On a class of embedded Markov processes and recurrence

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Abstract. By means of a general type of embedded process we shall give a short deduction of some recurrence properties of the Markov shift.

1. Preliminaries

Let (X,Σ,m) be a σ-finite measure space. Let M^+ be the space of (equivalence classes of almost everywhere equal) nonnegative extended real valued measurable functions on X. A Markov operator is a mapping P of M^+ into itself such that $P1 \le 1$, and

$$P(\sum_{n=1}^{\infty} \alpha_n f_n) = \sum_{n=1}^{\infty} \alpha_n Pf_n \,, \quad f_n \in M^+, \ \alpha_n \ge 0.$$

The domain of P can be extended to L_∞ such that P is a positive linear contraction in L_∞ Such an operator is always the adjoint of a positive linear contraction in L^1, which we shall also denote by P, but now written to the right of the function symbol. The action of this positive linear contraction P on L^1 can, by means of monotone approximation, be extended to the space M^+. It follows that

$$\langle fP,g\rangle = \langle f,Pg\rangle \quad \text{for all } f,\ g \in M^+ .$$

Here $\langle f,g\rangle$ stands for $\int fg\, dm$.

With respect to a given Markov operator P on (X,Σ,m) we can decompose the space X into a conservative part C and a dissipative part D. This decomposition is due to E. Hopf, and can e.g. be found in [1], chapter II, [5], chapter 4, § 2. For later use we collect the results which we shall need in two lemma's and some corollaries.

Lemma 1. The following statements are equivalent.

i. The conservative part of X with respect to P is C.

ii. The set C is the (mod m) largest set such that for all subsets A we have

$$\sum_{n=1}^{\infty} P^n 1_A = \infty \qquad \text{on } A.$$

iii. If $0 \le f < \infty$ and $Pf \le f$ on C, then $Pf = f$ on C.

Lemma 2. The following statements are equivalent.

i. The dissipative part of X with respect to P is D.

ii. The set D is the (mod m) largest set such that there exists a function $g \geq 0$, with $\{g > 0\} = D$ and $\sum_{n=0}^{\infty} P^n g$ is bounded.
(This equivalence can be obtained e.g. from (2,5) in [1] and the maximum principle, chapter 2, theorem 1.12 in [5]).

Corollaries.

1. $P1 = P1_C = 1$ on C, and therefore $P1_D = 0$ on C.

2. For every $f \in M^+$ we have $\sum_{n=1}^{\infty} P^n f = 0$ or ∞ on C.
If $C_0 = \{\sum_{n=1}^{\infty} P^n f = 0\} \cap C$, and $C_1 = \{\sum_{n=1}^{\infty} P^n f = \infty\} \cap C$, then $P1_{C_i} = 0$ on C_{1-i}, $i = 0,1$.

3. If $A \subset C$ and $m(A) > 0$, then $< 1_C, P1_A > \, > 0$. It follows that $\{1_C P > 0\} = C$.

4. For every $s \geq 1$ the conservative part of X with respect to P^s equals C.

2. The embedded processes

Let P, H and H' be Markov operators, and assume $H + H' = P^{s-1}$ for some $s \geq 1$. We define the operator Q by

$$Q = \sum_{n=0}^{\infty} (PH')^n PH$$

Obviously Q is a σ-additive mapping of M^+ onto itself. The next lemma implies by substituting $f = 1$ that Q is a Markov operator.

Lemma 3. If for $f \in M^+$ we have $Pf \leq f$, then we also have $Qf \leq f$.

Proof. Using $P^s f \leq f$, we easily verify by writing out

$$\sum_{r=0}^{n-1} (PH')^r PHf + (PH')^n f \leq f.$$

Hence it follows that $Qf \leq f$.

If H is the multiplication by the characteristic function of a set A, H' the multiplication by $1_{A'}$, then Q is the embedded process on the set A. The situation that H is multiplication by a function f, $0 \le f \le 1$, and H' multiplication by the function $1 - f$, is studied in [2], [4].
In both cases we have $H + H' = P^0$, hence $s = 1$. The situation with $s > 1$ occurs when we are investigating recurrence properties of the Markov shift, as we shall see in the next section.

Theorem 1. For every $f \in M^+$ we have

$$\sum_{n=1}^{\infty} Q^n f = \sum_{n=0}^{\infty} P^{ns+1} Hf.$$

Proof. $P^{(n-1)s+1}Hf$

$$= P(H' + H)P \ldots P(H' + H)PHf$$

$$= \sum_{n_1+\ldots+n_k+k=n} (PH')^{n_1} PH (PH')^{n_2} PH \ldots (PH')^{n_k} PHf ,$$

and therefore

$$\sum_{n=0}^{\infty} P^{ns+1} Hf$$

$$= \sum_{n=1}^{\infty} \sum_{n_0+\ldots+n_k+k=n} (PH')^{n_1} PH \ldots (PH')^{n_k} PHf$$

$$= \sum_{k=1}^{\infty} \sum_{n_0=0}^{\infty} \ldots \sum_{n_k=0}^{\infty} (PH')^{n_1} PH \ldots (PH')^{n_k} PHf$$

$$= \sum_{k=1}^{\infty} Q^k f .$$

Theorem 2. The conservative part of X with respect to Q is $\{1_C H > 0\} \cap C$, where C is the conservative part of X with respect to P.

Proof.

i. Suppose $A \subset \{1_c H > 0\} \cap C$.

By theorem 1 we have

$$\sum_{n=1}^{\infty} Q^n 1_A = \sum_{n=0}^{\infty} P^{ns} PH1_A .$$

It follows by corollary 4 and corollary 2 that this sum is 0 or ∞ on C.

Put

$$C_0 = \{\sum_{n=1}^{\infty} Q^n 1_A = 0\} \cap C, \quad A_0 = A \cap C_0$$

$$C_1 = \{\sum_{n=1}^{\infty} Q^n 1_A = \infty\} \cap C, \, .$$

then $P^s 1_{C_0} = 0$ on C_1, and since $PH1_{A_0} \le P^s 1_{C_0}$, we obtain

$$\sum_{n=0}^{\infty} P^{ns} PH1_{A_0} \begin{cases} \le \sum_{n=0}^{\infty} P^{ns} PH1_A = 0 \text{ on } C_0 \\ \\ \le \sum_{n=0}^{\infty} P^{ns} P^s 1_{C_0} = 0 \text{ on } C_1 \, , \end{cases}$$

hence in particular $PH1_{A_0} = 0$ on C.

This means $<1_C PH1_{A_0}> = 0$, and therefore by corollary 3

$$<1_C, H1_{A_0}> = <1_C H, 1_{A_0}> = 0 \, .$$

Since $A_0 \subset A$, we have $m(A_0) = 0$, and

$$\sum_{n=1}^{\infty} Q^n 1_A = \infty \qquad \text{on } A.$$

Hence by lemma 1 the set $\{1_C H > 0\} \cap C$ is a subset of the conservative part of X with respect to Q.

ii. If $A = \{1_C H = 0\} \cap C$, then $\langle 1_C H 1_A \rangle = 0$, hence $H 1_A = 0$ on C. It follows that

$$\sum_{n=1}^{\infty} Q^n 1_A = \sum_{n=0}^{\infty} P^{ns} P H 1_A = 0 \text{ on } C ,$$

hence A is a subset of the dissipative part of X with respect to Q.

iii. Let g be a function with $\{g > 0\} = D$ and $\sum_{n=0}^{\infty} P^n g$ is bounded. Then

$$\sum_{n=1}^{\infty} Q^n g = \sum_{n=0}^{\infty} P^{ns} P H g \leq \sum_{n=1}^{\infty} P^{ns} g \leq \sum_{n=0}^{\infty} P^n g < \infty ,$$

and D is a subset of the dissipative part of X with respect to Q.

<u>Corollary</u>. If $s = 1$ and H is multiplication by a function h with $0 \leq h \leq 1$, then $\{1_C H > 0\} \cap C = \{h > 0\} \cap C$ and we obtain a result of *Lin* [4].

3. <u>Recurrence properties for the Markov shift</u>

In this section we shall give, with the aid of the operator Q of the previous section, a fast deduction of some recurrence properties of the Markov shift. These results go back to a paper of Harris and Robbins [3].

Let S be a measurable transformation on a measure space (Ω, F, M). A set A is said to be *wandering* under S if no points of A return to A under the action of S, and *recurrent* if M-almost all points of A return to A under the action of S. A set is said to be *dissipative* if it is (mod M) the union of countably many wandering sets, and *conservative* if every subset is recurrent.

Obviously, if A_n is the subset of A of the points which return exactly n times under S to A then A_n is wandering. Hence, if almost all points of A return finitely many times to A, then A is dissipative.

Now let (Ω, F, M) be the realization space of Markov proces P on $(X, \sum)$ with initial probability m, i.e. $(\Omega, F) = \Pi_{n=0}^{\infty} (X, \sum)$, and

$$M(X_0 \in A_0, \ldots, X_n \in A_n) = \langle 1 I_{A_0} P I_{A_1} \ldots P I_{A_n} 1 \rangle$$

for all $A_0,\dots,A_n \in \Sigma$, where X_i denotes projection on the i-th coordinate. Let F_n denote the σ-algebra generated by $X_0^{-1}\Sigma$,..., $X_n^{-1}\Sigma$, and let S be the shift transformation in (Ω,F).
Suppose that the initial measure is such that P can also be considered as a Markov operator on $M^+(X,\Sigma,m)$. (This is the case if and only if $m(A) = 0 \Rightarrow$ $\Rightarrow P(\cdot,A) = 0$ m-a.e.) Let C be the conservative part of X with respect to P, and define $C^\infty = \{X_n \in C \text{ for all } n\}$.

Theorem 3.

i) The set $\Omega\backslash C^\infty$ is dissipative.

ii) For every n and every $A \in F_n$, the set $A \cap C^\infty$ is recurrent.

Proof. Because of $P1_D = 0$ on C, we have

$$\Omega\backslash C^\infty = \{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\} ,$$

where $D_1,D_2,\dots$ is a partition of D such that $\sum_{n=0}^{\infty} P^n 1_{D_i}$ is bounded. The existence of such a partition easily follows from lemma 2. We then have

$$\sum_{n=0}^{\infty} M(X_n \in D_i) = \sum_{n=0}^{\infty} \langle 1 P^n 1_{D_i} \rangle < \infty ,$$

hence by the Borel-Cantelli lemma we obtain $M\{X_n \in D_i \text{ i.o.}\} = 0$. Almost all points of $\{X_0 \in D_i\}$ return to this set under S only finitely many times, so $\{X_0 \in D_i\}$ is dissipative, and therefore $\{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\}$ is also disdipative.

ii) Without loss of generality we may assume that X = C, and therefore $C^\infty = \Omega$. Choose $A \in F_{s-1}$, and define for every $f \in M^+$

$$Hf = E(1_A . f(X_{s-1}) \mid F_0)$$

$$H'f = E(1_{\Omega\backslash A} . f(X_{s-1}) \mid F_0) ,$$

where we consider the F_0-measurable functions on Ω in the right-hand side as functions on X.
The operators H and H' are Markov operators on (X,Σ,m) and satisfy $(H + H')f = E(f(X_{s-1}) \mid F_0) = P^{s-1}f$.

Let A_0 be the set of points of A which return to A under S^s at least once. Then

$$A_0 = \bigcup_{n=1}^{\infty} \{\omega \in A \mid S^{is}\omega \in \Omega\backslash A \text{ for } 1 \leq i \leq n,\ S^{ns}\omega \in A\}$$

$$M(A_0) = \sum_{n=1}^{\infty} \langle 1H(PH')^{n-1}PH1\rangle = \langle 1HQ1\rangle \ .$$

Since by theorem 2 Q is conservative on $\{1H > 0\}$, we have $Q1 = 1$ on $\{1H > 0\}$, and therefore $M(A_0) = \langle 1H1\rangle = M(A) < \infty$, hence $A_0 = A$ (mod M). Hence the set A is recurrent under S^s, and therefore under S.

Remark 1. Let P be conservative. If $A \in F_s$, then $A \in F_t$ for every $t \geq s$, and we have actually shown that A is recurrent under S^t for every t. Hence almost all points of A return to A infinitely many times under S.

Remark 2. The crucial point in the paper of Harris and Robbins is that if there exists an algebra of recurrent sets generating F and a finite or σ-finite equivalent invariant measure, then Ω must be conservative. Therefore, if there exists a function u with $0 < u < \infty$ on C, $u = 0$ on D and $uP = u$, then the measure M' defined by $\frac{dM'}{dM} = u(X_0)$ is (σ-)finite and invariant under S, and the set C^∞ is conservative.

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