

## On a class of embedded Markov processes and recurrence

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On a class of embedded Markov processes and recurrence

by

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Eindhoven, July 1976

The Netherlands

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Abstract. By means of a general type of embedded process we shall give a short deduction of some recurrence properties of the Markov shift.

1. Preliminaries

Let  $(X, \mathcal{J}, m)$  be a  $\sigma$ -finite measure space. Let  $M^+$  be the space of (equivalence classes of almost everywhere equal) nonnegative extended real valued measurable functions on  $X$ . A Markov operator is a mapping  $P$  of  $M^+$  into itself such that  $P1 \leq 1$ , and

$$P\left(\sum_{n=1}^{\infty} \alpha_n f_n\right) = \sum_{n=1}^{\infty} \alpha_n P f_n, \quad f_n \in M^+, \alpha_n \geq 0.$$

The domain of  $P$  can be extended to  $L_{\infty}$  such that  $P$  is a positive linear contraction in  $L_{\infty}$ . Such an operator is always the adjoint of a positive linear contraction in  $L^1$ , which we shall also denote by  $P$ , but now written to the right of the function symbol. The action of this positive linear contraction  $P$  on  $L^1$  can, by means of monotone approximation, be extended to the space  $M^+$ . It follows that

$$\langle fP, g \rangle = \langle f, Pg \rangle \quad \text{for all } f, g \in M^+.$$

Here  $\langle f, g \rangle$  stands for  $\int fg \, dm$ .

With respect to a given Markov operator  $P$  on  $(X, \mathcal{J}, m)$  we can decompose the space  $X$  into a conservative part  $C$  and a dissipative part  $D$ . This decomposition is due to E. Hopf, and can e.g. be found in [1], chapter II, [5], chapter 4, § 2. For later use we collect the results which we shall need in two lemma's and some corollaries.

Lemma 1. The following statements are equivalent.

- i. The conservative part of  $X$  with respect to  $P$  is  $C$ .
- ii. The set  $C$  is the (mod  $m$ ) largest set such that for all subsets  $A$  we have

$$\sum_{n=1}^{\infty} P^n 1_A = \infty \quad \text{on } A.$$

- iii. If  $0 \leq f < \infty$  and  $Pf \leq f$  on  $C$ , then  $Pf = f$  on  $C$ .

Lemma 2. The following statements are equivalent.

- i. The dissipative part of  $X$  with respect to  $P$  is  $D$ .
- ii. The set  $D$  is the (mod  $m$ ) largest set such that there exists a function  $g \geq 0$ , with  $\{g > 0\} = D$  and  $\sum_{n=0}^{\infty} P^n g$  is bounded.  
(This equivalence can be obtained e.g. from (2,5) in [1] and the maximum principle, chapter 2, theorem 1.12 in [5]).

Corollaries.

1.  $Pl = Pl_C = 1$  on  $C$ , and therefore  $Pl_D = 0$  on  $C$ .

2. For every  $f \in M^+$  we have  $\sum_{n=1}^{\infty} P^n f = 0$  or  $\infty$  on  $C$ .

If  $C_0 = \{ \sum_{n=1}^{\infty} P^n f = 0 \} \cap C$ , and  $C_1 = \{ \sum_{n=1}^{\infty} P^n f = \infty \} \cap C$ , then  $Pl_{C_i} = 0$  on  $C_{1-i}$ ,  $i = 0, 1$ .

3. If  $A \subset C$  and  $m(A) > 0$ , then  $\langle 1_C, Pl_A \rangle > 0$ . It follows that  $\{1_C P > 0\} = C$ .

4. For every  $s \geq 1$  the conservative part of  $X$  with respect to  $P^s$  equals  $C$ .

## 2. The embedded processes

Let  $P$ ,  $H$  and  $H'$  be Markov operators, and assume  $H + H' = P^{s-1}$  for some  $s \geq 1$ . We define the operator  $Q$  by

$$Q = \sum_{n=0}^{\infty} (PH')^n PH$$

Obviously  $Q$  is a  $\sigma$ -additive mapping of  $M^+$  onto itself. The next lemma implies by substituting  $f = 1$  that  $Q$  is a Markov operator.

Lemma 3. If for  $f \in M^+$  we have  $Pf \leq f$ , then we also have  $Qf \leq f$ .

Proof. Using  $P^s f \leq f$ , we easily verify by writing out

$$\sum_{r=0}^{n-1} (PH')^r PHf + (PH')^n f \leq f.$$

Hence it follows that  $Qf \leq f$ .

If  $H$  is the multiplication by the characteristic function of a set  $A$ ,  $H'$  the multiplication by  $1_A$ , then  $Q$  is the embedded process on the set  $A$ . The situation that  $H$  is multiplication by a function  $f$ ,  $0 \leq f \leq 1$ , and  $H'$  multiplication by the function  $1 - f$ , is studied in [2], [4]. In both cases we have  $H + H' = P^0$ , hence  $s = 1$ . The situation with  $s > 1$  occurs when we are investigating recurrence properties of the Markov shift, as we shall see in the next section.

Theorem 1. For every  $f \in M^+$  we have

$$\sum_{n=1}^{\infty} Q^n f = \sum_{n=0}^{\infty} P^{ns+1} Hf.$$

Proof.  $P^{(n-1)s+1} Hf$

$$\begin{aligned} &= P(H' + H)P \dots P(H' + H)PHf \\ &= \sum_{n_1 + \dots + n_k + k = n} (PH')^{n_1} PH (PH')^{n_2} PH \dots (PH')^{n_k} PHf, \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{n=0}^{\infty} P^{ns+1} Hf \\ &= \sum_{n=1}^{\infty} \sum_{n_0 + \dots + n_k + k = n} (PH')^{n_1} PH \dots (PH')^{n_k} PHf \\ &= \sum_{k=1}^{\infty} \sum_{n_0=0}^{\infty} \dots \sum_{n_k=0}^{\infty} (PH')^{n_1} PH \dots (PH')^{n_k} PHf \\ &= \sum_{k=1}^{\infty} Q^k f. \end{aligned}$$

Theorem 2. The conservative part of  $X$  with respect to  $Q$  is  $\{1_C H > 0\} \cap C$ , where  $C$  is the conservative part of  $X$  with respect to  $P$ .

Proof.

i. Suppose  $A \subset \{l_C^H > 0\} \cap C$ .

By theorem 1 we have

$$\sum_{n=1}^{\infty} Q^n l_A = \sum_{n=0}^{\infty} P^{ns} PH l_A .$$

It follows by corollary 4 and corollary 2 that this sum is 0 or  $\infty$  on C.

Put

$$C_0 = \left\{ \sum_{n=1}^{\infty} Q^n l_A = 0 \right\} \cap C, \quad A_0 = A \cap C_0$$

$$C_1 = \left\{ \sum_{n=1}^{\infty} Q^n l_A = \infty \right\} \cap C, .$$

then  $P^s l_{C_0} = 0$  on  $C_1$ , and since  $PH l_{A_0} \leq P^s l_{C_0}$ , we obtain

$$\sum_{n=0}^{\infty} P^{ns} PH l_{A_0} \left\{ \begin{array}{l} \leq \sum_{n=0}^{\infty} P^{ns} PH l_A = 0 \text{ on } C_0 \\ \leq \sum_{n=0}^{\infty} P^{ns} P^s l_{C_0} = 0 \text{ on } C_1, \end{array} \right.$$

hence in particular  $PH l_{A_0} = 0$  on C.

This means  $\langle l_C, PH l_{A_0} \rangle = 0$ , and therefore by corollary 3

$$\langle l_C, H l_{A_0} \rangle = \langle l_C, H, l_{A_0} \rangle = 0 .$$

Since  $A_0 \subset A$ , we have  $m(A_0) = 0$ , and

$$\sum_{n=1}^{\infty} Q^n l_A = \infty \quad \text{on } A.$$

Hence by lemma 1 the set  $\{l_C^H > 0\} \cap C$  is a subset of the conservative part of X with respect to Q.

ii. If  $A = \{1_C^H = 0\} \cap C$ , then  $\langle 1_C^H 1_A \rangle = 0$ , hence  $H 1_A = 0$  on  $C$ . It follows that

$$\sum_{n=1}^{\infty} Q^n 1_A = \sum_{n=0}^{\infty} P^{ns} P H 1_A = 0 \text{ on } C,$$

hence  $A$  is a subset of the dissipative part of  $X$  with respect to  $Q$ .

iii. Let  $g$  be a function with  $\{g > 0\} = D$  and  $\sum_{n=0}^{\infty} P^n g$  is bounded. Then

$$\sum_{n=1}^{\infty} Q^n g = \sum_{n=0}^{\infty} P^{ns} P H g \leq \sum_{n=1}^{\infty} P^{ns} g \leq \sum_{n=0}^{\infty} P^n g < \infty,$$

and  $D$  is a subset of the dissipative part of  $X$  with respect to  $Q$ .

Corollary. If  $s = 1$  and  $H$  is multiplication by a function  $h$  with  $0 \leq h \leq 1$ , then  $\{1_C^H > 0\} \cap C = \{h > 0\} \cap C$  and we obtain a result of Lin [4].

### 3. Recurrence properties for the Markov shift

In this section we shall give, with the aid of the operator  $Q$  of the previous section, a fast deduction of some recurrence properties of the Markov shift. These results go back to a paper of Harris and Robbins [3].

Let  $S$  be a measurable transformation on a measure space  $(\Omega, F, M)$ . A set  $A$  is said to be *wandering* under  $S$  if no points of  $A$  return to  $A$  under the action of  $S$ , and *recurrent* if  $M$ -almost all points of  $A$  return to  $A$  under the action of  $S$ . A set is said to be *dissipative* if it is (mod  $M$ ) the union of countably many wandering sets, and *conservative* if every subset is recurrent.

Obviously, if  $A_n$  is the subset of  $A$  of the points which return exactly  $n$  times under  $S$  to  $A$  then  $A_n$  is wandering. Hence, if almost all points of  $A$  return finitely many times to  $A$ , then  $A$  is dissipative.

Now let  $(\Omega, F, M)$  be the realization space of Markov process  $P$  on  $(X, \mathcal{J})$  with initial probability  $m$ , i.e.  $(\Omega, F) = \Pi_{n=0}^{\infty} (X, \mathcal{J})$ , and

$$M(X_0 \in A_0, \dots, X_n \in A_n) = \langle 1_{A_0} P 1_{A_1} \dots P 1_{A_n} \rangle$$

for all  $A_0, \dots, A_n \in \Sigma$ , where  $X_i$  denotes projection on the  $i$ -th coordinate. Let  $F_n$  denote the  $\sigma$ -algebra generated by  $X_0^{-1}\Sigma, \dots, X_n^{-1}\Sigma$ , and let  $S$  be the shift transformation in  $(\Omega, F)$ .

Suppose that the initial measure is such that  $P$  can also be considered as a Markov operator on  $M^+(X, \Sigma, m)$ . (This is the case if and only if  $m(A) = 0 \Rightarrow P(\cdot, A) = 0$  m-a.e.) Let  $C$  be the conservative part of  $X$  with respect to  $P$ , and define  $C^\infty = \{X_n \in C \text{ for all } n\}$ .

Theorem 3.

- i) The set  $\Omega \setminus C^\infty$  is dissipative.
- ii) For every  $n$  and every  $A \in F_n$ , the set  $A \cap C^\infty$  is recurrent.

Proof. Because of  $P1_D = 0$  on  $C$ , we have

$$\Omega \setminus C^\infty = \{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\},$$

where  $D_1, D_2, \dots$  is a partition of  $D$  such that  $\sum_{n=0}^{\infty} P^n 1_{D_i}$  is bounded. The existence of such a partition easily follows from lemma 2. We then have

$$\sum_{n=0}^{\infty} M(X_n \in D_i) = \sum_{n=0}^{\infty} \langle 1_{D_i} P^n \rangle < \infty,$$

hence by the Borel-Cantelli lemma we obtain  $M\{X_n \in D_i \text{ i.o.}\} = 0$ . Almost all points of  $\{X_0 \in D_i\}$  return to this set under  $S$  only finitely many times, so  $\{X_0 \in D_i\}$  is dissipative, and therefore  $\{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\}$  is also dissipative.

ii) Without loss of generality we may assume that  $X = C$ , and therefore  $C^\infty = \Omega$ . Choose  $A \in F_{s-1}$ , and define for every  $f \in M^+$

$$Hf = E(1_A \cdot f(X_{s-1}) \mid F_0)$$

$$H'f = E(1_{\Omega \setminus A} \cdot f(X_{s-1}) \mid F_0),$$

where we consider the  $F_0$ -measurable functions on  $\Omega$  in the right-hand side as functions on  $X$ .

The operators  $H$  and  $H'$  are Markov operators on  $(X, \Sigma, m)$  and satisfy  $(H + H')f = E(f(X_{s-1}) \mid F_0) = P^{s-1}f$ .



Let  $A_0$  be the set of points of  $A$  which return to  $A$  under  $S^s$  at least once. Then

$$A_0 = \bigcup_{n=1}^{\infty} \{ \omega \in A \mid S^{is} \omega \in \Omega \setminus A \text{ for } 1 \leq i \leq n, S^{ns} \omega \in A \}$$

$$M(A_0) = \sum_{n=1}^{\infty} \langle 1H(PH')^{n-1}PH1 \rangle = \langle 1HQ1 \rangle .$$

Since by theorem 2  $Q$  is conservative on  $\{1H > 0\}$ , we have  $Q1 = 1$  on  $\{1H > 0\}$ , and therefore  $M(A_0) = \langle 1H1 \rangle = M(A) < \infty$ , hence  $A_0 = A \pmod{M}$ . Hence the set  $A$  is recurrent under  $S^s$ , and therefore under  $S$ .

Remark 1. Let  $P$  be conservative. If  $A \in F_s$ , then  $A \in F_t$  for every  $t \geq s$ , and we have actually shown that  $A$  is recurrent under  $S^t$  for every  $t$ . Hence almost all points of  $A$  return to  $A$  infinitely many times under  $S$ .

Remark 2. The crucial point in the paper of Harris and Robbins is that if there exists an algebra of recurrent sets generating  $F$  and a finite or  $\sigma$ -finite equivalent invariant measure, then  $\Omega$  must be conservative. Therefore, if there exists a function  $u$  with  $0 < u < \infty$  on  $C$ ,  $u = 0$  on  $D$  and  $uP = u$ , then the measure  $M'$  defined by  $\frac{dM'}{dM} = u(X_0)$  is ( $\sigma$ -)finite and invariant under  $S$ , and the set  $C^\infty$  is conservative.

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