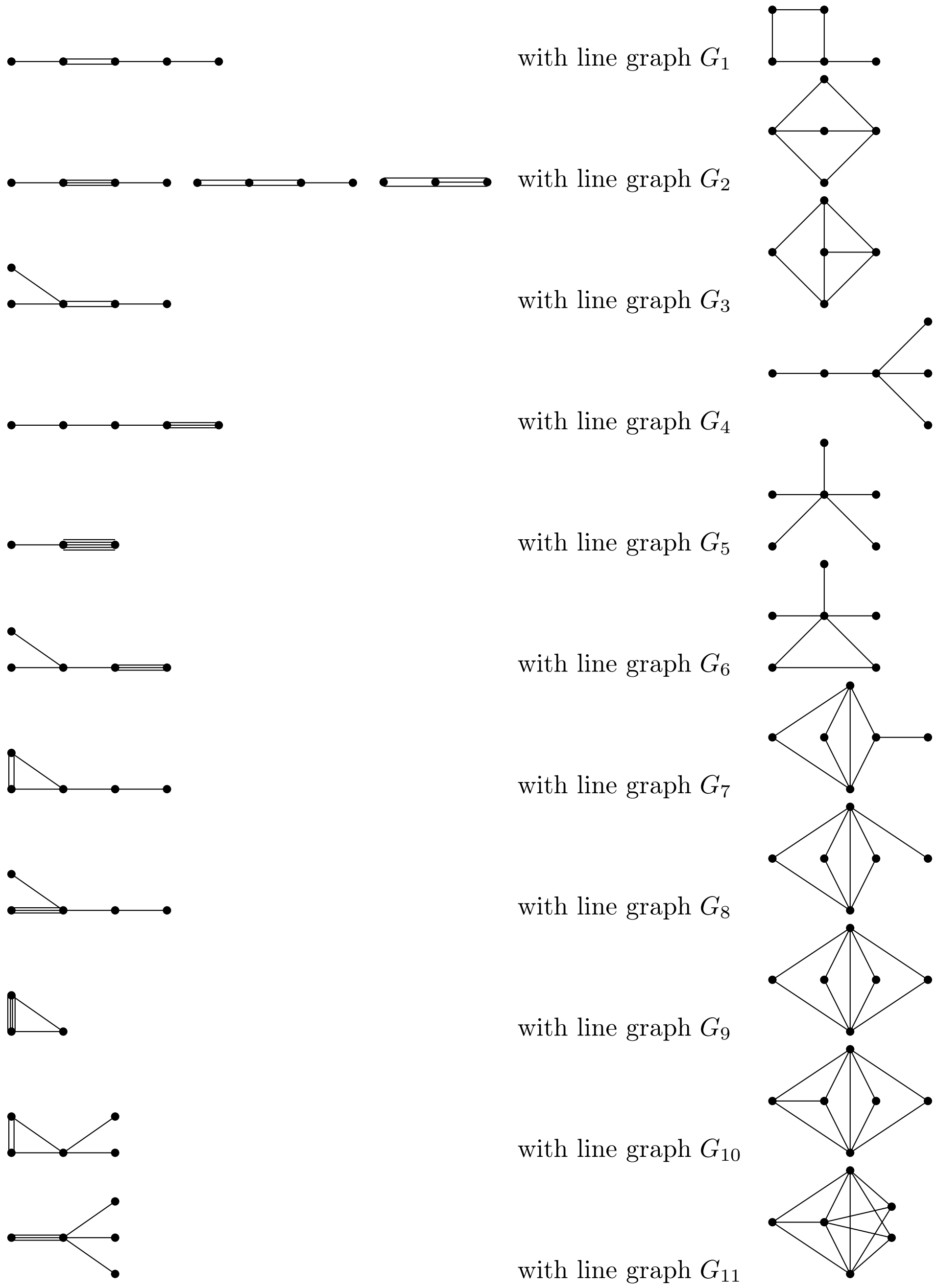


FIGURE 3. Some forbidden graphs Δ and their line graphs $G_1, \ldots, G_{11}$.

(i) By Theorem 1.1 checking whether a finite graph is the line graph of a multi-graph, only requires to consider all subgraphs on 6 vertices. Hence this can be done in polynomial time. An algorithm to find a multi-graph Δ with prescribed line graph $L(\Delta)$ is given in [6]. Notice that Δ need not be uniquely determined.

(ii) In [14], Vijayakumar considers root lattices generated by sets of roots and proves that a set of roots of equals length, which can not be embedded in a root system of type D, contains six roots generating the root lattice E_6. See

[14, Theorem 16]. This result can easily be shown to be a consequence of Seven's Lemma 4.2.

(iii) The infimum of the least eigenvalues of all finite induced subgraphs of an infinite graph is defined to be its least eigenvalue. Finite generalized line graphs are graphs with least eigenvalue $\geqslant -2$. By our results, together with the fact that any finite connected graph with least eigenvalue $\geqslant -2$ is a generalized line graph or can be represented by roots in the root system E_8 and has at most 36 vertices, see [4], we immediately find that a connected infinite graph with least eigenvalue $\geqslant -2$ is a generalized line graph (see [15]). Indeed, every finite subgraph of an infinite graph Γ can be embedded into a connected subgraph on more than 36 vertices and therefor is a generalized line graph and hence contains none of the 31 graphs of Theorem 1.3. But then Theorem 1.3 also shows that Γ itself is a generalized line graph.

(iv) In [13] Vijayakumar considers edge signed graphs that can be embedded in a (possibly infinite dimensional) root lattice of type D. He characterizes these graphs by a list of 49 forbidded subgraphs. Theorem 1.1 and methods similar to those used in Section 5 and Section 6 can be used to give a new proof for this characterization of these graphs.

(v) Given a finite simply laced Coxeter diagram Γ, we find, by Theorem 3.3, that the corresponding Coxeter group $(W(\Gamma), S)$ admits a quotient that maps the generators in S to transpositions in a symmetric group if and only if the graph Γ is a line graph of a multi-graph. If the corresponding Coxeter group $(W(\Gamma), S)$ does not have such quotient, then there are six generators $s_1, \ldots, s_6$ such that $\langle s_1, \ldots, s_6 \rangle$ admits a quotient $W(E_6)$ which maps the six generators to reflections in the Weyl group $W(E_6)$.

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