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Singular Fuel-optimal Control of the Velocity and Power-split of Hybrid Electric Vehicles via Pontryagin’s Minimum Principle

Thijs van Keulen and Mauro Salazar

Abstract—This paper studies the eco-driving problem of finding the fuel-optimal velocity and power-split control for parallel hybrid electric vehicles that must drive over a finite distance in limited time. Specifically, we combine Pontryagin’s minimum principle and singular control theory to derive the minimum-fuel control policy for the engine and the electric motor, and leverage it to rewrite the original optimal control problem into a Hamiltonian boundary value problem that can be efficiently solved with standard numerical methods. We showcase our findings with numerical simulations, revealing the fuel-optimal strategies favoring a faster, electrically-assisted, initial acceleration so that the required travel distance can be covered at the lowest possible cruising speed.

Keywords—Energy management strategy, hybrid electric vehicles, velocity optimization, Pontryagin’s minimum principle.

I. INTRODUCTION

With the advent of automated driving technologies, velocity trajectory optimization is drawing increasing attention as a means to reduce the energy consumption of road vehicles [1]. Thereby, optimal trajectories computed based on real-life route and vehicle information can be used to assist the driver or autopilot in so-called eco-driving strategies. When it comes to Hybrid Electric Vehicles (HEVs) as the one shown in Fig. 1, the energy consumption of the vehicle can be further improved by optimizing the energy management strategies coordinating the energy flows between the powertrain components. As the driven speed-trajectory and the power-split influence each other, they should be jointly optimized. Against this backdrop, this paper studies the eco-driving problem for HEVs, providing a numerical framework to efficiently compute the fuel-optimal speed-profile and power-split in a joint fashion.

Related literature: This paper contributes to the following research streams: The optimal operation of vehicles with a single power converter such as conventional or pure electric powertrains is well known. Mathematically, the problem is closely related to the fuel-optimal flight of a rocket which has been studied for over a century [2]. The solution involves determining extremal trajectories by solving the Hamiltonian Boundary Value Problem (HBVP) posed by first order optimality conditions extended with singular optimal control theory [3, 4, 5]. Extensions to the optimal velocity control of conventional road vehicles can be found in [6, 7, 8].

HEVs combine a prime mover such as a Fuel Cell (FC) or an Internal Combustion Engine (ICE) with an Electric Machine (EM) and battery to recover energy during braking and assist the prime mover to propel the vehicle. The optimal power-split control problem for vehicles with a hybrid electric powertrain has been studied extensively, see, e.g., [9] for an overview. It is well understood that Pontryagin’s Minimum Principle (PMP) provides a suitable framework to derive both online and offline control policies. These power-split strategies, however, assume a prescribed velocity and power trajectory.

The optimal velocity control of HEVs adds a challenge: The time instances where the recovered energy is used does not only affect the efficiency of the power converters but also the aerodynamic drag and rolling friction losses. In this context, finding the time-optimal operation for the hybrid electric powertrain of the Formula 1 car results into a problem that is a paradoxically simpler version of the eco-driving problem. In particular, its minimum-time objective allows to perform convex relaxations to the nonlinear time-speed relationship, which would not be lossless in conventional applications, and to efficiently solve it numerically with convex programming [10], from which the costate trajectories can be extracted and leveraged to implement the optimal speed-control and energy-management policy via PMP [11] and effectively adapt them in real-time [12, 13]. However, these methods are not directly applicable to the time constrained problem as the objective is different and the inequalities would inevitably relax, resulting in a non-physical solution.

Current approaches for optimal velocity control of HEVs subject to a time constraint are based on approximated [14, 15] or nested optimization approaches [1, 16, 17, 18]. Hereby, the optimal velocity trajectory is first computed for a vehicle with a conventional drive-train disregarding the energy recovery options, and the optimal power-split is determined in a second stage, thereby ignoring the coupling between power-split and road load losses.

In conclusion, to the best of the authors’ knowledge, whilst important progress has been made in the optimal operation of road vehicles in terms of eco-driving or power-split, there are no provably optimal solutions that can solve the problem.
jointly for a fixed distance and time.

Statement of contributions: This article provides an analytical solution to the joint optimal velocity and power-split control problem for HEVs based on PMP and singular control, which we leverage to devise an efficient numerical solution method.

Organization: Section II provides a system description, whilst the optimal control problem is formulated is stated in Section III. Section IV studies the necessary conditions for optimal power-split control on prescribed power trajectories. In Section V, the main results are presented extending the power-split results to include also velocity trajectory optimization. Section VI provides a numerical example, and we draw the conclusions in Section VII.

II. System Description

Vehicles with a hybrid powertrain as shown in Fig. 1 employ at least two power converters, 1) a prime mover which provides output power \( P_p \), consuming fuel \( P_f \) via an irreversible process, e.g., an ICE or FC, and 2) a secondary power converter which converts mechanical power \( P_m \) reversibly into a power quantity suitable for a storage device \( P_s \), e.g., an EM in combination with a battery. Furthermore, they are equipped with service brakes that can provide braking power \( 0 \leq P_d \leq P_f \). The vehicle itself can be viewed as a storage device which can acquire kinetic energy \( P_k \), depending on which the vehicle suffers from power loss \( P_f \). This section presents a dynamical model of the powertrain shown in Fig. 1, whereby we omit time-dependence whenever clear from the context.

A. Prime Mover

The relation between fuel power \( P_f \geq 0 \) and mechanical power \( P_{p,0} \leq P_p \leq P_f \) is given by the relation

\[
P_f = \gamma_P P_p + P_{p,0},
\]

which, for ICES, is known as the Willans line, where \( \gamma_P > 1 \) is the inverse of the fuel conversion efficiency and \( P_{p,0} \) is the engine drag loss power, as shown in Fig. 2a.

\[\text{Fig. 2. Schematic cost functions of a) the prime mover, and b) the EM.}\]

B. Electric Machine

The EM is described by a piece-wise continuous linear relation:

\[
P_b = \begin{cases} \gamma_m P_m, & \text{if } P_m \leq 0, \\ \frac{1}{\gamma_m} P_m, & \text{if } P_m > 0. \end{cases}
\]

Hereby, \( P_b \geq 0 \) is the electric power, \( 0 < \gamma_m \leq 1 \) the EM efficiency, and \( P_{m} \leq P_m \leq P_{m} \) the bounded EM mechanical power, as shown in Fig. 2b.

C. Battery

The battery open-circuit voltage \( U_{oc} \) is approximated with an affine function of the State-of-Energy (SOE) \( E_s \) of the battery:

\[
U_{oc} = U_0 + \phi E_s.
\]

Here, \( \phi > 0 \) is the battery open-circuit increase factor, and \( U_0 > 0 \) the voltage of a depleted battery, as shown in Fig. 3a. The energy stored in the battery is governed by the differential equation

\[
\dot{E}_s(t) = -P_s(t),
\]

where the battery storage power \( P_s \) can be modeled with an internal resistance model, see Fig. 3b:

\[
P_s = I_s^2 R + P_b,
\]

\[
= I_s U_{oc},
\]

in which \( I_s \) is the current and \( R \) the internal battery resistance. The battery terminal voltage \( U_t \) is computed by

\[
U_t = U_{oc} - I_s R. \tag{7}
\]

Note that a fully charged battery has a higher open-circuit voltage \( U_{oc} \), so that a power \( P_f \) can be generated with a lower current and hence lower ohmic losses \( I_s^2 R \).

\[\text{Fig. 3. Battery characteristics: a) the open-circuit voltage as function of state-of-energy, b) the equivalent circuit for the battery.}\]

The current \( I_s \) can be obtained from (5) and (6) as

\[
I_s = \frac{U_{oc} - \sqrt{U_{oc}^2 - 4RF_b}}{2R}. \tag{8}
\]

Therefore, the storage power \( P_s \) can be expressed as a function of mechanical power \( P_m \) of the EM:

\[
P_s = \begin{cases} \frac{U_{oc}}{2R} \left( U_{oc} - \sqrt{U_{oc}^2 - 4RF_b} \right), & \text{if } P_m \leq 0, \\ \frac{U_{oc}}{2R} \left( U_{oc} - \sqrt{U_{oc}^2 - 4RF_b} \right), & \text{if } P_m > 0. \end{cases} \tag{9}
\]

D. Vehicle Dynamics

The vehicle kinetic energy is described by

\[
E_k = \frac{1}{2} m v^2, \tag{10}
\]

in which \( m > 0 \) is the vehicle mass and \( v \geq 0 \) is the longitudinal velocity.
Assuming a flat road, the loss power $P_l$ stemming from rolling and aerodynamic resistance is

$$P_l = c_r m g v + \frac{1}{2} \rho A c_d v^3,$$

where $c_r > 0$ is a dimensionless coefficient for the rolling resistance, $g$ the gravitational constant, $\rho$ the air density, $A$ the vehicle frontal area, and $c_d > 0$ a dimensionless aerodynamic drag coefficient.

To conclude the system description, the vehicle kinetic energy $E_k$ and traveled distance $s$ are governed by the following differential equations:

$$\dot{E}_k(t) = P_p(t) + P_m(t) - P_d(t) - P_l(t),$$

$$\dot{s}(t) = \sqrt{\frac{2}{m} E_k(t)}.$$

### III. Problem Description

This section studies two problems with the objective of minimizing the fuel consumption of HEVs: first, the problem where the power request $P_r(t)$ and service brakes $P_d(t)$ trajectories are known a priori and solely the optimal split of the requested power between ICE and EM is to be determined; second, the problem of driving a HEV over a given distance within a limited travel time $t_f$ and using a minimal amount of fuel.

#### A. Power-split Control

This subsection describes the optimal power-split control problem for a known power demand trajectory.

For convenience, we define a power request trajectory which includes the service brake operation $P_d(t) = P_r(t) - P_d(t)$ such that, at each time instant, the power-split equality constraint holds:

$$P_p(t) = P_r(t) - P_m(t),$$

in which the power request is bounded by $P_{p,0} \leq P_r(t) \leq P_B$. Only power requests above $P_{p,0}$ are considered because the solution for negative power requests below the engine drag is determined based on physical insight alone: The optimal strategy is to generate electricity and store it in the battery if $P_r$ drops below the level where $P_l = 0$, and is referred to as energy recovery. Moreover, it is assumed that the requested power does not exceed the maximum power of the ICE such that the constraint $P_B$ can be dropped from the problem formulation.

The resulting optimal control problem is summarized by:

$$P_A := \left\{ \begin{array}{l}
\min_{P_m \in [P_{m,0}, P_B]} \int_{t_0}^{t_f} P_l(P_m) dt, \\
\text{s.t. Eq. (4), (14), } E_s(t_0) = E_{s,0}, E_s(t_f) = E_{s,f}.
\end{array} \right.$$  

Here, the optimal mechanical EM power is selected as optimizer. Furthermore, the battery SOE at the start $E_{s,0}$ and the desired battery level at the end $E_{s,f}$ are assumed to be known.

### B. Power-split and Velocity Control

Next, this subsection extends the power-split control problem $P_A$ to velocity control. In this problem formulation, $P_r$ and $P_d$ are unknown a priori so (14) does no longer hold. Instead, the problem has three decision variables: the EM power $P_m$, the ICE power $P_p$ and the brake power $P_d$.

$$P_B := \begin{cases}
\min_{P_m, P_p, P_d} \int_{t_0}^{t_f} P_l(P_m, P_p, P_d) dt, \\
\text{s.t. Eq. (4), (12), (13), } \\
E_s(t_0) = E_{s,0}, E_s(t_f) = E_{s,f}, E_k(t_f) = E_{k,f}, s(t_0) = 0, s(t_f) = S.
\end{cases}$$  

Beside the boundary conditions on the battery SOE, $E_{s,0}$ and $E_{s,f}$, the kinetic energy at the start and end of the trajectory $(E_{k,0}$ and $E_{k,f}$, respectively) is a boundary condition along with the distance to travel $S$.

### IV. Optimal Power-split Control

This section addresses the first problem formulated: the optimal power-split control for a HEV.

#### A. Necessary Conditions of Optimality

The Hamiltonian associated to problem $P_A$ is given by

$$H_A = \gamma_p (P_r(t) - P_m(t)) + P_{p,0} + \lambda(t) P_r(t).$$  

Hereby, the fuel power is described by (1), (2), and (14), and $\lambda$ denotes the Lagrange multiplier, $P_r$ is described by (9). The sign of $\lambda$ is opposite to the conventional description of the Hamiltonian in optimal control such that $\lambda$ has a physical meaning as a fuel equivalent weighting factor. Note that, $\lambda$ is dimensionless and $H_A$ has the dimension of power. The necessary conditions for optimality are

$$P^* = \arg \min_{P_m \in [P_{m,0}, P_B]} H_A(t, E_s, P_m, \lambda),$$

$$\dot{\lambda} = \frac{\partial H_A}{\partial E_s}.$$

Ignoring the input constraints on $P_m$ for the moment, the optimal control can be found by setting the first derivative of $H_A$ with respect to the control to zero:

$$\frac{\partial H_A}{\partial P_m} = 0.$$

The first condition then gives

$$\begin{cases}
-\gamma_p + \frac{U_{oc}}{\sqrt{U_{oc}^2 - 4\gamma_p U_{oc} P_m}} = 0, & \text{if } P_m \leq 0, \\
-\gamma_p + \frac{U_{oc}}{\sqrt{U_{oc}^2 - 4\gamma_p U_{oc} P_m}} = 0, & \text{if } P_m > 0,
\end{cases}$$

which can be solved for $P_m$ to obtain an analytical expression for the optimal control in the state and costate variable:

$$P_m^* = \begin{cases}
\frac{U_{oc}^2}{4\gamma_p} \left(1 - \lambda^2 \frac{\gamma_p}{U_{oc}}\right), & \text{if } \lambda > \frac{2\gamma_p}{U_{oc}}, \\
0, & \text{if } \gamma_p \gamma_m \leq \lambda \leq \frac{2\gamma_p}{U_{oc}}, \\
\frac{2\gamma_p U_{oc}^2}{4R} \left(1 - \lambda \frac{1-\gamma_p}{\gamma_p \gamma_m}\right), & \text{if } \lambda < \gamma_p \gamma_m.
\end{cases}$$
Due to the non-smoothness in the EM description and the losses associated with charging the battery, a dead-zone is obtained in the relation between optimal control and costate value where the EM is not used.

The second condition for optimality (19) yields
\[
\dot{\lambda} = \lambda \frac{\partial U_{\text{oc}}}{2R} \psi, \tag{23}
\]
whereby
\[
\psi = \begin{cases} 
2 - \sqrt{\frac{2}{4R \gamma_m P_m}} - \frac{U_{\text{oc}}}{U_{\text{oc}} - 2R \gamma_m P_m} & \text{if } P_m < 0, \\
0, & \text{if } P_m = 0, \\
2 - \sqrt{\frac{2}{4R \gamma_m P_m}} - \frac{U_{\text{oc}}}{U_{\text{oc}} - 2R \gamma_m P_m} & \text{if } P_m > 0.
\end{cases}
\tag{24}
\]
The costate dynamics for \( P_m \neq 0 \) have the special form
\[
\dot{\lambda} = \lambda \frac{\partial U_{\text{oc}}}{2R} \left( 2 - \frac{\lambda'}{\lambda'} \right), \tag{25}
\]
where \( \lambda' \) is a dimensionless parameter. By design, the open-circuit voltage is positive \( U_{\text{oc}} > 0 \) and a real valued solution requires \( \frac{U_{\text{oc}}^2}{4R} \geq 4RP_0 \). Therefore, a physically meaningful solution is only obtained for \( \lambda' \geq 0 \) and, since \( \gamma_p > 1 \) and \( \gamma_m > 0 \), it follows that \( \lambda > 0 \).

Furthermore, the sum of the three terms between the brackets in (25) never exceeds zero and \( \dot{\lambda} = 0 \) if and only if \( \lambda' = 1 \), so for \( P_m > 0 \) and \( \gamma_p \gamma_m \leq \lambda \leq \frac{\gamma_m}{\gamma_m} \). Moreover, \( 0 < \lambda' < 1 \) corresponds to \( P_m > 0 \) and \( \lambda' > 1 \) to \( P_m < 0 \). Also, the sign of (25) will always oppose the sign for costate \( \lambda \), so \( \lambda \dot{\lambda} \leq 0 \). Hence, the costate \( \lambda \) obeys a differential equation whose solution is always directed towards \( \lambda = 0 \), see Fig. 4a.

\[
\begin{array}{ccc}
\lambda & \gamma_p \gamma_m & \gamma_m \\
P_m & P_m & \lambda \\
\end{array}
\]

Fig. 4. Schematic indicating optimality conditions: a) costate dynamics, b) optimal power-split control as function of the costate.

Given the explicit relation between the optimal control \( P_m^* \) and costate \( \lambda \), it is possible to determine the state-dependent upper bound
\[
\bar{\lambda}(E_s) = \frac{\gamma_p}{\gamma_m U_{\text{oc}}(E_s)} \sqrt{\frac{2}{4R \gamma_m P_m}} E_s - 4R \gamma_m P_m, \tag{26}
\]
and lower bound
\[
\underline{\lambda}(E_s) = \frac{\gamma_p \gamma_m}{U_{\text{oc}}(E_s)} \sqrt{\frac{2}{4R \gamma_m P_m}} E_s - 4R \frac{1}{\gamma_m} P_m, \tag{27}
\]
on the costate value at which the optimal control \( P_m^* \) is clipped as shown in Fig. 4b. Hence, the constraints on \( P_m \) can be added to (22) to give the optimal control policy including the input constraints:
\[
P_m^* = \begin{cases} 
\frac{P_m}{U_{\text{oc}}^2} \left( 1 - \lambda^2 \frac{\lambda^2}{\gamma^2} \right), & \text{if } \lambda > \bar{\lambda}, \\
0, & \text{if } \frac{\gamma_m}{\gamma_m} < \lambda < \bar{\lambda}, \\
\frac{2\gamma_m P_m}{4R} \left( 1 - \lambda^2 \frac{1}{\gamma^2} \right), & \text{if } \lambda < \gamma_p \gamma_m, \tag{28} \\
\frac{\gamma_m}{\gamma_m}, & \text{if } \lambda < \gamma_p \gamma_m.
\end{cases}
\]
Here it is assumed that \( P_m \) and \( \bar{P}_m \) are selected such that the battery terminal voltage \( U_t \) in (7) remains positive.

B. Structure of the Solution

The state-costate vector field has a set \( \lambda \in [\gamma_p \gamma_m, \gamma_m] \) where \( E_s = \dot{\lambda} = 0 \). Therefore, it is not possible to find a costate trajectory with values \( \lambda < \gamma_p \gamma_m \) and \( \lambda < \gamma_p \gamma_m \) or \( \lambda < \gamma_p \gamma_m \). It can thus be concluded that either the optimal solution includes charging the battery using the ICE, or the EM is used to assist the engine providing driving power. However, in a depleting solution trajectory it is not profitable to charge the battery on any occasion with the ICE, i.e., the EM is only used to employ the energy recovered during braking.

A shooting algorithm can be employed [9] to derive a numerical solution for the two-point HBVP. This means that the initial costate value \( \lambda(t_0) \) is estimated, the state and costate dynamics are solved forward in time, and it is evaluated whether the final state \( E_s(t_f) \) meets the boundary condition. If the final state does not coincide with the boundary condition, the estimate of the initial costate \( \lambda(t_0) \) is adjusted and this process is repeated in an iterative fashion.

V. OPTIMAL VELOCITY CONTROL OF HYBRID VEHICLES

This section studies the optimal velocity control for vehicles equipped with a hybrid powertrain as stated in problem \( \mathcal{P}_B \).

A. Necessary Conditions of Optimality

The Hamiltonian associated to problem \( \mathcal{P}_B \) is given by
\[
H_B = \gamma_p P_p(t) + P_{p,0} + \lambda_e(t) P_s(t) - \lambda_k(t) [P_p(t) + P_m(t) - P_m(t) - P_k(t, E_k)] - \lambda_s(t) \sqrt{\frac{2}{m} E_k(t)}. \tag{29}
\]
Here, \( \lambda_e \) denotes the costate associated to the battery SOE, \( \lambda_k \) the costate related to kinetic energy, and \( \lambda_s \), the costate related to the traveled distance. The sign of \( \lambda_s \), \( \lambda_k \), and \( \lambda_e \) is the opposite of the conventional description of the Hamiltonian in optimal control such that the costate has the physical meaning of a fuel equivalent weighting factor. Note that \( \lambda_e \) and \( \lambda_k \) are dimensionless, \( \lambda_s \) has dimension force, whilst the Hamiltonian \( H_B \) has dimension power.
The necessary conditions for optimality result in
\[ P_p^* = \arg \min_{P_p} H_B, \quad (30) \]
\[ P_m^* = \arg \min_{P_m} H_B, \quad (31) \]
\[ P_d^* = \arg \min_{[0, \tau_p]} H_B, \quad (32) \]
\[ \dot{\lambda}_k = \frac{\partial H_B}{\partial E_k}, \quad \dot{\lambda}_c = \frac{\partial H_B}{\partial E_c}, \quad (33) \]
\[ \dot{\lambda}_s = \frac{\partial H_B}{\partial s}. \quad (34) \]

Before deriving the optimal control \( P_p^* \) and \( P_d^* \), it is convenient to first derive the costate dynamics associated to kinetic energy (34) and position (35):
\[ \dot{\lambda}_k = \lambda_k \left( \frac{gc}{v} + \frac{3\rho A c_d}{2m} v \right) - \frac{\lambda_s}{mv}, \quad (36) \]
\[ \dot{\lambda}_s = 0. \quad (37) \]

The Hamiltonian (29) is affine with respect to the control \( P_p \). To solve (30) we rewrite (29) as
\[ H_B = p(t) + q(t) P_p(t), \quad (38) \]
where \( p \) is a function independent of \( P_p \), and \( q \) is a so-called switching function described by
\[ q = \gamma_p - \lambda_k. \quad (39) \]

To solve (30), we minimize the Hamiltonian with respect to \( P_p \), yielding
\[ q P_p^* \leq q P_p. \quad (40) \]

A particular situation occurs when \( q \) becomes zero. In that case, the Hamiltonian \( H_B \) does not depend upon \( P_p \) explicitly. Although the control arc satisfies the first order conditions, the optimal control solution cannot be found directly by minimizing \( H_B \), as it must satisfy additional higher order necessary conditions for optimality [3, 4, 5], also called singular optimal control conditions. These additional conditions require that all of the derivatives of \( q \), along the optimal trajectory, must vanish in this time interval as well, i.e., \( \dot{q} = 0, \ddot{q} = 0, \dot{\gamma} = 0 \), and so on. Setting (39) equal to zero leads to
\[ \dot{\lambda}_k = \gamma_p. \quad (41) \]

Hence, along the singular arc, the costate \( \lambda_k \) attains a constant value equal to the prime mover incremental fuel cost \( \gamma_p \). Since setting \( \dot{q} = 0 \) does not provide a condition that explicitly contains \( P_p^* \), the first time-derivative of the switching function is set equal to zero as \( \dot{q} = 0 \) as
\[ \dot{q} = -\lambda_k = 0, \quad (42) \]

which, combined with (36) and (41), leads to the condition
\[ \dot{\lambda}_s(s) = \gamma_p \left( c, mg + \frac{3\rho A c_d}{m} \dot{E}_k \right). \quad (43) \]

This way, along the singular arc, the costate associated to position attains values that are equal to the speed-derivative of the vehicle loss power multiplied with the incremental fuel cost.

The condition on the second derivative of \( q \) with respect to time becomes
\[ \ddot{q} = -\lambda_k \frac{3\rho A c_d}{m} v = 0. \quad (44) \]

It follows that the second derivative of \( q \) can only vanish when \( \dot{v} = \dot{E}_k = 0 \), which explicitly contains the control variable \( P_p \), so \( \dot{v} \) and thus \( \dot{E}_k \) are constant and the optimal singular control is given by the power required to maintain a constant velocity:
\[ \ddot{P}_p = -P_d^* + P_1(\dot{E}_k). \quad (45) \]

The optimal control of the prime mover has the following set of subarcs:
\[ P_p^* = \begin{cases} T_p, & \text{if } \lambda_k > \gamma_p, \\ \{ [P_p, T_p], & \text{if } \lambda_k = \gamma_p, \\ P^*_{ps}, & \text{if } \lambda_k < \gamma_p. \end{cases} \quad (46) \]

Next, the optimal control of the service brakes can be derived as below. Again the Hamiltonian is rewritten as
\[ H_B = p_d(t) + q_d(t) P_d(t). \quad (47) \]

Setting the first time derivative of the switching function equal to zero \( \ddot{q}_d = \lambda_k = 0 \) leads to
\[ \ddot{q}_d = -\lambda_k. \quad (48) \]

Due to the non-triviality condition, a singular solution in \([0, T_d]\) does not occur. Hence, the optimal service brake operation is governed by
\[ P_d^* = \begin{cases} 0, & \text{if } \lambda_k > 0, \\ T_d, & \text{if } \lambda_k \leq 0. \end{cases} \quad (49) \]

The necessary conditions for optimality for the unconstrained control of \( P_m \), i.e., solving (31), yield
\[ \frac{\partial H_B}{\partial P_m} = -\lambda_k + \lambda_c \frac{\partial P_s}{\partial P_m} = 0, \quad (50) \]

providing the condition
\[ \begin{aligned} &-\lambda_k + \lambda_c \frac{U_{\text{m}, \gamma_m}}{\sqrt{U_{\text{m}, \gamma_m}^2 - 4R_m^2 P_m}} = 0, & \text{if } P_m \leq 0, \\ &-\lambda_k + \lambda_c \frac{U_{\text{m}, \gamma_m}}{\sqrt{U_{\text{m}, \gamma_m}^2 - 4R_m^2 P_m}} = 0, & \text{if } P_m > 0. \end{aligned} \quad (51) \]

Note that (51) is similar to (21) with the difference that \( \gamma_p \) is replaced by \( \lambda_k \). Hence, similar to (28), an analytical expression for the optimal control in the state and costate variables can be obtained as
\[ P_m^* = \begin{cases} P_m, & \text{if } \lambda_c > \lambda_m, \\ \frac{U_{\text{m}, \gamma_m}^2}{4R_m} \left( 1 - \lambda_c^2 \frac{\lambda_m}{\lambda_k \gamma_m} \right), & \text{if } \frac{\lambda_m}{\gamma_m} < \lambda_c < \lambda_m, \\ 0, & \text{if } \lambda_c < \lambda_m, \end{cases} \quad (52) \]

where
\[ \text{if } \lambda_m < \lambda_c < \lambda_m, \quad \text{if } \lambda_c < \lambda_m. \]
Where the upper and lower bound on $\lambda_e$ at which the optimal control $P_m^*$ is clipped depends on both the state $E_s$ and costate $\lambda_k$:

$$\lambda_e(E_s, \lambda_k) = \frac{\lambda_k}{7mU_{oc}(E_s)} \sqrt{U_{oc}^2(E_s) - 4R_m^2 P_m},$$

$$\lambda_e(E_s, \lambda_k) = \frac{\lambda_k 7m}{U_{oc}(E_s)} \sqrt{U_{oc}^2(E_s) - 4R_m^2 P_m}.$$

Next, we derive the differential conditions on the costate $\lambda_e$ associated to energy storage:

$$\dot{\lambda}_e = \lambda_e \phi U_{oc} \left(2 - \lambda - \frac{1}{\lambda} \right),$$

in which the dimensionless parameter $\lambda'$ is given by

$$\lambda' = \begin{cases} \frac{\lambda e}{\lambda_{km}}, & \text{if } P_m < 0, \\ 1, & \text{if } P_m = 0, \\ \frac{\lambda e}{\lambda_{km}^2}, & \text{if } P_m > 0. \end{cases}$$

Again, the dynamics of $\lambda_e$ in (55) are closely related to $\lambda$ in (25) with the difference that $\lambda_k$ replaces $\gamma_p$.

Given (41), it follows that the optimal EM power at the singular solution described by (52) is reduced to (28). Therefore, during the singular solution, the EM power is governed by $\lambda_e$ only, and the battery is either charging, discharging or not used, whilst a combination of charging and discharging is not possible.

B. Structure of the Solution

The structure of the solution includes an acceleration phase, a singular arc phase, and a deceleration phase. During the acceleration phase the maximum prime mover power is used and also the EM can provide power to propel the vehicle governed by (52). During the deceleration phase the EM and service brakes are governed by (52). From the results derived in this section it follows that the singular arc phase has the following features:

- the velocity $\dot{v}$ and kinetic energy $\dot{E}_k$ are constant,
- the costate $\lambda_k$ equals the incremental fuel cost $\gamma_p$,
- the costate $\lambda_e$ is constant,
- the costate dynamics $\lambda_k$ and EM power $P_m$ are decoupled from the other state and costate values,
- the prime mover output power $P_p$ and EM power $P_m$ are in equilibrium with the vehicle losses,
- the switching between the singular solution and the non-singular solution is governed by the costate $\lambda_k$.

C. Numerical Solution Method

Leveraging our results and the structure of the solution, we can now devise a numerical solution method for the joint velocity and power-split eco-driving problem. The presence of a singular arc, at which the costate values are prescribed, allows to split the HBVP into two BVPs with a shorter horizon which are numerically better conditioned.

To compute a numerical solution we take the following steps:

1) make an initial guess for the singular velocity $\dot{v}$ and $\dot{\lambda}_e$, e.g., based on the average traveling velocity $\dot{v} = \frac{v(t_f)}{t_f}$ and $\gamma_p \gamma_m < \dot{\lambda}_e < \frac{2m}{3R_m}$. Note that, with this guess the boundary conditions $\lambda_k = \gamma_p$ and $\lambda_k$ are given by (43) at the start and end of the singular arc;

2) guess the duration of the acceleration $\tau_a$ and deceleration phase $\tau_d$ and solve the HBVP for the acceleration phase and deceleration phase, now defined by the state values at $t_0$ and $t_f$ and costate values at the singular arc;

3) iterate on the duration $\tau_a$ and $\tau_d$ until reaching the singular velocity;

4) iterate on $\dot{v}$ and $\dot{\lambda}_e$ until the required travel distance $s$ is reached and $E_s(t) = E_s(t_f)$ is met.

VI. SIMULATION EXAMPLE

Using the vehicle parameters shown in Table I, we compute the optimal trajectories using the solution method presented in Section V-C for a distance $s = 4$ km and travel time $t_f = 200$ s.

Fig. 5 shows that it is more beneficial to make use of any recovered energy during the acceleration phase of the velocity trajectory than during the cruising phase, i.e., to reach a higher acceleration such that the required travel distance is reached with a lower singular velocity.

VII. CONCLUSION

This article studied a method to efficiently compute the fuel-optimal velocity and power-split control strategies for a hybrid electric vehicle in a joint fashion. To this end, we first derived necessary conditions for optimality combining Pontryagin’s minimum principle with singular optimal control theory, and subsequently solved the optimal control problem as a Hamiltonian boundary value problem. A simulation study with a hybrid truck showed that it is beneficial to use any recovered energy during the acceleration phase to maintain a low cruising velocity. Future work will focus on extending the problem formulation to include road-grade and state constraints on the battery state of energy and the vehicle velocity.

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<th>Name</th>
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<td>kW</td>
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REFERENCES

Fig. 5. Optimal state and costate trajectories.