EINDHOVEN UNIVERSITY OF TECHNOLOGY

Analyticity spaces of self-adjoint operators subjected to perturbations with applications to Hankel invariant distribution spaces

Citation for published version (APA):
Eijndhoven, van, S. J. L., & Graaf, de, J. (1986). Analyticity spaces of self-adjoint operators subjected to perturbations with applications to Hankel invariant distribution spaces. *SIAM Journal on Mathematical Analysis*, *17*(2), 485-494. https://doi.org/10.1137/0517038

DOI:
10.1137/0517038

Document status and date:
Published: 01/01/1986

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Download date: 19. May. 2025

SIAM J. MATH. ANAL.
Vol. 17, No. 2, March 1986

© 1986 Society for Industrial and Applied Mathematics
016

ANALYTICITY SPACES OF SELF-ADJOINT OPERATORS SUBJECTED TO PERTURBATIONS WITH APPLICATIONS TO HANKEL INVARIANT DISTRIBUTION SPACES*

S. J. L. VAN EIJNDHOVEN† AND J. DE GRAAF†

Abstract. A new theory of generalized functions has been developed by one of the authors (de Graaf). In this theory the analyticity domain of each positive self-adjoint unbounded operator $\mathscr{A}$ in a Hilbert space X is regarded as a test space denoted by $\mathscr{S}_{X,\mathscr{A}}$. In the first part of this paper, we consider perturbations $\mathscr{P}$ on $\mathscr{A}$ for which there exists a Hilbert space Y such that $\mathscr{A}+\mathscr{P}$ is a positive self-adjoint operator in Y. In particular, we investigate for which perturbations $\mathscr{P}$ and for which $\nu>0$, $\mathscr{S}_{X,\mathscr{A}^\nu}\subset\mathscr{S}_{Y,(\mathscr{A}+\mathscr{P})^\nu}$. The second part is devoted to applications. We construct Hankel invariant distribution spaces. The corresponding test spaces are described in terms of the $\mathscr{S}_\alpha^\beta$-spaces introduced by Gel'fand and Shilov. It turns out that the modified Laguerre polynomials establish an uncountable number of bases for the space of even entire functions in $\mathscr{S}_\mu^\mu$ ($\frac{1}{2}\leqq\mu\leqq1$). For an even entire function φ we give necessary and sufficient conditions on the coefficients in the Fourier expansion with respect to each basis such that $\varphi\in\mathscr{S}_\mu^\mu$.

AMS(MOS)subject classifications. Primary 46F12, 46F05, 33A65

Introduction. Let X be a separable infinitely dimensional Hilbert space and let $\mathscr{L}$ be a linear operator in X. Then $\mathfrak{D}^\omega(\mathscr{L})$, the analyticity domain of $\mathscr{L}$, consists of all vectors $v\in\bigcap_{n=1}^\infty\mathfrak{D}(\mathscr{L}^n)$ satisfying

$$\exists_{a>0}\exists_{b>0}\forall_{n\in\mathbf{N}}:\|\mathscr{L}^n v\|\leqq n!a^n b.$$

For a positive self-adjoint operator $\mathscr{A}$ in X, Nelson [13] proved that $\mathscr{D}^\omega(\mathscr{A})$ can also be described as

$$\mathfrak{D}^\omega(\mathscr{A})=\bigcup_{t>0}e^{-t\mathscr{A}}(X)=\{e^{-t\mathscr{A}}w\mid w\in X,t>0\}.$$

Instead of $\mathfrak{D}^\omega(\mathscr{A})$ we use the notation $\mathscr{S}_{X,\mathscr{A}}$ introduced by de Graaf. The spaces of type $\mathscr{S}_{X,\mathscr{A}}$ are called analyticity spaces. They are nonstrict inductive limits of Hilbert spaces. Together with their strong duals $\mathscr{T}_{X,\mathscr{A}}$ they establish the functional analytic description of the distribution theory in [7].

For each positive constant ν the operator $\mathscr{A}^\nu$ is well defined, positive and self-adjoint in X. So it makes sense to write $\mathscr{S}_{X,\mathscr{A}^\nu}$. The question arises for which perturbations $\mathscr{P}$ on $\mathscr{A}$ there can be found a Hilbert space Y such that $\mathscr{A}+\mathscr{P}$ is a positive self-adjoint operator in Y and $\mathscr{S}_{X,\mathscr{A}^\nu}\subset\mathscr{S}_{Y,(\mathscr{A}+\mathscr{P})^\nu}$. In the paper [1] the case $\nu=1$ has been considered. Also some results concerning analytic dominancy can be found there.

In the second part of this paper we study a class of Hankel invariant test and distribution spaces, and also their relations to the $\mathscr{S}_\alpha^\beta$-spaces of Gel'fand and Shilov [9]. With our papers [2] and [4] we have started this study. There we have shown that the space of even functions in $\mathscr{S}_{1/2}^{1/2}$ remains invariant under the modified Hankel transforms $\mathbf{H}_\alpha$, $\alpha>-1$, defined by

$$(\mathbf{H}_\alpha f)(x)=\int_0^\infty(xy)^{-\alpha}J_\alpha(xy)f(y)y^{2\alpha+1}dy.$$

Moreover, for each $\alpha>-1$ the space of even functions in $\mathscr{S}_{1/2}^{1/2}$ equals the analyticity space $\mathscr{S}_{X_\alpha,\mathscr{A}_\alpha}$ where $X_\alpha=\mathfrak{L}_2((0,\infty),x^{2\alpha+1}dx)$ and $\mathscr{A}_\alpha=-d^2/dx^2+x^2-(2\alpha+1)xd/dx$.

*Received by the editors December 6, 1983.

† Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, the Netherlands.

The operator $\mathscr{A}_\alpha$ has an orthonormal basis of eigenvectors $(\mathbf{L}_n^{(\alpha)})_{n=0}^\infty$ with eigenvalues $4n+2\alpha+2$. So for each even $f\in\mathscr{S}_{1/2}^{1/2}$ there exists an l_2-sequence $(\omega_n)_{n=0}^\infty$ and $t>0$ such that $f=\sum_{n=0}^\infty \exp(-(4n+2\alpha+2)t)\omega_n \mathbf{L}_n^{(\alpha)}$. Here we prove similar results for the spaces $\mathscr{S}_{X_\alpha,(\mathscr{A}_\alpha)^\nu}$ with $\nu\geqq\frac{1}{2}$ and $\alpha>-1$. It will follow that for all $\alpha,\beta>-1$ and all $\nu\geqq\frac{1}{2}$

$$\mathscr{S}_{X_\alpha,(\mathscr{A}_\alpha)^\nu}=\mathscr{S}_{X_\beta,(\mathscr{A}_\beta)^\nu}.$$

For $\nu\in[\frac{1}{2},1]$ the analyticity space $\mathscr{S}_{X_{-1/2},(\mathscr{A}_{-1/2})^\nu}$ contains just the even functions in $\mathscr{S}_{1/2\nu}^{1/2\nu}$.

1. General theory. Let $\mathscr{A}$ be a positive self-adjoint operator in a Hilbert space X and let $\nu>0$. It makes sense to write $\mathscr{A}^\nu$ and the operator $\mathscr{A}^\nu$ is positive and self-adjoint in X. So the space $\mathscr{S}_{X,\mathscr{A}^\nu}$ is well defined. Its elements are characterized by

LEMMA 1.1. *For each $f\in\mathfrak{D}(\mathscr{A}^\infty)\subset X$ the following statements are equivalent*:

(i) $\exists_{a>0}\exists_{b>0}\forall_{k\in\mathbf{N}}:\|\mathscr{A}^k f\|\leqq(k!)^{1/\nu}a^k b$.

(ii) $f\in\mathscr{S}_{X,\mathscr{A}^\nu}$.

Proof. (i)$\Rightarrow$(ii). Let $N\in\mathbf{N}$ and let $\tau>0$. Consider the following estimation

$$(*)\qquad \sum_{k=0}^N \frac{\tau^k}{k!}\|\mathscr{A}^{\nu k}f\|\leqq\sum_{k=0}^N\frac{\tau^k}{k!}\|\mathscr{A}^{-1+\nu k-[\nu k]}\|\|\mathscr{A}^{[\nu k]+1}f\|$$

$$\leqq b_1\sum_{k=0}^N\frac{\tau^k}{k!}(([\nu k]+1)!)^{\nu k}a^k$$

where $b_1=b\sup_{k\in\mathbf{N}\cup\{0\}}(\|\mathscr{A}^{-1+\nu k-[\nu k]}\|)$. The following inequalities are valid:

$$([\nu k]+1)!\leqq([\nu k]+1)([\nu k]+1)^{[\nu k]}\leqq e([\nu k]+1)(\nu k)^{\nu k}.$$

So $([\nu k]+1)!\leqq(e([\nu k]+1))^{1/\nu}(\nu e)^k k!$, and for $\tau<(\nu e a)^{-1}$ the series $(*)$ converges. It implies that $f\in\exp(-\tau\mathscr{A}^\nu)(X)$.

(ii)$\Rightarrow$(i). Suppose $g\in\mathscr{S}_{X,\mathscr{A}^\nu}$. Then there exists $s>0$ and $w\in X$ such that $g=\exp(-s\mathscr{A}^\nu)w$. Let $k\in\mathbf{N}$. Then we estimate as follows

$$\|\mathscr{A}^k f\|\leqq\|\mathscr{A}^k\exp(-s\mathscr{A}^\nu)\|\|w\|=\|w\|\left(\frac{k}{\nu s}\right)^{k/\nu}e^{-k/\nu}$$

$$\leqq\|w\|(1/\nu s)^{k/\nu}\cdot(k!)^{1/\nu}.$$

With $a=(\nu s)^{-1/\nu}$ and $b=\|w\|$ the implication (ii)$\Rightarrow$(i) has been proved. $\square$

Let $\mathscr{L}$ be an unbounded linear operator in X. Then the operators $\mathscr{L}^2,\mathscr{L}^3,\cdots$ are well defined. As a corollary of the previous theorem we get the following.

COROLLARY 1.2. *Let $n\in\mathbf{N}$ and let $f\in\mathfrak{D}^\omega(\mathscr{L})$. The following statements are equivalent.*

(i) $\exists_{a>0}\exists_{b>0}\forall_{k\in\mathbf{N}}:\|\mathscr{L}^k f\|\leqq(k!)^{1/n}a^k b$.

(ii) $f\in\mathfrak{D}^\omega(\mathscr{L}^n)$.

As mentioned in the introduction we investigate perturbations $\mathscr{P}$ on $\mathscr{A}$ such that $\mathfrak{D}^\omega((\mathscr{A}+\mathscr{P})^\nu)\supset\mathscr{S}_{X,\mathscr{A}^\nu}$. For $\nu=1$ the following result has been proved in [1]. Here we consider general $\nu>0$.

THEOREM 1.3. *Let $\mathscr{P}$ be a linear operator in X with $\mathfrak{D}(\mathscr{P})\supset\mathscr{S}_{X,\mathscr{A}^\nu}$. Suppose the following conditions are satisfied.*

(i) *There exists a Hilbert space Y such that* $\exp(-t\mathscr{A}^\nu)$ *maps X into Y for all* $t>0$.

(ii) *In addition,* $\mathscr{A}+\mathscr{P}$ *defined on* $\mathscr{S}_{X,\mathscr{A}^\nu}$ *is positive and essentially self-adjoint in Y.*

(iii) *There exists an everywhere defined, monotone nonincreasing function* φ *on* $(0,1)$ *such that*

$$\forall_{r:0<r<1}:\|\exp(r\mathscr{A}^\nu)\mathscr{P}\mathscr{A}^{-1}\exp(-r\mathscr{A}^\nu)\|_X\leqq\varphi(r).$$

Then $\mathscr{S}_{X,\mathscr{A}^\nu}\subset\mathscr{S}_{Y,(\mathscr{A}+\mathscr{P})^\nu}$.

Proof. We note first that $\mathscr{S}_{X,\mathscr{A}^\nu}=\bigcup_{0<t<1}\exp(-t\mathscr{A}^\nu)(X)$. So let $0<t<1$, and let $0<\tau<t$. Put $s=t-\tau$. We want to estimate the norm of the operator $\exp(\tau\mathscr{A}^\nu)\cdot(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^\nu)$ for each $k\in\mathbf{N}$. Therefore we factor as follows

$$\exp(\tau\mathscr{A}^\nu)(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^\nu)$$
$$=\prod_{j=0}^{k-1}\left\{\exp\left(\left(\tau+\frac{j}{k}s\right)\mathscr{A}^\nu\right)(\mathscr{I}+\mathscr{P}\mathscr{A}^{-1})\exp\left(-\left(\tau+\frac{j}{k}s\right)\mathscr{A}^\nu\right)\mathscr{A}\exp\left(-\frac{s}{k}\mathscr{A}^\nu\right)\right\}.$$

This factoring yields the estimate

$$\|\exp(\tau\mathscr{A}^\nu)(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^\nu)\|$$
$$\leqq\left\|\mathscr{A}\exp\left(-\frac{s}{k}\mathscr{A}^\nu\right)\right\|^k\prod_{j=0}^{k-1}\left\|\exp\left(\left(\tau+\frac{j}{k}s\right)\mathscr{A}^\nu\right)(\mathscr{I}+\mathscr{P}\mathscr{A}^{-1})\exp\left(-\left(\tau+\frac{j}{k}s\right)\mathscr{A}^\nu\right)\right\|$$
$$\leqq(k!)^{1/\nu}\left(\frac{1}{\nu s}\right)^{k/\nu}\prod_{j=0}^{k-1}\left(1+\varphi\left(\tau+\frac{j}{k}s\right)\right).$$

Since $\varphi(\tau+js/k)\leqq\varphi(\tau)$ for all $j=0,1,\cdots,k-1$, we get

$$\prod_{j=0}^{k-1}\left(1+\varphi\left(\tau+\frac{j}{k}s\right)\right)\leqq(1+\varphi(\tau))^k.$$

Thus we have proved that

$$\forall_{t>0}\forall_{\tau,0<\tau<t}\exists_{a>0}\forall_{k\in\mathbf{N}\cup\{0\}}:\|\exp(\tau\mathscr{A}^\nu)(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^\nu)\|\leqq(k!)^{1/\nu}a^k.$$

Let $t>0$ and let $w\in X$. Set $f=\exp(-t\mathscr{A}^\nu)w$. Then for $0<\tau<t$ fixed there exists $a>0$ such that

$$\|(\mathscr{A}+\mathscr{P})^kf\|_Y\leqq\|\exp(-\tau\mathscr{A}^\nu)\|_{X\to Y}\|\exp(\tau\mathscr{A}^\nu)(\mathscr{A}+\mathscr{P})^kf\|_X$$
$$\leqq\|\exp(-\tau\mathscr{A}^\nu)\|_{X\to Y}\|w\|_Xa^k(k!)^{1/\nu}.$$

From Lemma 1.1 it follows that $f\in\mathscr{S}_{Y,(\mathscr{A}+\mathscr{P})^\nu}$. □

Remark. Suppose there exists $k\in\mathbf{N}$ such that the operator $\mathscr{A}^{-k}$ maps X continuously into Y. Then Condition (iii) of Theorem 1.3 is fulfilled because

$$\|\exp(-t\mathscr{A}^\nu)\|_{X\to Y}\leqq\|\mathscr{A}^{-k}\|_{X\to Y}\|\mathscr{A}^k\exp(-t\mathscr{A}^\nu)\|_X.$$

COROLLARY 1.4. *Let $\mathscr{P}$ be an operator in X and let $n \in \mathbf{N}$ with $\mathfrak{D}(\mathscr{P}) \supset \mathscr{S}_{X,\mathscr{A}^n}$. Suppose there exists an everywhere defined monotone nonincreasing function φ on $(0,1)$ such that*

$$\forall_{0<r<1}: \|\exp(r\mathscr{A}^n)\mathscr{P}\mathscr{A}^{-1}\exp(-r\mathscr{A}^n)\| \leqq \varphi(r).$$

Then $\mathscr{S}_{X,\mathscr{A}^n} \subset \mathfrak{D}^{\omega}((\mathscr{A}+\mathscr{P})^n)$.

Proof. As in the proof of the previous theorem: $\forall_{t>0}\forall_{\tau,0<\tau<t}\exists_{a>0}\forall_{k\in\mathbf{N}}$:

$$\|\exp(\tau\mathscr{A}^n)(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^n)\| \leqq (k!)^{1/n}a^k.$$

So for $f=\exp(-t\mathscr{A}^n)w$, $t>0$, $w \in X$, we get

$$\|(\mathscr{A}+\mathscr{P})^k f\|_X \leqq \|\exp(\tau\mathscr{A}^n)(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^n)\|\,\|w\| \leqq (k!)^{1/n}a^k\|w\|. \qquad \square$$

Remark. If $\mathscr{P}$ satisfies the conditions in Corollary 1.4, then $\mathscr{A}^n$ analytically dominates $(\mathscr{A}+\mathscr{P})^n$. (For the terminology, see [6].)

In order to prove the converse statement of Theorem 1.3, i.e.,

$$\mathscr{S}_{Y,(\mathscr{A}+\mathscr{P})^\nu} \subset \mathscr{S}_{X,\mathscr{A}^\nu}$$

we have to interchange the roles of $\mathscr{A}$ and $\mathscr{A}+\mathscr{P}$. Put differently, if we write $\mathscr{B}=\mathscr{A}+\mathscr{P}$ and hence $\mathscr{A}=\mathscr{B}-\mathscr{P}$, then we have to check whether the pair $\mathscr{B}$, $\mathscr{P}$ satisfies the conditions required in Theorem 1.3.

2. Hankel invariant distribution spaces. In our papers [2], [4] on Hankel invariant distribution spaces the following results have been proved.

Let $\mathscr{A}_\gamma$ denote the differential operator $-d^2/dx^2+x^2-(2\gamma+1)/x\,d/dx$ and let X_γ denote the Hilbert space $\mathfrak{L}_2((0,\infty), x^{2\gamma+1}dx)$ where we take $\gamma>-1$. Then for every $\alpha,\beta>-1$ we have shown that

$$\mathscr{S}_{X_\alpha,\mathscr{A}_\alpha}=\mathscr{S}_{X_\beta,\mathscr{A}_\beta}.$$

Moreover, $f \in \mathscr{S}_{X_\gamma,\mathscr{A}_\gamma}$ if and only if f is extendible to an even function in $\mathscr{S}^{1/2}_{1/2}$. Also, it has been proved that the space $\mathscr{S}_{X_\gamma,\mathscr{A}_\gamma}$ remains invariant under the modified Hankel transform $\mathbf{H}_\gamma$ defined by

$$(\mathbf{H}_\gamma f)(x)=\int_0^\infty f(y)(xy)^{-\gamma}J_\gamma(xy)\,y^{2\gamma+1}\,dy.$$

Here J_γ denotes the Bessel function of the first kind and of order γ. The Hankel transform $\mathbf{H}_\gamma$ extends to a unitary operator on X_γ and $\mathbf{H}_\gamma\mathscr{A}_\gamma=\mathscr{A}_\gamma\mathbf{H}_\gamma$. It follows that for all $\alpha,\beta>-1$, $\mathbf{H}_\alpha$ maps the space $\mathscr{S}_{X_\beta,\mathscr{A}_\beta}$ onto itself. By duality, each $\mathbf{H}_\alpha$ leaves invariant each space of generalized functions $\mathscr{T}_{X_\beta,\mathscr{A}_\beta}$ corresponding to $\mathscr{S}_{X_\beta,\mathscr{A}_\beta}$. The functions $\mathbf{L}_n^{(\gamma)}$ defined by

$$\mathbf{L}_n^{(\gamma)}(x)=\left(\frac{2\Gamma(n+1)}{\Gamma(n+\gamma+1)}\right)^{1/2} e^{-x^2/2}\mathscr{L}_n^{(\gamma)}(x^2), \qquad n\in\mathbf{N}\cup\{0\}, \quad x>0$$

establish an orthonormal basis in X_γ and they are the eigenfunctions of the self-adjoint operator $\mathscr{A}_\gamma$ with respective eigenvalues $4n+2\gamma+2$. Here $\mathscr{L}_n^{(\gamma)}$ denotes the nth generalized Laguerre polynomial of order γ. We note that $\mathbf{H}_\gamma\mathbf{L}_n^{(\gamma)}=(-1)^n\mathbf{L}_n^{(\gamma)}$. We recall that for each $\alpha,\beta>-1$ the functions $f\in\mathscr{S}_{X_\alpha,\mathscr{A}_\alpha}$ can be written as $f=\sum_{n=0}^\infty \omega_n\mathbf{L}_n^{(\beta)}$ where $\omega_n=O(e^{-nt})$ for some $t>0$.

With the aid of the theory presented in the first part of this paper we extend the mentioned results and prove that

$$\mathcal{S}_{X_\alpha,(\mathcal{A}_\alpha)^\nu}=\mathcal{S}_{X_\beta,(\mathcal{A}_\beta)^\nu}$$

for all $\nu\geqq\frac{1}{2}$ and all $\alpha,\beta>-1$. In addition, we show that for each $\nu\in[\frac{1}{2},1]$ and all $\alpha>-1$ the space $\mathcal{S}_{X_\alpha,(\mathcal{A}_\alpha)^\nu}$ contains just the even functions of the Gel'fand–Shilov space $\mathcal{S}^{1/2\nu}_{1/2\nu}$. So each even function $f\in\mathcal{S}^{1/2\nu}_{1/2\nu}$ admits Fourier expansions $f=\sum_{n=0}^{\infty}\rho_n^{(\alpha)}\mathbf{L}_n^{(\alpha)}$ with $\rho_n^{(\alpha)}=O(\exp(-n^\nu t))$.

Let $\alpha,\beta>-1$. Then $\mathcal{A}_\alpha$ can be written as

$$\mathcal{A}_\alpha=\mathcal{A}_\beta+2(\alpha-\beta)\mathcal{R}$$

where we put $\mathcal{R}=(1/x)d/dx$. Obviously, $\mathcal{A}_\alpha$ can be obtained from $\mathcal{A}_\beta$ by means of the "perturbation" $2(\alpha-\beta)\mathcal{R}$, and $\mathcal{A}_\beta$ from $\mathcal{A}_\alpha$ by means of $2(\beta-\alpha)\mathcal{R}$. In order to show that $\mathcal{R}$ and hence $c\mathcal{R}$, $c\in\mathbf{C}$, is a perturbation in the sense of Theorem 1.3 we compute the matrix of $\mathcal{R}$ with respect to the orthonormal basis $(\mathbf{L}_n^{(\gamma)})_{n=0}^{\infty}$. To this end, we mention that

$$\mathcal{R}\mathbf{L}_n^{(\gamma)}=-\mathbf{L}_n^{(\gamma+1)}-2\mathbf{L}_{n-1}^{(\gamma+1)}$$

where the relation $d\mathcal{L}_n^{(\gamma)}/dx=-\mathcal{L}_{n-1}^{(\gamma+1)}$ is used.

Now $\mathcal{L}_k^{(\gamma+1)}=\sum_{j=0}^{k}\mathcal{L}_j^{(\gamma)}$ and hence

$$\mathcal{R}\mathbf{L}_n^{(\gamma)}=-\left(\frac{2\Gamma(n+1)}{\Gamma(n+\gamma+1)}\right)^{1/2}\left[\left(\frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)}\right)^{1/2}\mathbf{L}_n^{(\gamma)}+2\sum_{m=0}^{n-1}\left(\frac{\Gamma(m+\gamma+1)}{2\Gamma(m+1)}\right)^{1/2}\mathbf{L}_m^{(\gamma)}\right].$$

Thus we obtain the matrix of $\mathcal{R}$ with respect to the basis $(\mathbf{L}_n^{(\gamma)})_{n=0}^{\infty}$

$$(\mathcal{R}\mathbf{L}_k^{(\gamma)},\mathbf{L}_l^{(\gamma)})_\gamma=\begin{cases}-1 & \text{if } l=k,\quad k\in\mathbf{N},\\ 0 & \text{if } l>k,\quad k\in\mathbf{N}\cup\{0\},\\ -2\left(\dfrac{\Gamma(k+1)}{\Gamma(k+\gamma+1)}\,\dfrac{\Gamma(l+\gamma+1)}{\Gamma(l+1)}\right)^{1/2} & \text{if } 0\leqq l<k,\quad k\in\mathbf{N}.\end{cases}$$

The inequality (cf. [11])

$$n^{1-s}\leqq\frac{\Gamma(n+1)}{\Gamma(n+s)}\leqq(n+1)^{1-s},\qquad 0\leqq s\leqq 1,\quad n\in\mathbf{N}$$

yields

$$|(\mathcal{R}\mathbf{L}_k^{(\gamma)},\mathbf{L}_l^{(\gamma)})|\leqq\begin{cases}2 & \text{if } \gamma\geqq 0,\ 0\leqq l<k,\ k\in\mathbf{N}\cup\{0\},\\ 2k^{-\gamma/2} & \text{if } -1<\gamma<0,\ 0\leqq l<k,\ k\in\mathbf{N}\cup\{0\}.\end{cases}$$

For each $\nu\geqq\frac{1}{2}$, the operator $\exp(r(\mathcal{A}_\gamma)^\nu)\mathcal{R}(\mathcal{A}_\gamma)^{-1}\exp(-r(\mathcal{A}_\gamma)^\nu)$ has to satisfy Condition (iii) of Theorem (1.3). We define the weighted shift operators $\mathcal{W}_{\gamma,\nu}^{(n)}(r)$, $n\in\mathbf{N}\cup\{0\}$,

$$(\mathcal{W}_{\gamma,\nu}^{(n)}(r)\mathbf{L}_k^{(\gamma)},\mathbf{L}_l^{(\gamma)})_\gamma=\begin{cases}0 & \text{if } k\neq l+n,\\ (\mathcal{R}\mathbf{L}_{l+n}^{(\gamma)},\mathbf{L}_l^{(\gamma)})\dfrac{\exp(-r(4(l+n)+2\gamma+2))^\nu-(4l+2\gamma+2)^\nu}{4(l+n)+2\gamma+2}\end{cases}$$

with norms

(*)

$$\|\mathscr{W}_{\gamma,\nu}^{(n)}(r)\|_{X_\gamma}=\sup_{l\in\mathbf{N}\cup\{0\}}|(\mathscr{R}\mathbf{L}_{l+n}^{(\gamma)},\mathbf{L}_l^{(\gamma)})|\frac{\exp(-r(4(l+n)+2\gamma+2))^\nu-(4l+2\gamma+2)^\nu}{4(l+n)+2\gamma+2}.$$

So $\|\mathscr{W}_{\gamma,\nu}^{(0)}(r)\|\leqq 1/(2\gamma+2)$. Now let $n\in\mathbf{N}$. The inequality

$$(4(l+n)+2\gamma+2)^\nu-(4l+2\gamma+2)^\nu\geqq(l+n)^{1/2}-l^{1/2}$$

is valid for all $l\in\mathbf{N}\cup\{0\}$ and all $\nu\geqq\frac{1}{2}$. In addition, the matrix elements $|(\mathscr{R}\mathbf{L}_{l+n}^{(\gamma)},\mathbf{L}_l^{(\gamma)})|$ are smaller than $2(l+n)^{-\gamma/2}$ for $-1<\gamma<0$ and smaller than 2 for $\gamma\geqq 0$. If $-1<\gamma\leqq 0$ we therefore get

$$\begin{aligned}\|\mathscr{W}_{\gamma,\nu}^{(n)}(r)\|&\leqq\sup_{l\in\mathbf{N}\cup\{0\}}\frac{2(l+n)^{-\gamma/2}}{4(l+n)+2\gamma+2}\exp\left(-r\left((l+n)^{1/2}-l^{1/2}\right)\right)\\&\leqq\sup_{l\in\mathbf{N}\cup\{0\}}\left(\frac{1}{2}(l+n)^{-1/2\gamma-1}\exp\left(-\frac{1}{2}rn(l+n)^{-1/2}\right)\right)\\&\leqq\frac{1}{2}\left(1+\frac{1}{2}\gamma\right)^{2+\gamma}\left(\frac{1}{r}\right)^{2+\gamma}\left(\frac{1}{n}\right)^{2+\gamma}\exp(2+\gamma)=:d_1\left(\frac{1}{r}\right)^{2+\gamma}\left(\frac{1}{n}\right)^{2+\gamma}.\end{aligned}$$

Since

$$\exp\left(r(\mathscr{A}_\gamma)^\nu\right)\mathscr{R}(\mathscr{A}_\gamma)^{-1}\exp\left(-r(\mathscr{A}_\gamma)^\nu\right)=\sum_{n=0}^{\infty}\mathscr{W}_{\gamma,\nu}^{(n)}(r)$$

we can use the following straightforward estimate for all $r>0$

$$\begin{aligned}\left\|\exp\left(r(\mathscr{A}_\gamma)^\nu\right)\mathscr{R}(\mathscr{A}_\gamma)^{-1}\exp\left(-r(\mathscr{A}_\gamma)^\nu\right)\right\|&\leqq\sum_{n=0}^{\infty}\|\mathscr{W}_{\gamma,\nu}^{(n)}(r)\|\\&\leqq\frac{1}{2\gamma+2}+d_1\left(\frac{1}{r}\right)^{2+\gamma}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{2+\gamma}\\&\leqq d_\gamma\left(\frac{1}{r}\right)^{2+\gamma}+\frac{1}{2\gamma+2}\end{aligned}$$

where $d_\gamma=d_1\sum_{n=1}^{\infty}(1/n)^{2+\gamma}$. Summarized we have

LEMMA 2.1. *Let $\gamma>-1$. Then there exist constants $d_\gamma>0$ and $p_\gamma>0$ such that*

$$\forall_{r>0}:\left\|\exp\left(r(\mathscr{A}_\gamma)^\nu\right)\mathscr{R}\mathscr{A}_\gamma^{-1}\exp\left(-r(\mathscr{A}_\gamma)^\nu\right)\right\|\leqq d_\gamma\left(\frac{1}{r}\right)^{p_\gamma}+\frac{1}{2\gamma+2}.$$

Proof. For $-1<\gamma\leqq 0$ the assertion has already been proved. For $\gamma>0$ it follows from the matrix expressions for $\mathscr{R}$ that

$$\left\|\exp r(\mathscr{A}_\gamma)^\nu\mathscr{R}\mathscr{A}_\gamma^{-1}\exp\left(-r(\mathscr{A}_\gamma)^\nu\right)\right\|\leqq d_0\left(\frac{1}{\gamma}\right)^{p_0}+\frac{1}{2\gamma+2}.\qquad\square$$

In addition, we show that given $r>0$, $\gamma,\delta>-1$, the operator $\exp(-r(\mathscr{A}_\gamma)^\nu)$ maps X_γ into X_δ. In [2, p. 17], the following result has been proved

$$\forall_{s\in\mathbf{N}}\exists_{l\in\mathbf{N}}: \left\|\mathscr{Q}^{2s}(\mathscr{A}_\gamma)^{-l}\right\|_\gamma<\infty.$$

Here $\mathscr{Q}$ denotes the multiplication operator in X_γ given by

$$(\mathscr{Q}f)(x)=xf(x).$$

Now let $\delta>-1$ and let $f\in X_\gamma$. Put $s:=[\max\{0,(\delta-\gamma)/2\}]+1$. Then there exists $l_0\in\mathbf{N}$ such that $\|\mathscr{Q}^{2s}\mathscr{A}_\gamma^{-l}\|_\gamma<\infty$ for all $l\geqq l_0$. So we derive

$$(*)\qquad \int_1^\infty\left|\left((\mathscr{A}_\gamma)^{-l}f\right)(x)\right|^2x^{2\delta+1}\,dx=\int_1^\infty x^{2(\delta-\gamma)}\left|\left((\mathscr{A}_\gamma)^{-l}f\right)(x)\right|^2x^{2\gamma+1}\,dx$$
$$\leqq\int_1^\infty x^{4s}\left|\left((\mathscr{A}_\gamma)^{-l}f\right)(x)\right|^2x^{2\gamma+1}\,dx$$
$$\leqq\left\|\mathscr{Q}^{2s}(\mathscr{A}_\gamma)^{-l}\right\|_\gamma^2\|f\|_\gamma^2.$$

Following [12, p. 248], there exists $l_1\in\mathbf{N}$ and $d>0$ such that

$$\max_{x\in[0,1]}|\mathbf{L}_k^{(\gamma)}(x)|\leqq d(k+1)^{l_1}.$$

For $l>l_1$ it yields

$$(**)\qquad \int_0^1\left|\left((\mathscr{A}_\gamma)^{-2}f\right)(x)\right|^2x^{2\delta+1}\,dx$$
$$\leqq\left(\max_{x\in[0,1]}\left|\left((\mathscr{A}_\gamma)^{-l}f\right)(x)\right|\right)^2\int_0^1x^{2\delta+1}\,dx$$
$$\leqq\frac{1}{2\delta+2}\left(\sum_{k=0}^\infty(f,\mathbf{L}_k^{(\gamma)})_\gamma\left(\frac{1}{4k+2\gamma+2}\right)^l\max_{x\in[0,1]}\left|\mathbf{L}_k^{(\gamma)}(x)\right|\right)^2$$
$$\leqq\frac{1}{2\delta+2}\left(d^2\sum_{k=0}^\infty\frac{(k+1)^{2l_1}}{(4k+2\gamma+2)^{2l}}\right)\|f\|_\gamma^2.$$

From $(*)$ and $(**)$ we get

$$\forall_{\gamma>-1}\forall_{\delta>-1}\exists_{l\in\mathbf{N}}\exists_{c>0}\forall_{f\in X_\gamma}:$$
$$\left\|(\mathscr{A}_\gamma)^{-l}f\right\|_\delta^2=\int_0^\infty\left|\left((\mathscr{A}_\gamma)^{-l}f\right)(x)\right|^2x^{2\delta+1}\,dx\leqq c\|f\|_\gamma^2$$

i.e. $(\mathscr{A}_\gamma)^{-l}$ is a continuous linear operator from X_γ into X_δ.

LEMMA 2.2. *Let $\gamma>-1$. Then for every $r>0$, $\nu>0$ and $\delta>-1$ the operator $\exp(-r(\mathscr{A}_\gamma)^\nu)$ is a continuous linear operator from X_γ into X_δ.*

Proof. Let $r>0$, $\nu>0$ and let $\delta>-1$. Then there exists $l\in\mathbf{N}$ such that $(\mathscr{A}_\gamma)^{-l}$ is a continuous linear mapping from X_γ into X_δ. Hence $\exp(-r(A_\gamma)^\nu)=(\mathscr{A}_\gamma)^{-l}\{(\mathscr{A}_\gamma)^l\exp(-r(\mathscr{A}_\gamma)^\nu)\}$ is also a continuous linear mapping from X_γ into X_δ. □

Lemmas 2.1 and 2.2 yield the following important result.

THEOREM 2.3. *Let $\alpha, \beta > -1$. Then for every $\nu \geqq \frac{1}{2}$*

$$\mathcal{S}_{X_\alpha,(\mathcal{A}_\alpha)^\nu} = \mathcal{S}_{X_\beta,(\mathcal{A}_\beta)^\nu}.$$

Proof. Let $\nu \geqq \frac{1}{2}$. We have shown that:

–$\exp(-t(\mathcal{A}_\alpha)^\nu)$, $t>0$, maps X_α continuously into X_β.

–$\mathfrak{D}(\mathcal{R}) \supset \mathcal{S}_{S_\alpha,(\mathcal{A}_\alpha)^\nu}$, and $\mathcal{A}_\beta = \mathcal{A}_\alpha + 2(\alpha-\beta)\mathcal{R}$ is positive and self-adjoint in X_β.

–There exist constants $d_\alpha, p_\alpha > 0$ such that for all $r>0$

$$\left\| \exp\left(r(\mathcal{A}_\alpha)^\nu\right)\mathcal{R}(\mathcal{A}_\alpha)^{-1}\exp\left(-r(\mathcal{A}_\alpha)^\nu\right)\right\|_\alpha \leqq d_\alpha\left(\frac{1}{r}\right)^{p_\alpha} + \frac{1}{2\alpha+2}.$$

So by Theorem 1.3, $S_{X_\alpha,(\mathcal{A}_\alpha)^\nu} \subset \mathcal{S}_{X_\beta,(\mathcal{A}_\beta)^\nu}$. Interchanging α and β we get the wanted result. □

Let $\alpha > -1$. Since $\mathbf{H}_\alpha \mathcal{A}_\alpha = \mathcal{A}_\alpha \mathbf{H}_\alpha$, also $\mathbf{H}_\alpha(\mathcal{A}_\alpha)^\nu = (\mathcal{A}_\alpha)^\nu \mathbf{H}_\alpha$. So the Hankel transform $\mathbf{H}_\alpha$ is a continuous bijection on the space $\mathcal{S}_{X_\alpha,(\mathcal{A}_\alpha)^\nu}$, $\nu \geqq \frac{1}{2}$, and hence on the spaces $\mathcal{S}_{X_\beta,(\mathcal{A}_\beta)^\nu}$, $\nu \geqq \frac{1}{2}$, $\beta > -1$. By duality each transform $\mathbf{H}_\alpha$ leaves invariant the spaces of generalized functions $\mathcal{T}_{X_\beta,(\mathcal{A}_\beta)^\nu}$. For $\alpha = -\frac{1}{2}$ we get $X_{-1/2} = \mathfrak{L}_2((0,\infty))$ and $\mathcal{A}_{-1/2} = -(d^2/dx^2) + x^2$. The functions $\mathbf{L}_k^{(-1/2)}$ are the even Hermite functions. With the aid of the papers [8] and [10] the following characterization of the spaces $\mathcal{S}_{X_{-1/2},(\mathcal{A}_{-1/2})^\nu}$, $\nu \in [\frac{1}{2}, 1]$, can be obtained,

$$f \in \mathcal{S}_{X_{-1/2}},\left(\mathcal{A}_{-1/2}\right)^\nu : \Leftrightarrow f \text{ is extendible to an even function in the space } \mathcal{S}_{1/2\nu}^{1/2\nu}.$$

The spaces $\mathcal{S}_p^q$, $p+q \geqq 1$, $p,q \geqq 0$, are introduced by Gel'fand and Shilov in [9]. In this connection we note that in our paper [5] we have proved that the spaces $\mathcal{S}_{1/k+1}^{k/k+1}$ are analyticity spaces; explicitly

$$\mathcal{S}_{1/k+1}^{k/k+1} = \mathcal{S}_{\mathfrak{L}_2(\mathbf{R}),\mathcal{B}_k} \quad \text{with } \mathcal{B}_k = \left(-\frac{d^2}{dx^2} + x^{2k}\right)^{(k+1)/2k}.$$

Relevant for the present paper are the spaces $\mathcal{S}_\mu^\mu$, $\frac{1}{2} \leqq \mu \leqq 1$. We have

> $\varphi \in \mathcal{S}_\mu^\mu$, $\frac{1}{2} \leqq \mu \leqq 1$ if and only if φ is an entire function satisfying $\exists_{A,B,C>0} : |\varphi(x+iy)| \leqq C\exp(-A|x|^{1/\mu} + B|y|^{1/1-\mu})$

and

> $\varphi \in \mathcal{S}_1^1$ if and only if φ is analytic on a strip about the real axis say of width $r>0$ and satisfying $\exists_{A,C>0} : \sup_{|y|<r} |\varphi(x+iy)| \leqq C\exp(-A|x|)$.

Now Theorem 2.3 leads to the following important results.

COROLLARY 2.4. *Let $\alpha > -1$ and let $\nu \in [\frac{1}{2}, 1]$. Then $f \in \mathcal{S}_{X_\alpha,(\mathcal{A}_\alpha)^\nu}$ if and only if f is extendible to an even function in the space $\mathcal{S}_{1/2\nu}^{1/2\nu}$.*

COROLLARY 2.5. *Let $f \in \mathcal{S}_{1/2\nu}^{1/2\nu}$ be even, with $\nu \in [\frac{1}{2}, 1]$. Then for each $\gamma > -1$, there exists an l_2-sequence $(\omega_n^{(\gamma)})_{n=0}^\infty$ and $t>0$ such that $f = \sum_{n=0}^\infty \exp(-n^\nu t)\omega_n^{(\gamma)}\mathbf{L}_n^{(\gamma)}$ where the series converges pointwise.*

Appendix. The set of so-called entire vectors for a positive self-adjoint operator $\mathscr{A}$ in a Hilbert space X is equal to

$$\mathfrak{D}^{\infty}(e^{\mathscr{A}})=\bigcap_{t>0} e^{-t\mathscr{A}}(X).$$

In [3], van Eijndhoven has used the Fréchet space $\mathfrak{D}^{\infty}(e^{\mathscr{A}})$ as the test space in a theory of generalized functions which is a kind of reverse of the theory in [7]. The space $\mathfrak{D}^{\infty}(e^{\mathscr{A}})$ is denoted by $\tau(X,\mathscr{A})$ and it may be called the entireness space. To our opinion the well-known theory of tempered distributions is considerably generalized in [3]. (Put $\mathscr{A}=\log(-d^2/dx^2+x^2+1)$. Then $\tau(\mathfrak{L}_2(\mathbf{R}), \mathscr{A})$ is the space $\mathscr{S}(\mathbf{R})$ of functions of rapid decrease.)

Similar to Theorem 1.3 we prove

THEOREM A.1. *Let $\mathscr{P}$ be a linear operator in X with $\mathfrak{D}(\mathscr{P})\supset\exp(-\sigma\mathscr{A}^{\nu})(X)$ for some $\sigma>0$ sufficiently large. Suppose the following conditions are satisfied.*

(i) *There exists a Hilbert space Y such that $\exp(-t\mathscr{A}^{\nu})$ maps X into Y for all $t>0$.*

(ii) *Also, $\mathscr{A}+\mathscr{P}$ defined on $\exp(-\sigma\mathscr{A}^{\nu})(X)$ is a positive essentially self-adjoint operator in Y.*

(iii) *There exist positive constants $r_0\geqq 1$, $d>0$ and $0\leqq q<1/\nu$ such that for all $r>r_0$*

$$\|\exp(r\mathscr{A}^{\nu})\mathscr{P}\mathscr{A}^{-1}\exp(-r\mathscr{A}^{\nu})\|_X<dr^q.$$

Then $\tau(X,\mathscr{A}^{\nu})\subset\tau(Y,(\mathscr{A}+\mathscr{P})^{\nu})$.

Proof. Since $\tau(X,\mathscr{A}^{\nu})=\bigcap_{t>r_0}\exp(-t\mathscr{A}^{\nu})(X)$, we consider $t>r_0$ only. Let $0<\tau<1$ with $s=t-\tau>1$. The factoring used in Theorem 1.3 yields the following estimate

$$\|\exp(\tau\mathscr{A}^{\nu})(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^{\nu})\|\leqq k!\left(\frac{1}{\nu s}\right)^{k/\nu}\prod_{j=0}^{k-1}\left(1+d(\tau+js/k)^q\right).$$

Put $b_\tau=1+d\tau^q$. Then

$$\prod_{j=0}^{k-1}\left(1+d(\tau+js/k)^q\right)\leqq b_\tau\prod_{j=1}^{k-1}(1+d)\left(\frac{k+js}{k}\right)^q\leqq b_\tau(1+d)^k2^{qk}s^{qk}.$$

Set $a=(1+d)2^q(\frac{1}{\nu})^{1/\nu}$. Then

$$\|\exp(\tau\mathscr{A}^{\nu})(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^{\nu})\|_X\leqq(k!)^{1/\nu}\left(\frac{1}{s}\right)^{(-q+1/\nu)k}a^kb_\tau.$$

For $f\in\exp(-t\mathscr{A}^{\nu})(X)$ it yields

$$\|(\mathscr{A}+\mathscr{P})^kf\|_Y\leqq\|\exp(-\tau A^{\nu})\|_{X\to Y}\|\exp(\tau\mathscr{A}^{\nu})(\mathscr{A}+\mathscr{P})^k\exp(-t\mathscr{A}^{\nu})\|_X\|\exp(t\mathscr{A}^{\nu})f\|$$

$$\leqq(k!)^{1/\nu}\left(a\cdot\left(\frac{1}{s}\right)^{1/\nu-q}\right)^k b_\tau\|\exp(-\tau\mathscr{A}^{\nu})\|_{X\to Y}\|\exp(t\mathscr{A}^{\nu})f\|_X.$$

Thus we find that $f\in\exp(-r(\mathscr{A}+\mathscr{P})^{\nu})(Y)$ for all $r<(1/\nu ae)s^{-q+1/\nu}$. Now put $r(t)=(1/(\nu ae+1))s^{-q+1/\nu}$ with $s=t+1/t-1$ for instance. Then we get

$$\tau(X,\mathscr{A}^{\nu})=\bigcap_{t>r_0}(\exp(-t\mathscr{A}^{\nu})(X))\subset\bigcap_{t>r_0}\Big(\exp(-r(t)(\mathscr{A}+\mathscr{P})^{\nu})(Y)$$

$$=\bigcap_{r>0}(\exp(-r(\mathscr{A}+\mathscr{P})^{\nu})(Y))\Big)=\tau(Y,(\mathscr{A}+\mathscr{P})^{\nu}).\qquad\square$$

It is not hard to see that the spaces $\tau(X_\alpha, (\mathscr{A}_\alpha)^\nu)$, $\alpha > -1$, are Hankel invariant, and hence their strong duals $\sigma(X_\alpha, (\mathscr{A}_\alpha)^\nu)$. The previous theorem and the Lemmas 2.1 and 2.2 lead to the following classification.

THEOREM A.2. *Let* $\alpha, \beta > -1$ *and let* $\nu \geqq \frac{1}{2}$. *Then*

$$\tau\left(X_\alpha, (\mathscr{A}_\alpha)^\nu\right) = \tau\left(X_\beta, (\mathscr{A}_\beta)^\nu\right).$$

By [2] *and* [8] *we obtain the following characterizations*

$f \in \tau(X_{-1/2}, \mathscr{A}_{-1/2})$ *iff f is extendible to an even entire function for which* $\forall_{0<a<1} \exists_{C>0} \forall_{x+iy \in C} : |f(x+iy)| \leqq C \exp(-\frac{1}{2}ax^2 + \frac{1}{2a}y^2)$

and

$f \in \tau(X_{-1/2}, (\mathscr{A}_{-1/2})^{1/2})$ *iff f is extendible to an even entire function for which* $\forall_{r>0}$: $\sup_{|y|<r,\, -\infty<x<\infty} e^{r|x|} |f(x+iy)| < \infty$.

Finally, Theorem A.2 gives the characterization in classical analytic terms of the elements in each $\tau(X_\alpha, \mathscr{A}_\alpha)$, respectively $\tau(X_\alpha, (\mathscr{A}_\alpha)^{1/2})$, $\alpha > -1$.

REFERENCES

[1] S. J. L. VAN EIJNDHOVEN, *Invariance of the analyticity domain of self-adjoint operators subjected to perturbations*, preprint, 1982.

[2] ———, *On Hankel invariant distribution spaces*, EUT-Report 82-WSK-01, Eindhoven Univ. of Technology, Eindhoven, the Netherlands, 1982.

[3] ———, *A theory of generalized functions based on one-parameter groups of unbounded self-adjoint operators*, TH-Report 81-WSK-03, Eindhoven Univ. of Technology, Eindhoven, the Netherlands, 1981.

[4] S. J. L. VAN EIJNDHOVEN AND J. DE GRAAF, *Some results on Hankel invariant distribution spaces*, Proc. Koninklijke Nederlands Akademie van Wetenschappen, A(86)1, 1983.

[5] S. J. L. VAN EIJNDHOVEN, J. DE GRAAF AND R. S. PATHAK, *A characterization of the spaces* $\mathscr{S}_{1/k+1}^{k/k+1}$ *by means of holomorphic semigroups*, this Journal, 14 (1983), pp. 1180–1187.

[6] W. G. FARIS, *Self-Adjoint Operators*, Lecture Notes in Mathematics 433, Springer, Berlin, 1974.

[7] J. DE GRAAF, *A theory of generalized functions based on holomorphic semigroups*, (three papers), Proc. Koninklijke Nederlandse Akademie van Wetenschappen. A 86(4), 1983, pp. 407–420; A 87(2), 1984, pp. 155–171; A 87(2), 1984, pp. 173–187.

[8] R. GOODMAN, *Analytic and entire vectors for representation of Lie groups*, Trans. Amer. Math. Soc., 143 (1969), pp. 5–76.

[9] I. M. GEL'FAND AND G. E. SHILOV, *Generalized Functions*, *Vol.* II, Academic Press, New York, 1968.

[10] ZHANG GONG-ZHING, *Theory of distributions of S-type and expansions*, Chinese Math., 4(2) (1963), pp. 211–221.

[11] D. S. MITRINOVIC, *Analytic Inequalities*, first ed., Springer, Berlin, 1970.

[12] W. MAGNUS, F. OBERHETTINGER AND R. P. SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics*, third ed., Springer, Berlin, 1966.

[13] E. NELSON, *Analytic vectors*, Ann. Math., 70 (1959), pp. 572–615.