

On a class of embedded Markov processes and recurrence

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On a class of embedded Markov processes and recurrence

by

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Eindhoven, July 1976

The Netherlands

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Abstract. By means of a general type of embedded process we shall give a short deduction of some recurrence properties of the Markov shift.

1. Preliminaries

Let (X, \mathcal{J}, m) be a σ -finite measure space. Let M^+ be the space of (equivalence classes of almost everywhere equal) nonnegative extended real valued measurable functions on X . A Markov operator is a mapping P of M^+ into itself such that $P1 \leq 1$, and

$$P\left(\sum_{n=1}^{\infty} \alpha_n f_n\right) = \sum_{n=1}^{\infty} \alpha_n P f_n, \quad f_n \in M^+, \alpha_n \geq 0.$$

The domain of P can be extended to L_{∞} such that P is a positive linear contraction in L_{∞} . Such an operator is always the adjoint of a positive linear contraction in L^1 , which we shall also denote by P , but now written to the right of the function symbol. The action of this positive linear contraction P on L^1 can, by means of monotone approximation, be extended to the space M^+ . It follows that

$$\langle P f, g \rangle = \langle f, P g \rangle \quad \text{for all } f, g \in M^+.$$

Here $\langle f, g \rangle$ stands for $\int fg \, dm$.

With respect to a given Markov operator P on (X, \mathcal{J}, m) we can decompose the space X into a conservative part C and a dissipative part D . This decomposition is due to E. Hopf, and can e.g. be found in [1], chapter II, [5], chapter 4, § 2. For later use we collect the results which we shall need in two lemma's and some corollaries.

Lemma 1. The following statements are equivalent.

- i. The conservative part of X with respect to P is C .
- ii. The set C is the (mod m) largest set such that for all subsets A we have

$$\sum_{n=1}^{\infty} P^n 1_A = \infty \quad \text{on } A.$$

- iii. If $0 \leq f < \infty$ and $P f \leq f$ on C , then $P f = f$ on C .

Lemma 2. The following statements are equivalent.

- i. The dissipative part of X with respect to P is D .
- ii. The set D is the (mod m) largest set such that there exists a function $g \geq 0$, with $\{g > 0\} = D$ and $\sum_{n=0}^{\infty} P^n g$ is bounded.
(This equivalence can be obtained e.g. from (2,5) in [1] and the maximum principle, chapter 2, theorem 1.12 in [5]).

Corollaries.

1. $P1 = P1_C = 1$ on C , and therefore $P1_D = 0$ on C .

2. For every $f \in M^+$ we have $\sum_{n=1}^{\infty} P^n f = 0$ or ∞ on C .

If $C_0 = \{ \sum_{n=1}^{\infty} P^n f = 0 \} \cap C$, and $C_1 = \{ \sum_{n=1}^{\infty} P^n f = \infty \} \cap C$, then $P1_{C_i} = 0$ on C_{1-i} , $i = 0, 1$.

3. If $A \subset C$ and $m(A) > 0$, then $\langle 1_C, P1_A \rangle > 0$. It follows that $\{1_C P > 0\} = C$.

4. For every $s \geq 1$ the conservative part of X with respect to P^s equals C .

2. The embedded processes

Let P , H and H' be Markov operators, and assume $H + H' = P^{s-1}$ for some $s \geq 1$. We define the operator Q by

$$Q = \sum_{n=0}^{\infty} (PH')^n PH$$

Obviously Q is a σ -additive mapping of M^+ onto itself. The next lemma implies by substituting $f = 1$ that Q is a Markov operator.

Lemma 3. If for $f \in M^+$ we have $Pf \leq f$, then we also have $Qf \leq f$.

Proof. Using $P^s f \leq f$, we easily verify by writing out

$$\sum_{r=0}^{n-1} (PH')^r PHf + (PH')^n f \leq f.$$

Hence it follows that $Qf \leq f$.

If H is the multiplication by the characteristic function of a set A , H' the multiplication by 1_A , then Q is the embedded process on the set A . The situation that H is multiplication by a function f , $0 \leq f \leq 1$, and H' multiplication by the function $1 - f$, is studied in [2], [4]. In both cases we have $H + H' = P^0$, hence $s = 1$. The situation with $s > 1$ occurs when we are investigating recurrence properties of the Markov shift, as we shall see in the next section.

Theorem 1. For every $f \in M^+$ we have

$$\sum_{n=1}^{\infty} Q^n f = \sum_{n=0}^{\infty} P^{ns+1} Hf.$$

Proof. $P^{(n-1)s+1} Hf$

$$\begin{aligned} &= P(H' + H)P \dots P(H' + H)PHf \\ &= \sum_{n_1 + \dots + n_k + k = n} (PH')^{n_1} PH (PH')^{n_2} PH \dots (PH')^{n_k} PHf, \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{n=0}^{\infty} P^{ns+1} Hf \\ &= \sum_{n=1}^{\infty} \sum_{n_0 + \dots + n_k + k = n} (PH')^{n_1} PH \dots (PH')^{n_k} PHf \\ &= \sum_{k=1}^{\infty} \sum_{n_0=0}^{\infty} \dots \sum_{n_k=0}^{\infty} (PH')^{n_1} PH \dots (PH')^{n_k} PHf \\ &= \sum_{k=1}^{\infty} Q^k f. \end{aligned}$$

Theorem 2. The conservative part of X with respect to Q is $\{1_C H > 0\} \cap C$, where C is the conservative part of X with respect to P .

Proof.

i. Suppose $A \subset \{l_C^H > 0\} \cap C$.

By theorem 1 we have

$$\sum_{n=1}^{\infty} Q^n l_A = \sum_{n=0}^{\infty} P^{ns} PHl_A .$$

It follows by corollary 4 and corollary 2 that this sum is 0 or ∞ on C.

Put

$$C_0 = \left\{ \sum_{n=1}^{\infty} Q^n l_A = 0 \right\} \cap C, \quad A_0 = A \cap C_0$$

$$C_1 = \left\{ \sum_{n=1}^{\infty} Q^n l_A = \infty \right\} \cap C, .$$

then $P^s l_{C_0} = 0$ on C_1 , and since $PHl_{A_0} \leq P^s l_{C_0}$, we obtain

$$\sum_{n=0}^{\infty} P^{ns} PHl_{A_0} \left\{ \begin{array}{l} \leq \sum_{n=0}^{\infty} P^{ns} PHl_A = 0 \text{ on } C_0 \\ \leq \sum_{n=0}^{\infty} P^{ns} P^s l_{C_0} = 0 \text{ on } C_1, \end{array} \right.$$

hence in particular $PHl_{A_0} = 0$ on C.

This means $\langle l_C PHl_{A_0} \rangle = 0$, and therefore by corollary 3

$$\langle l_C, Hl_{A_0} \rangle = \langle l_C^H, l_{A_0} \rangle = 0 .$$

Since $A_0 \subset A$, we have $m(A_0) = 0$, and

$$\sum_{n=1}^{\infty} Q^n l_A = \infty \quad \text{on } A.$$

Hence by lemma 1 the set $\{l_C^H > 0\} \cap C$ is a subset of the conservative part of X with respect to Q.

ii. If $A = \{1_C^H = 0\} \cap C$, then $\langle 1_C^H 1_A \rangle = 0$, hence $H 1_A = 0$ on C . It follows that

$$\sum_{n=1}^{\infty} Q^n 1_A = \sum_{n=0}^{\infty} P^{ns} P H 1_A = 0 \text{ on } C,$$

hence A is a subset of the dissipative part of X with respect to Q .

iii. Let g be a function with $\{g > 0\} = D$ and $\sum_{n=0}^{\infty} P^n g$ is bounded. Then

$$\sum_{n=1}^{\infty} Q^n g = \sum_{n=0}^{\infty} P^{ns} P H g \leq \sum_{n=1}^{\infty} P^{ns} g \leq \sum_{n=0}^{\infty} P^n g < \infty,$$

and D is a subset of the dissipative part of X with respect to Q .

Corollary. If $s = 1$ and H is multiplication by a function h with $0 \leq h \leq 1$, then $\{1_C^H > 0\} \cap C = \{h > 0\} \cap C$ and we obtain a result of Lin [4].

3. Recurrence properties for the Markov shift

In this section we shall give, with the aid of the operator Q of the previous section, a fast deduction of some recurrence properties of the Markov shift. These results go back to a paper of Harris and Robbins [3].

Let S be a measurable transformation on a measure space (Ω, F, M) . A set A is said to be *wandering* under S if no points of A return to A under the action of S , and *recurrent* if M -almost all points of A return to A under the action of S . A set is said to be *dissipative* if it is (mod M) the union of countably many wandering sets, and *conservative* if every subset is recurrent.

Obviously, if A_n is the subset of A of the points which return exactly n times under S to A then A_n is wandering. Hence, if almost all points of A return finitely many times to A , then A is dissipative.

Now let (Ω, F, M) be the realization space of Markov process P on (X, \mathcal{J}) with initial probability m , i.e. $(\Omega, F) = \Pi_{n=0}^{\infty} (X, \mathcal{J})$, and

$$M(X_0 \in A_0, \dots, X_n \in A_n) = \langle 1_{A_0} P 1_{A_1} \dots P 1_{A_n} \rangle$$

for all $A_0, \dots, A_n \in \Sigma$, where X_i denotes projection on the i -th coordinate. Let F_n denote the σ -algebra generated by $X_0^{-1}\Sigma, \dots, X_n^{-1}\Sigma$, and let S be the shift transformation in (Ω, F) .

Suppose that the initial measure is such that P can also be considered as a Markov operator on $M^+(X, \Sigma, m)$. (This is the case if and only if $m(A) = 0 \Rightarrow P(\cdot, A) = 0$ m-a.e.) Let C be the conservative part of X with respect to P , and define $C^\infty = \{X_n \in C \text{ for all } n\}$.

Theorem 3.

- i) The set $\Omega \setminus C^\infty$ is dissipative.
- ii) For every n and every $A \in F_n$, the set $A \cap C^\infty$ is recurrent.

Proof. Because of $P1_D = 0$ on C , we have

$$\Omega \setminus C^\infty = \{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\},$$

where D_1, D_2, \dots is a partition of D such that $\sum_{n=0}^{\infty} P^n 1_{D_i}$ is bounded. The existence of such a partition easily follows from lemma 2. We then have

$$\sum_{n=0}^{\infty} M(X_n \in D_i) = \sum_{n=0}^{\infty} \langle 1_{D_i} P^n \rangle < \infty,$$

hence by the Borel-Cantelli lemma we obtain $M\{X_n \in D_i \text{ i.o.}\} = 0$. Almost all points of $\{X_0 \in D_i\}$ return to this set under S only finitely many times, so $\{X_0 \in D_i\}$ is dissipative, and therefore $\{X_0 \in D\} = \bigcup_{i=1}^{\infty} \{X_0 \in D_i\}$ is also dissipative.

ii) Without loss of generality we may assume that $X = C$, and therefore $C^\infty = \Omega$. Choose $A \in F_{s-1}$, and define for every $f \in M^+$

$$Hf = E(1_A \cdot f(X_{s-1}) \mid F_0)$$

$$H'f = E(1_{\Omega \setminus A} \cdot f(X_{s-1}) \mid F_0),$$

where we consider the F_0 -measurable functions on Ω in the right-hand side as functions on X .

The operators H and H' are Markov operators on (X, Σ, m) and satisfy $(H + H')f = E(f(X_{s-1}) \mid F_0) = P^{s-1}f$.

Let A_0 be the set of points of A which return to A under S^s at least once. Then

$$A_0 = \bigcup_{n=1}^{\infty} \{ \omega \in A \mid S^{is} \omega \in \Omega \setminus A \text{ for } 1 \leq i \leq n, S^{ns} \omega \in A \}$$

$$M(A_0) = \sum_{n=1}^{\infty} \langle 1H(PH')^{n-1}PH1 \rangle = \langle 1HQ1 \rangle .$$

Since by theorem 2 Q is conservative on $\{1H > 0\}$, we have $Q1 = 1$ on $\{1H > 0\}$, and therefore $M(A_0) = \langle 1H1 \rangle = M(A) < \infty$, hence $A_0 = A \pmod{M}$. Hence the set A is recurrent under S^s , and therefore under S .

Remark 1. Let P be conservative. If $A \in F_s$, then $A \in F_t$ for every $t \geq s$, and we have actually shown that A is recurrent under S^t for every t . Hence almost all points of A return to A infinitely many times under S .

Remark 2. The crucial point in the paper of Harris and Robbins is that if there exists an algebra of recurrent sets generating F and a finite or σ -finite equivalent invariant measure, then Ω must be conservative. Therefore, if there exists a function u with $0 < u < \infty$ on C , $u = 0$ on D and $uP = u$, then the measure M' defined by $\frac{dM'}{dM} = u(X_0)$ is (σ -)finite and invariant under S , and the set C^∞ is conservative.

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