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Asymmetric Ising Model on the Configuration Model

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Abstract

In this thesis, we investigated an asymmetrical version of the Ising model on random graphs and specifically on the configuration model. Each vertex on the graphs can now have one of the states $-d, c$, where we define $0 < d < c$. The goal was to compare the critical behaviour between the asymmetric model and the symmetric model for the quenched and annealed setting. First, we considered the quenched and annealed Ising model mathematically and then using a Monte Carlo simulation. It turns out that for the asymmetrical model, it is important to define a critical curve of the external magnetic field as a function of temperature, at which the magnetization is zero. The spontaneous magnetization is then defined as the magnetization at this critical curve.

First of all, we examined the asymmetric model on a complete graph. In this case, the asymmetry only caused a linear shift of the critical curve and therefore the critical behaviour did not change. Then the results for the quenched Ising model on the configuration model could be obtained by considering locally tree-like graphs. In order to determine the spontaneous magnetization in this case, we used a recursion equation on the trees. It turned out that if the degrees in the configuration model were i.i.d random variables, it was very difficult to determined the critical curve at which the phase change occurs. Some mathematical bounds were found for the critical curve, but the curve itself could only be determined by a numerical simulation.

After considering the quenched Ising model, we looked at the annealed version on the configuration model. There, the known results on the symmetric model were used to determine the pressure per particle (or Helmholtz free energy) for the asymmetric model.

In the last part of this thesis, we used a Metropolis Monte Carlo simulation to determined the critical temperature and some critical exponents for the magnetization and the susceptibility of both the quenched and the annealed Ising model. Here, it turned out that the critical temperature in the asymmetric model for random degrees, is very close to the corresponding result for the symmetric model. However, the critical exponents seem to be different than in the symmetric model. Furthermore, there were some problems with the Monte Carlo simulations. It turned out that the critical curves that were determined using the mathematical relations, did not always give good results in the simulations. This might be caused by a difference in the critical curve in the thermodynamic limit and in finite systems.

Contents

1 Introduction

This thesis is inspired by a paper from Jeroen Bruggeman et al. [1], where a well-known model, the Ising model, is put into a different perspective. The Ising model is a model that describes interacting particle systems, where each particle can have one of two states, either plus or minus, called 'spin up' or 'spin down'. This model is used a lot in statistical physics, where it describes a situation where each particle favours a spin state, depending on the spin states of its neighbours. In the situation that is considered here, the particles prefer to be in the same direction as its neighbours. In [1], a proposal is made to use the Ising model in social situations, where a group of people can either cooperate or defect on a matter of public good. They also suggest that in that case, the Ising model should not be symmetric in the two states anymore, because cooperating and defecting do not contribute in the same way to the system. Therefore, an asymmetric Ising model is introduced. This is an interesting idea, because a population of people can be described by a graph, where each person represents a vertex in the graph and an edge between two people is made whenever they are related in some way. In this way, the situation is perfectly laid out to apply the Ising model.

In this project, the asymmetric Ising model will be investigated. The standard Ising model gives each vertex (or person) a state of ± 1. In the asymmetric setting, this is generalized by giving each vertex a state of $-d$ or c, where d stands for defecting and c stands for cooperating. We assume that $0 < d < c$. Using this definition, we will see that the system prefers people to be in the same state, but with a stronger preference for cooperating. From the standard Ising model, we know that the system allows for so-called phase changes, where the expected average state of the system can suddenly change by shifting the temperature. In sociological context, this could result in a sudden cooperation in a group of people, when for example the amount of rumours is decreased. We explore the behaviour in critical regions where a phase change occurs. There are many known results on the symmetric Ising model, but it will turn out that adding the asymmetry really adds complications in finding the critical values and critical exponents of the system, which explain the behaviour around the phase change.

In this thesis, first of all, the asymmetric model will be defined and the general concepts will be explained in Section 2. Then an introduction to the configuration model will be given in Section 3. This model is a specific model to generate graphs where the degrees of the vertices can be given as an input. This makes the model also more applicable in real life situation. To analyze the Ising model on the configuration model, first the Ising model will be explored on a complete graph in Section 4. Here every person in a group of people is connected to every other person. This is also the situation that is considered by Bruggeman et al. Then, our view will be broadened by looking into locally tree-like graphs in Section 5, which can be applied on the configuration model. The behaviour of the model will be analyzed in the quenched setting and in the annealed setting. In the former, the networks are fixed, while in the latter the networks are allowed to change during the optimization. The annealed setting will be discussed in Section 6. Furthermore, some numerical methods will be used to obtain results for which no exact mathematical expression can be found. One of these methods is a Monte Carlo simulation that will be used to analyze the behaviour of the asymmetric Ising model in the quenched setting, as well as in the annealed setting. This is explained in Section 7. The overall goal is to find the critical temperature and the critical exponents in the asymmetric model and analyze the differences with the symmetric model. The definitions of these quantities will be given in the Section 2.

2 Introduction to the Ising model

A model that is often used to describe interacting particle systems is the Ising model. This model uses a given configuration of the particles where some particles have an interaction and some do not. Such a configuration can be written down mathematically as a graph. In this graph the vertices represent the particles and two vertices are connected by an edge if the two particles can interact. An example that is often encountered in literature about the Ising model is a lattice of particles where each particle has either an up or a down spin. This spin is denoted by $\sigma_i = \pm 1$. As explained before, in this paper we will consider an asymmetric version of the Ising model. Before explaining the asymmetric model, we will define the symmetric model. For this symmetric system, the Hamiltonian can be defined by

$$H_n(\vec{\sigma}) = -\sum_{\{i,j\}\in E_n} J_{ij}\sigma_i\sigma_j - \sum_{i\in V_n} B_i\sigma_i. \tag{2.1}$$

Here $G_n = (V_n, E_n)$ is the graph that describes a system with n particles, where V_n is the set of vertices and E_n the set of edges. In this paper, we will define $V_n = \{1, 2, ..., n\}$ and $\{i, j\} \in E_n$, if i and j are connected in the graph. We will always take the interaction energies J_{ij} positive, in which case we have a ferromagnetic system where it is more energy efficient for the particles to align their spin than to have opposite spins. Furthermore, the coefficients B_i represent the external magnetic field for each vertex. In the sociological context, the external magnetic field can be seen as an external influence that pushes the opinion of every individual more towards one side. The interaction energy describes how important it is for people to have the same opinion as their connections. In this paper, B_i and J_{ij} are taken equal for all $i, j \in V_n$. With this simplification the Hamiltonian for the standard Ising model can be written as

$$H_n(\vec{\sigma}) = -J\sum_{(i,j)\in E_n} \sigma_i\sigma_j - B\sum_{i\in V_n} \sigma_i. \tag{2.2}$$

Using this Hamiltonian, the Boltzman distribution is used as the probability measure for the Ising model. This distribution is given by

$$\mu(\vec{\sigma}) = \frac{1}{Z_n(\beta, B)} \exp(-\beta H_n(\vec{\sigma})). \tag{2.3}$$

Here $\beta \propto 1/T$ denotes the inverse temperature. In the sociological context, this temperature denotes how easily people can change their opinion. We can for example think of the amount of rumours. Furthermore, the partition function $Z_n(\beta, B)$, that is used to normalize the probability measure, is given by

$$Z_n(\beta, B) = \sum_{\vec{\sigma}\in\{-1,1\}^n} \exp(-\beta H_n(\vec{\sigma})). \tag{2.4}$$

Using this normalization of the measure, the sum over all spin configurations of $\mu(\vec{\sigma})$ is equal to one.

2.1 Asymmetric Ising model

In a paper from Jeroen Bruggeman et al. [1], a version of the Ising model is used to analyse a sociological situation. They are using this model to find out the influence of spreading rumours in a group of people. In some cases a group of people can suddenly start to cooperate. They approach this problem with an asymmetric version of the Ising model, because in this setting the two possible opinions do not have to be equally valued. This version will be further analysed in this report and compared to the symmetric version. First of all, the assumption still holds that each vertex in the graph can occupy one of two states. However instead of ± 1, each vertex can have state c or $-d$, where $0 < d < c$. Here c stands for the 'cooperating', while d stands for 'defecting'. In this way, where the normal Ising model only favours states where neighbouring vertices have the same state, this model has a preference for cooperating over deflecting, as now (c, c) is more efficient than $(-d, -d)$. An example of such a situation is the transfer of believes. In this case c represents a 'good' belief and $-d$ a 'worse' believe. Now, in society it is possible that everyone has the worse idea in a stable situation, while actually it would be better if everyone changed their idea to the good one.

From now on, when the asymmetric Ising model is considered, x_i denotes the state of vertex $i \in V_n$, with $x_i \in \{-d, c\}$. Often it is convenient to rewrite the asymmetric system in a symmetric form again. This is done by defining $A = (c-d)/2$ and $\Delta = (c+d)/2$. In this way $x_i = A + \Delta\sigma_i$, where again $\sigma_i = \pm 1$. Therefore, we can see $\vec{x}(\vec{\sigma})$ as a function $\vec{\sigma}$. However, for simplicity, we will only write $\vec{x}$ from now on, which implicitly dependents on $\vec{\sigma}$. Furthermore, by defining d_i as the degree of vertex i, we can write the Hamiltonian as

$$
\begin{aligned}
H_n(\vec{x}) &= -J \sum_{(i,j)\in E_n} x_i x_j - B \sum_{i\in V_n} x_i \\
&= -J \left(|E|A^2 + A\Delta \sum_{i\in V_n} d_i\sigma_i + \Delta^2 \sum_{(i,j)\in E_n} \sigma_i\sigma_j \right) - BnA - B\Delta \sum_{i\in V_n} \sigma_i. \\
&= -(JA^2|E| + BAn) - J\Delta^2 \sum_{\{i,j\}\in E_n} \sigma_i\sigma_j - \sum_{i\in V_n} (JA\Delta d_i + B\Delta)\sigma_i.
\end{aligned}
\tag{2.5}
$$

In the second line, the substitution of $x_i = A + \Delta\sigma_i$ is used and in the last step, the terms are grouped in a similar way as the Hamiltonian of the symmetric system. By rewriting the Hamiltonian like this, it becomes clear that the asymmetric system looks very similar to the symmetric system. The main differences are an extra term that is independent of σ_i and an extra contribution of $JA\Delta d_i$ to the magnetic field. It will turn out that this last contribution has very interesting effects on the behavior of the system. The first term in the Hamiltonian will be of less importance, as it will drop out when the probability measure is determined, as in equation (2.3). From this analysis we find that the addition to the magnetic field is vertex dependent, which was not the case in the original Ising model that we considered. Using this Hamiltonian, the partition function is defined in a similar way as in the symmetric model, by

$$Z_n(\beta, B) = \sum_{\vec{x} \in \{-d,c\}^n} \exp(-\beta H_n(\vec{x})). \tag{2.6}$$

2.2 Thermodynamic quantities

Before analysing the asymmetric system in more detail, we will need some definitions of quantities that follow directly from the partition function. First of all, the partition function has a direct relation to the physical Helmholtz free energy. In the mathematical literature this energy is usually called the 'pressure per particle'. To be consistent with the mathematical literature, we will only use the term pressure per particle. Note that to get the Helmholtz free energy from this pressure per particle, you need to divide by the inverse temperature β and multiply by the system size n. The pressure per particle is defined by

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B). \tag{2.7}$$

If we want to know the expectation of some function $f(\vec{x})$ over the Ising measure, we can use

$$\langle f(\vec{x}) \rangle_{\mu_n} = \sum_{\vec{x} \in \{-d,c\}^n} f(\vec{x}) \mu_n(\vec{x}). \tag{2.8}$$

However, it can be very difficult to calculate the sum over all possible spin configurations. Luckily, it turns out that some important quantities can be calculated by taking derivatives of the pressure per particle to β or B. In this way, if we are able to find a nice expression for the pressure per particle, we can often also find these quantities. The two quantities that will be considered in this report are the magnetization and the susceptibility. These quantities are defined by respectively

$$M_{x,n}(\beta, B) = \langle \frac{1}{n} \sum_{i \in V_n} x_i \rangle_{\mu_n} = \frac{1}{\beta} \frac{\partial}{\partial B} \psi_n(\beta, B) \tag{2.9}$$

and

$$\chi_{x,n}(\beta, B) = \frac{1}{n} \sum_{\{i,j\} \in E_n} \Big(\langle x_i x_j \rangle_{\mu_n} - \langle x_i \rangle_{\mu_n} \langle x_j \rangle_{\mu_n} \Big) = \frac{1}{\beta^2} \frac{\partial^2}{\partial B^2} \psi_n(\beta, B). \tag{2.10}$$

The subscripts of x denote that these quantities are defined using the values of $\vec{x}$. These equalities follow directly from the definition of the Hamiltonian. However, as we will often compare the asymmetric model with the symmetric model, we will use slightly different definitions of the magnetization and susceptibility. These are given by respectively

$$M_n(\beta, B) = \langle \frac{1}{n} \sum_{i \in V_n} \sigma_i \rangle_{\mu_n} = \frac{1}{\Delta} (M_{x,n}(\beta, B) - A) \tag{2.11}$$

and

$$\chi_n(\beta, B) = \frac{1}{n} \sum_{\{i,j\} \in E_n} \Big(\langle \sigma_i \sigma_j \rangle_{\mu_n} - \langle \sigma_i \rangle_{\mu_n} \langle \sigma_j \rangle_{\mu_n} \Big). \tag{2.12}$$

Now, if the magnetization is equal to zero, it means that there is an equal amount of vertices with c as with $-d$, which is found to be the more relevant definition of zero magnetization. For example, if β is taken very close to zero, the magnetization will tend to zero using this definition, which is better comparable to the symmetric system.

2.3 Phase transitions and critical behaviour

One important aspect of the Ising model is that it can be used to describe phase transitions. When the inverse temperature β is gradually increased, at a certain point there is a sudden change in the average spin (magnetization) of the system. However, there is no real phase transition on finite systems. Therefore, many results will only hold in the limit of n to infinity, which is called the thermodynamic limit. One advantage is that in general, the system converges relatively fast. Therefore, also in finite systems, the phase transition can be seen. In Figure 1, an example is shown of the magnetization as a function of the inverse temperature β for different system sizes n. The meaning of $B_0(\beta)^+$ will be explained later in this section.

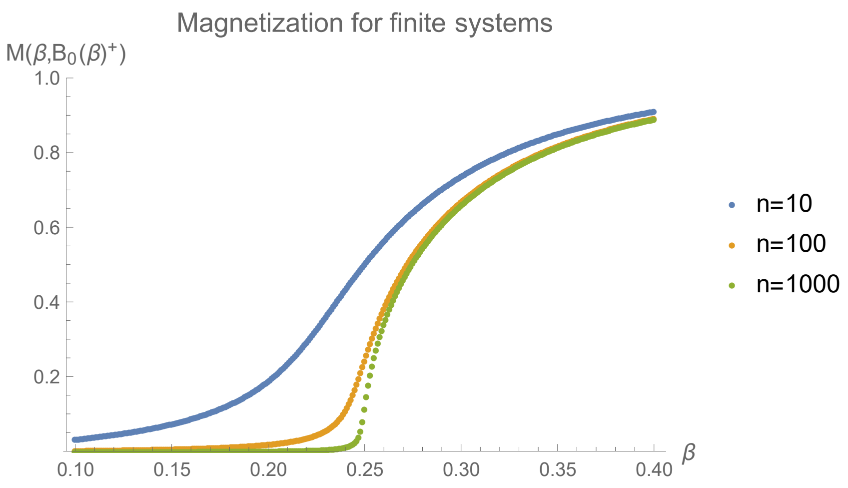


Figure 1: *Magnetization of a complete graph for the asymmetric Ising model with $c = 3$ and $d = 1$ for different system sizes. The limit is taken of the magnetic field to the critical curve from above.*

From this figure, it becomes clear that the sharp phase transition only happens for large enough values of n. Therefore, all quantities that are discussed above need to be defined in the thermodynamic limit. The most important quantity is the pressure per particle. It is not trivial that this always has a limit, but if it exists it is defined by

$$\phi(\beta, B) = \lim_{n\to\infty} \psi_n(\beta, B). \tag{2.13}$$

Now the magnetization in the thermodynamic limit is defined by

$$M(\beta, B) = \lim_{n\to\infty} M_n(\beta, B). \tag{2.14}$$

In the situations that are considered in this paper, this magnetization can also be expressed in terms of the derivative of $\phi(\beta, B)$ to B, similar to the magnetization for finite n. This relation is however not always trivial and needs to be proved in every situation. Luckily, for many cases this is already shown in literature. Therefore, we can combine this with equation (2.9) and equation (2.11) to get

$$M(\beta, B) = \frac{1}{\Delta}\left(\frac{1}{\beta}\frac{\partial}{\partial B}\phi(\beta, B) - A\right). \tag{2.15}$$

Taking the thermodynamic limit is not the only step in getting to the phase transition, as the phase transition only happens at the right value of the magnetic field. In the symmetric case, we can easily argue by symmetry that for $B = 0$, we must have $M(\beta, 0) = 0$. However for very small B, the magnetization can still be positive. This leads us to the concept of the spontaneous magnetization. This is the magnetization when the magnetic field is taken to zero from above ($B \searrow 0$). In this case, it is important that the limit of $n \to \infty$ is taken before the limit of $B \searrow 0$. If we now look at the spontaneous magnetization, as a function of β, it turns out that there is a so-called critical temperature β_c at which the spontaneous magnetization suddenly becomes positive.

However, in the asymmetric Ising model, the magnetization is in general not zero for $B = 0$. For a fixed temperature, we can find a magnetic field at which the magnetization is zero. Therefore, we can define a curve of the magnetic fields B as a function of β, at which the magnetization is equal to zero. However, above the critical temperature β_c, the magnetization will only be equal to zero if we are exactly on this critical curve, while a small deviation will give a positive or negative magnetization. Therefore, the critical curve is defined as

$$C = \{(\beta, B) \mid \forall \epsilon > 0 : M(\beta, B + \epsilon) > 0, M(\beta, B - \epsilon) < 0\}. \tag{2.16}$$

With this definition, a mathematical description of the critical curve is given. This critical curve will also be denoted as the function $B_0(\beta)$. Later, it will turn out that it can be quite difficult to determine this critical curve using simulations for high β, as this definition only formally holds in the thermodynamic limit. Therefore, in the simulations we will define the critical curve to be the curve at which the magnetization switches between positive and negative, given an initial condition of $M = 0$. The effect of these different definitions will be discussed in Section 7.6.

Using the critical curve, we can also give a formal definition of the spontaneous magnetization and the critical temperature. These are now given by respectively

$$M(\beta, B_0(\beta)^+) = \lim_{B \searrow B_0(\beta)} M(\beta, B) \tag{2.17}$$

and

$$\beta_c = \inf\{\beta \mid M(\beta, B_0(\beta)^+) > 0\}. \tag{2.18}$$

Now the critical point is defined as (β_c, B_c), where $B_c = B_0(\beta_c)$. One of the goals of this project is to find this critical point in different situations. Here it becomes clear that the asymmetry literally adds an extra dimension to the search for the critical point, as in the symmetric situation we always have $B_c = 0$.

One other interesting aspect of the Ising model is the behaviour of the magnetization and susceptibility close to the critical point. This can be described by critical exponents. In this project the critical exponents $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are considered. One other critical exponent that is related to the magnetization is $\boldsymbol{\delta}$. These exponents describe the behaviour of the magnetization and the susceptibility around the critical point. We use the definitions as defined by Van der Hofstad [2]. Then the critical exponents are given by

$$M(\beta, B_0(\beta)^+) \sim (\beta - \beta_c)^{\boldsymbol{\beta}}, \qquad \text{for } \beta \searrow \beta_c, \tag{2.19}$$

$$M(\beta_c, B) \sim (B - B_0(\beta_c))^{1/\boldsymbol{\delta}}, \qquad \text{for } B \searrow B_0(\beta_c), \tag{2.20}$$

$$\chi(\beta, B_0(\beta)^+) \sim (\beta_c - \beta)^{-\boldsymbol{\gamma}}, \qquad \text{for } \beta \nearrow \beta_c, \tag{2.21}$$

$$\chi(\beta, B_0(\beta)^+) \sim (\beta - \beta_c)^{-\boldsymbol{\gamma}'}, \qquad \text{for } \beta \searrow \beta_c. \tag{2.22}$$

Here the $\sim$ sign denotes that the ratio of the right and left hand side is bounded away from zero and infinity in the given limit. Furthermore, two different exponents are specified for the susceptibility. In the physics literature, these are usually assumed to be equal. In mathematical literature, it is proven for some cases that they are equal, while in other cases they are suspected to be equal. In this thesis, we will not make any distinction between the two and in some methods, the assumption will be made that they are equal, so $\boldsymbol{\gamma}' = \boldsymbol{\gamma}$. Furthermore, in this project we mainly attempt to find the critical exponents $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. For a further research, it would be interesting to make a better comparison between $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ or determine $\boldsymbol{\delta}$.

3 Configuration model and trees

The model that will be analyzed further in this report is the configuration model. This is a very basic model to generate graphs. For us, it will be interesting to see how we can apply the Ising model on configurations models. For example Bruggeman et al. [1] only considered the complete graph or graphs with many connections. First of all, the configuration model is more realistic than the complete graph. One other advantage of the configuration model is that the degrees of the vertices can be chosen beforehand. This gives the possibility to use a real life degree distribution for this model. In this section, the configuration model will be defined mathematically, following the description of the model by Van der Hofstad in his book Random Graphs and Complex Networks [3]. Then a degree distribution will be defined that will be used in the simulations that are explained in Section 7. The goal is to define a degree distribution that satisfies the requirements for the mathematical analysis, but also resembles the real world in some sense.

3.1 Configuration model

First of all, the idea of the configuration model is to fix a degree sequence for the vertices, where a degree is given for each vertex. Then all vertices are given that number of half-edges. Finally, all half-edges are randomly connected, in this way creating a graph. To formalize the method of generating a random graph using the configuration model, we first consider a degree sequence $(d_i)_{i \in V_n}$, where V_n is the set of all positive integers less than or equal to n. The goal is to generate a vertices simple graph where the vertices have degrees according to this degree sequence. A simple graph is defined as a graph where there is at most one edge between each pair of vertices and there are no self loops, i.e. edges connecting a vertex to itself. The method that will be described does, however, allow for double edges and self loops. Therefore we will use one of the several existing methods to deal with those.

First of all, we need to choose the degree sequence such that the total degree

$$\ell_n = \sum_{i \in V_n} d_i \tag{3.1}$$

is even. If the degree sequence is created randomly and turns out to have a odd total degree, we just increase the degree of the first vertex by one. For the considered degree sequences and large enough values of n this will have a negligible effect. This is to make sure that all half-edges can be paired. The next step is to start with the first half edge of the first vertex. This half edge is randomly paired with one of the remaining $\ell_n - 1$ half edges. Then the first half edge and the one it has connected to are removed from the set of half edges. Then the next half edge from this set is randomly paired with another half edge and this process continues until there are no half edges left. Using this method, a so-called multi-graph is created, where self loops and double edges are allowed. However, for simplicity, we will only use the erased configuration model, where all self loops and double edges are removed after the creation of a graph. In this way, the resulting degree sequence of the graph will be a bit lower in general. However, it can be shown that the difference in the degree distributions of the configuration model and the erased configuration model becomes very small for large enough system sizes [3]. The graph that is created using the erased configuration model will be called $\mathrm{CM}_n(\mathbf{d})$, where $\mathbf{d} = (d_i)_{i \in V_n}$ is the degree sequence.

One important aspect of graphs created by the configuration model is that they are locally tree like. This will be explained in Section 5. In Section 7, Monte Carlo simulations will be used on the configuration model to find out more information about the critical behaviour of the Ising model on these graphs. In each simulation, a graph needs to be created with the configuration model. One way to do this, is to follow the steps described in this section. However, it turns out that the described method is not very efficient. Therefore, when implementing a configuration model in a simulation, we use a more efficient method to create the graphs. Now we make a list of all half edges of all vertices. Then this list is randomly shuffled, keeping track of which half edge comes from which vertex. The half edges can then be paired by connecting each consecutive pair of half edges in this list. Note that the graphs that are created in this way are still made with the configuration model.

For these simulations it is also important to define a degree distribution to create the graphs. Therefore, in the next section, we will define a default degree distribution that will be used in this thesis.

3.2 Degree distribution

First of all a good choice for a degree distribution is a Poisson distribution. One advantage of a Poisson distribution is the fact that all of its moments are finite. This is important for some of the proofs on locally tree-like graphs that will follow later. One observation that we can make, is that for large degrees, the Poisson distribution will look like a discrete normal distribution. However, if the degree of a vertex represents the number of connections of a person in the real world, a power law distribution seems more reasonable. This is because, in the real world, the number of acquaintances of a person tends to follow a power law distribution [4]. In order to get both a power law distribution and the advantages of a Poisson distribution, a mixed Poisson distribution is a good solution. This is a Poisson distribution with a random variable as mean. In this case a Pareto distribution is used as mean for the Poisson distribution. The Pareto distribution is a much used distribution following a power law.

Now to get a mixed Poisson, the distribution of the degree will be denoted as $D \sim \mathrm{Poi}(P)$. This means that the random variable D is distributed with a Poisson distribution with as mean the random variable P. Now P will be a version of a Pareto distribution that dependents on parameters α and $\overline{D}$, where α denotes the strength of the power law and $\overline{D}$ is the average degree of the graph.

The cumulative distribution function of a random variable P with a standard (type 1) Pareto distribution is given by

$$F_P(x) = \mathbb{P}(P \leq x) = \begin{cases} 0 & \text{for } x < x_m, \\ 1 - \left(\frac{x_m}{x}\right)^\alpha & \text{for } x \geq x_m. \end{cases} \tag{3.2}$$

In this definition x_m is the minimum possible value for P. The expectation of the Pareto distribution is given by $\alpha x_m/(\alpha - 1)$. Since the mean of the random variable

P will result in the same mean of D, x_m is chosen in such a way that the mean will be equal to $\overline{D}$. This gives

$$x_m = \frac{(\alpha - 1)\overline{D}}{\alpha}. \tag{3.3}$$

This choice of x_m results in the probability density function of P, given by

$$f_P(x) = \frac{\alpha}{x^{\alpha+1}} x_m^\alpha \qquad \text{for } x \geq x_m. \tag{3.4}$$

When this random variable is combined with the Poisson distribution the probability mass function of D can be determined by

$$\begin{aligned} f_D(k) = \mathbb{P}(D = k) &= \int_{x_m}^{\infty} \mathbb{P}(D = k | P = x) f_P(x) dx \\ &= \int_{x_m}^{\infty} \frac{e^{-x} x^k}{k!} \frac{\alpha}{x^{\alpha+1}} x_m^\alpha dx \\ &= \frac{\alpha x_m^\alpha}{k!} \int_{x_m}^{\infty} e^{-x} x^{k-\alpha-1} dx \\ &= \frac{\alpha x_m^\alpha}{k!} \Gamma(k - \alpha, x_m), \end{aligned} \tag{3.5}$$

where $\Gamma(s, x)$ is the upper incomplete gamma distribution.

One last distribution that is needed later, is the size-biased version of this degree distribution. This will be needed to describe the degree of a vertex chosen uniformly from a uniformly chosen edge. The distribution of the size-biased degree D^* is given by

$$\mathbb{P}(D^* = k) = \frac{k\mathbb{P}(D = k)}{\sum_{i=0}^{\infty} k\mathbb{P}(D = k)} = \frac{k\mathbb{P}(D = k)}{\overline{D}}. \tag{3.6}$$

Here the sum in the denominator is by construction of D equal to the average degree. Now these definitions will be used later as the default distribution to analyze the random trees and the configuration model.

4 Ising model on the complete graph

A first step in analysing the asymmetric Ising model is to consider the complete graph, in which all vertices are connected to all other vertices. The goal of this section is to determine the critical curve and critical point for the complete graph. First of all, consider the complete graph $G_n = (V_n, E_n)$. The set of vertices is still defined as $V_n = \{1, 2, ..., n\}$ where $n \in \mathbb{N}$ denotes the system size. Now the set of edges contains every pair $\{i, j\}$, where $i, j \in V_n$ and $i \neq j$. In the first part of this section, we will give the results that we obtain on the complete graph. In the second part, we will derive the partition function using a mean field approximation, which turns out to give exact results in the thermodynamic limit.

4.1 Results on the complete graph

Using the setting as described above, it turns out that the partition function is given by

$$\begin{aligned} Z_n(\beta, B) =& \exp\left(\beta\left(\frac{1}{2}Jn(n-1)A^2 - \frac{1}{2}J\Delta^2 n + BnA - \frac{1}{2}J\Delta^2 n^2\langle\sigma\rangle^2\right)\right) \\ &\cdot \Big(2\cosh\left(\beta\Delta(JA(n-1) + Jn\Delta\langle\sigma\rangle + B)\right)\Big)^n. \end{aligned} \tag{4.1}$$

Here $\langle\sigma\rangle$ stands for the ensemble average of σ_i of all vertices. The derivation of this partition function is shown in Section 4.2. With the partition function, the pressure per particle can be calculated. Here we use $J = 1/n$, to make sure that in the thermodynamic limit, the Hamiltonian does not blow up. The idea of defining J as the reciprocal of the degree, will also be used later when we consider different kinds of graphs. In this way, the pressure per particle is given by

$$\begin{aligned} \psi_n(\beta, B) =& \beta\left(\frac{1}{2}(1-\frac{1}{n})A^2 - \frac{1}{2}\Delta^2\frac{1}{n} + BA - \frac{1}{2}\Delta^2\langle\sigma\rangle^2\right) \\ &+ \log(2) + \log\left(\cosh\left(\beta\Delta(A(1-\frac{1}{n}) + \Delta\langle\sigma\rangle + B)\right)\right). \end{aligned} \tag{4.2}$$

To get the pressure per particle in the thermodynamic limit, we take the limit of $n \to \infty$. This gives

$$\phi(\beta, B) = \beta\left(\frac{1}{2}A^2 + BA - \frac{1}{2}\Delta^2\langle\sigma\rangle^2\right) + \log(2) + \log(\cosh(\beta\Delta(A + \Delta\langle\sigma\rangle + B))). \tag{4.3}$$

Now, using equation (2.15), we can conclude that the magnetization $M(\beta, B)$ is equal to one of the solutions of

$$M(\beta, B) = \tanh(\beta\Delta(A + B + \Delta M(\beta, B))), \tag{4.4}$$

where we used the fact that $\langle\sigma\rangle$ is equal to the magnetization. This equation a very interesting result, because from this simple equation, a lot of conclusions can be drawn. First of all, we observe that if we want a solution where the magnetization is zero, we need to have $B + A = 0$, because the hyperbolic tangent equals zero if its argument is zero. This immediately gives us the critical curve for the complete graph

$$B_0(\beta) = -A. \tag{4.5}$$

Hence, the critical curve is independent of the temperature. This can be explained by realizing that in this situation, the asymmetry causes an addition to the external field of A. Furthermore, if the hyperbolic tangent in equation (4.4) is plotted against the line $y = \langle\sigma\rangle$, there can be up to three intersections. If the limit is taken of B from above, the highest intersection will be the solution. By changing the temperature β, the maximum slope of the hyperbolic tangent changes. In this way, we find the critical temperature $\beta_c = 1/\Delta^2$. At that temperature, the slope of the right hand side of equation (4.4) is exactly equal to one on the critical curve. For larger values of β, there will be three intersections. This difference is shown in Figure 2, where the right and left hand side of equation (4.4) are plotted.

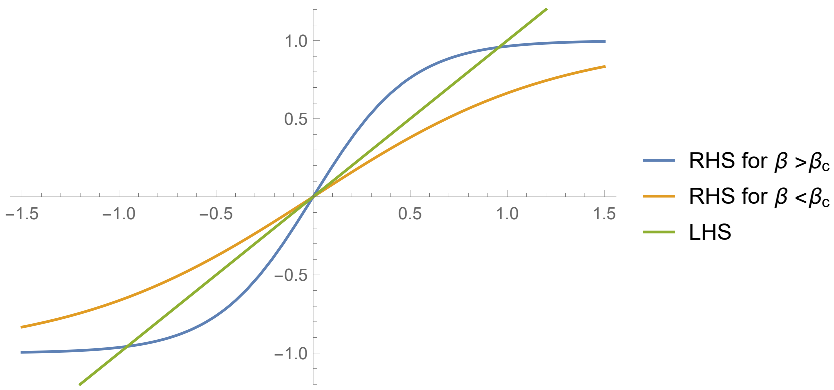
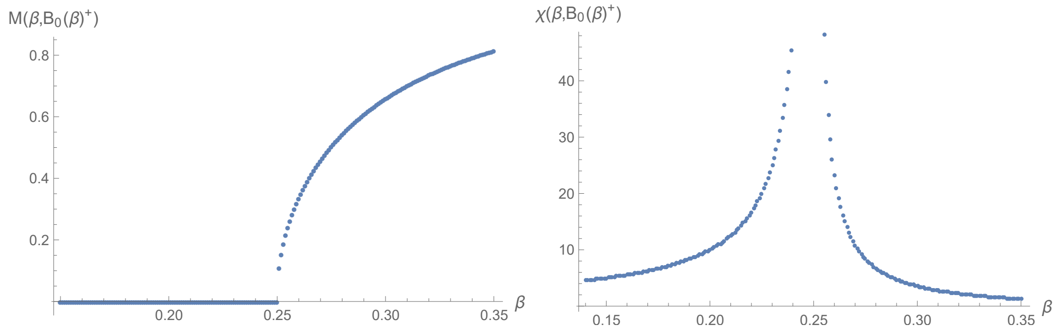


Figure 2: *Plot of the left hand side and the right hand side of the fixed point equation for the complete graph. Here the magnetic field is taken at the critical curve and the inverse temperature is taken above and below the critical temperature.*

All in all, we found the critical curve and the critical temperature. Furthermore, as the solution of the asymmetric model is only a translation of the solution of the symmetric model, the critical exponents are the same as in the symmetric case. Hence the critical exponents are $\boldsymbol{\beta} = 0.5$ and $\boldsymbol{\gamma} = 1$, as shown by Van der Hofstad [2, Theorem 5.7]. Now we can use equation (4.4) to get a graph of the spontaneous magnetization. This is done by using $B = -A$ and then take the largest solution to the fixed point equation. By taking the derivative of the magnetization to B, we also find the susceptibility. These are shown in Figure 3.



(a) *Spontaneous magnetization*

(b) *Susceptibility*

Figure 3: *Spontaneous magnetization and susceptibility for the asymmetric Ising model on the complete graph. Here $c = 3$ and $d = 1$ are used, resulting in a critical temperature $\beta_c = 0.25$*

4.2 Derivation partition function

In this section, a proof will be given of equation (4.1). To derive this partition function, a mean field approximation will be used. This method is often used in physics. For more information about this method, see for example the explanation in chapter 5.4 of Introduction to Modern Statistical Mechanics [5]. It will turn out that the results that are obtained using a mean field approximation are equal to the exact solution in the thermodynamic limit for the complete graph. It is also possible to prove the partition function in a more mathematically sound fashion [2].

To begin the proof, the Hamiltonian for the complete graph can be written as

$$H_n(\vec{x}) = -\frac{1}{2}J\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} x_i x_j - B\sum_{i=1}^{n} x_i, \tag{4.6}$$

where the factor of 1/2 is added to prevent us from summing over every edge twice. Using $x_i = A + \Delta\sigma_i$, we can rewrite this Hamiltonian to the symmetric model as

$$\begin{aligned} H_n(\vec{x}) =& -\frac{1}{2}J\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(A^2 + A\Delta(\sigma_i+\sigma_j) + \Delta^2\sigma_i\sigma_j) - B\sum_{i=1}^{n}(A+\Delta\sigma_i) \\ =& -\frac{1}{2}Jn(n-1)A^2 - BnA + \frac{1}{2}Jn\Delta^2 \\ &- \frac{1}{2}J\Delta^2\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j - \Delta(JA(n-1)+B)\sum_{i=1}^{n}\sigma_i. \end{aligned} \tag{4.7}$$

In this equation, the term $\frac{1}{2}Jn\Delta^2$ is added to let the double sum run over all i and j. Now this double sum can be rewritten using the mean field approximation. This is a method that is used in physics to simplify the Hamiltonian. We write $\sigma_i = \langle\sigma\rangle + \delta\sigma_i$, where we make use of the ensemble average $\langle\sigma\rangle$ and a fluctuation $\delta\sigma_i$. Note that this ensemble average is independent of the spin configuration, as it is the expectation over the Ising measure. Using this idea the double sum can be expressed as

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\langle\sigma\rangle+\delta\sigma_i\right)\left(\langle\sigma\rangle+\delta\sigma_j\right)\\ &= n^2\langle\sigma\rangle^2+2n\langle\sigma\rangle\sum_{i=1}^{n}\delta\sigma_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\delta\sigma_i\delta\sigma_j\\ &= n^2\langle\sigma\rangle^2+2n\langle\sigma\rangle\sum_{i=1}^{n}(\sigma_i-\langle\sigma\rangle)+\sum_{i=1}^{n}\sum_{j=1}^{n}\delta\sigma_i\delta\sigma_j\\ &= -n^2\langle\sigma\rangle^2+2n\langle\sigma\rangle\sum_{i=1}^{n}\sigma_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\delta\sigma_i\delta\sigma_j.\end{aligned} \tag{4.8}$$

Up to now, there has been no approximation. The mean field approximation assumes the double sum over the fluctuations vanishes when the system size is large enough. In the case of a complete graph, it turns out that this approximation is exact in the thermodynamic limit, as every vertex is connected to all other vertices. Using this result, the Hamiltonian becomes

$$\begin{aligned}H_n(\vec{x}) =& -\left(\frac{1}{2}Jn(n-1)A^2+BnA-\frac{1}{2}J\Delta^2n-\frac{1}{2}J\Delta^2n^2\langle\sigma\rangle^2\right)\\ &-\big(JA\Delta(n-1)+J\Delta^2n\langle\sigma\rangle+B\Delta\big)\sum_{i=1}^{n}\sigma_i\\ =& -Q(B)-R(B)\sum_{i=1}^{n}\sigma_i,\end{aligned} \tag{4.9}$$

where $Q(B)$ and $R(B)$ are defined to simplify the calculations. This Hamiltonian can be used to find that the partition function equals

$$\begin{aligned}Z_n(\beta,B) =& \sum_{\vec{x}\in\{-d,c\}^n}\exp(-H(\vec{x}))\\ =&\mathrm{e}^{-Q}\sum_{\vec{x}\in\{-1,1\}^n}\exp\left(-R\sum_{i=1}^{n}\sigma_i\right)\\ =&\mathrm{e}^{-Q}\prod_{i=1}^{n}\sum_{\sigma_i=\pm1}\exp(-R\sigma_i)\\ =&\mathrm{e}^{-Q}\left(\exp(R)+\exp(-R)\right)^n\\ =&\mathrm{e}^{-Q}(2\cosh(R))^n\\ =&\exp\left(\beta\left(\frac{1}{2}Jn(n-1)A^2+BnA-\frac{1}{2}J\Delta^2n-\frac{1}{2}J\Delta^2n^2\langle\sigma\rangle^2\right)\right)\\ &2^n(\cosh\big(\beta JA\Delta(n-1)+\beta J\Delta^2n\langle\sigma\rangle+\beta B\Delta\big))^n.\end{aligned} \tag{4.10}$$

This ends the proof of equation (4.1). Taking the logarithm and dividing by n directly results in the pressure per particle and taking the limit $n\to\infty$ gives equation (4.3). Note that by defining $\langle\sigma\rangle$ as the ensemble average of the spins, this is equal to the magnetization.

5 Ising model on locally tree-like graphs

As explained before, the goal of the project is to consider the Ising model on the configuration model. It will turn out that in the thermodynamic limit, the graphs that are created by the configuration model are locally tree-like. This helps a lot with analyzing the Ising model on these graphs, because there are a couple of tricks that we can use on trees. Therefore, in this section, the mathematical notion of a tree will be explained, as well as what it means for a graph to be locally tree-like. The Ising model will then be analyzed on these trees. Firstly, we will draw some general conclusions about these trees. Afterwards, a division will be made between regular graphs (with fixed degrees) and independent and identically distributed (i.i.d.) random graphs, where the degrees are drawn from a probability distribution.

5.1 Trees and locally tree-like graphs

First of all, a mathematical tree can be seen as a graph with no loops in it. We can generate a tree by considering a branching process. In a branching process, we start with one single vertex. This vertex gets connected to the first generation of vertices. This process continues with each vertex of the next generation. The graph that is obtained in this way is called a tree.

To make this a little more precise, we first define the root ϕ of the tree. Now let D be a random variable taking integer values k with probabilities p_k for $k \in \mathbb{N}$. We assume that $p_0 = 0$ and naturally, the sum of all p_k equals one. Furthermore, we define the so-called size biased degree distribution by the random variable D^* with as probability mass function

$$p_k^* = \frac{kp_k}{\sum_{l \geq 0} lp_l}. \tag{5.1}$$

Now the root gets degree distribution D, while every other vertex will have degree distribution D^*. This means that each vertex, except for the root, has an offspring of $k-1$ with probability p_k^*. The reason that the distribution changes when we move away from the root, comes from the fact that for all other vertices we already know that the degree is at least one. Therefore, we must use the degree distribution of a vertex that is connected to a random edge. It turns out that the size biased distribution is the distribution of a vertex that is connected to a randomly chosen edge. This is because for a vertex that is connected to a random edge, the probability that it has a certain degree increases for higher degrees, as the edge is more likely to be connected to a vertex with a higher degree. The tree defined in this way can be denoted by $T(p, p^*, t)$, where t is the number of generations from the root. This means that $t = 0$ implies a tree that only consists of the root. Note that $T(p, p^*, t)$ defines a random tree, as the degrees are still random variables and are thus not fixed. Sometimes we will write $T(t)$ to denote the tree. In that case the degree distribution is either not important, or it is obvious from the context.

In the rest of this chapter, the Ising model is analyzed on trees. However, before we do that, we will give an idea of why these results can also be applied on graphs that are not trees. The following definition of local convergence is taken from Dembo and Montanari [6]. First of all, let $G_n = (V_n, E_n)$ be a graph with vertex set V_n

consisting of n vertices and with edge set E_n. Now we define the distance $d(i, j)$ between two vertices $i, j \in V_n$ as the length of the shortest path along the graph between the vertices. With this notion of distance, we can define the ball $B_i(t)$ around vertex i to be the subgraph of G_n, consisting of vertex i and all vertices that have a distance to i of at most t. Lastly, we say that $T_1 \simeq T_2$ if the two trees can become identical by only relabeling the non-root vertices of one of the graphs.

Using these definitions, we say that the sequence of graphs $(G_n)_{n\in\mathbb{N}}$ converges locally to the random tree $T(p, p^*, \infty)$ if for all t and any (fixed) rooted tree $S(t)$

$$\lim_{n\to\infty} \mathbb{P}_n \{\mathrm{B}_i(t) \simeq S(t)\} = \mathbb{P}\{\top(P, \rho, t) \simeq S(t)\}, \tag{5.2}$$

where $B_i(t)$ is a ball in G_n centered around a uniformly chosen vertex $i \in V_n$. Furthermore, $\mathbb{P}$ and $\mathbb{P}_n$ are defined by the probability distributions on the random graphs $B_i(t)$ and $T(p, p^*, t)$. This definition tells us that any finite subgraph of G_n, can be seen as a tree for large enough n.

One other constriction that we need to add to the graphs that we consider is that they need to be uniformly sparse. This constriction makes sure that for increasing system sizes, the degrees of the vertices with the highest degrees, do not increase too fast. Let d_i be the degree of vertex i. Then we say that G_n is uniformly sparse if

$$\lim_{l\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{i\in V_n} d_i \mathbb{1}(d_i \geq l) = 0. \tag{5.3}$$

Here $\mathbb{1}$ denotes the indicator function. It turns out that using these definitions, it can be shown that the configuration model gives locally tree-like graphs. Therefore, we will now add the Ising model to trees.

5.2 Tree recursion

The next step is to see what happens if we add spins to the vertices of a tree. The approach of Dembo and Montanari [6] to find the pressure per particle uses a recursion relation on the expected spin in the generations of the tree. Van der Hofstad also uses this recursion to find the magnetization in Stochastic processes on random graphs [2]. Before we get into this recursion, we make a remark on finite trees. Suppose that we consider a finite tree with an average offspring of two vertices. If we have for example 10 generations from the root, about half of the vertices will lie on the boundary of the tree. This is because in each generation from the root, the total number of vertices added is doubled. Therefore, it might well be possible that the boundary of the tree has a large influence on the magnetization of the root. In their paper, Dembo and Montanari show that if the number of generations is taken large enough, the influence of the boundary vanishes. The recursive relation uses this idea. The recursion starts at the boundary of a finite tree and then in each step the effect of the boundary is considered as an addition to the magnetic field on the previous generation. In this way, we can get all the way to the root. In this section, we will give the recursive relation for the asymmetric model and discuss the results that follow from this. In the next sections, a more extensive discussion of the results on regular and i.i.d. random trees is given, followed by a proof of the recursion formula.

The recursive relation of interest, for the asymmetric Ising model, is given by

$$\beta\Delta(JA+h^{(t+1)}) = \beta\Delta(JAD^*+B)+\sum_{i=1}^{D^*-1} \operatorname{arctanh}\left[\tanh\big(\beta J\Delta^2\big)\tanh\Big(\beta\Delta(JA+h_i^{(t)})\Big)\right]. \tag{5.4}$$

Here D^* is a random variable with distribution p^*. Furthermore $h^{(t)}$ are random variables for $t \in \mathbb{N}$ and $h_i^{(t)}$ are i.i.d. copies of $h^{(t)}$. This random variable is related to the expected spin, given only the part of the tree towards the boundary. Every time that t increases, a step is taken towards the root of the tree. Let T be the finite tree that we are considering and let U be a subtree of T, containing the root. Then consider the 'rest' of the graph $W = T \setminus U$ and a vertex $u \in U$ that has a single-edge connection to at least one vertex in W. This means that vertex u is on the boundary of our subtree U. Now we take the maximal subtree of $W \cup \{u\}$, rooted at u, which results in a root magnetization of u, given by

$$\langle\sigma_u\rangle_W = \tanh\Big(\beta\Delta h^{(t)}\Big), \tag{5.5}$$

where $h^{(t)}$ is the random variable that denotes the effective magnetic field of vertex u induced by the tree in $W \cup \{u\}$. This can be understood by considering the graph, consisting only of vertex u. Then the Hamiltonian in equation (2.5) reduces to

$$H_n(\vec{x}) = h^{(t)}An - h^{(t)}\Delta\sigma_u. \tag{5.6}$$

Now if we calculate the expectation of σ_u with the Ising measure on this graph, we get

$$\langle\sigma_u\rangle = \sum_{\sigma_u\in\{-1,1\}} \sigma_u \frac{\beta\Delta h^{(t)}\sigma_u}{\mathrm{e}^{\beta\Delta h^{(t)}}+\mathrm{e}^{-\beta\Delta h^{(t)}}} = \frac{\mathrm{e}^{\beta\Delta h^{(t)}}-\mathrm{e}^{-\beta\Delta h^{(t)}}}{\mathrm{e}^{\beta\Delta h^{(t)}}+\mathrm{e}^{-\beta\Delta h^{(t)}}} = \tanh\Big(\beta\Delta h^{(t)}\Big). \tag{5.7}$$

The proof that from this definition of $h^{(t)}$, the recursion formula in equation (5.4) follows, will be given in the Section 5.5. Now, using the definitions

$$\hat{\beta} = \tanh\big(\beta J\Delta^2\big), \tag{5.8}$$

$$\tilde{h}^{(t)} = \beta\Delta h^{(t)} + \beta JA\Delta, \tag{5.9}$$

$$\xi(\tilde{h}) = \operatorname{arctanh}\left(\hat{\beta}\tanh\Big(\tilde{h}\Big)\right), \tag{5.10}$$

we can simplify the recursion in equation (5.4) to

$$\tilde{h}^{(t+1)} = \beta\Delta(JAD^*+B) + \sum_{i=1}^{D^*-1}\xi(\tilde{h}_i^{(t)}). \tag{5.11}$$

This recursive relation can now be used to give an expression for the magnetization. It turns out that it is not needed to determine the pressure per particle first. This will make use of the fixed point of the recursion. Dembo and Montanari show in their paper [6] that the recursion has a unique fixed point for the symmetric Ising model. In Section 5.3, it will be shown that this is also true on regular trees for the asymmetric Ising model. However, on i.i.d random trees, it is not certain that this fixed point always exists. In Section 5.4, we will show numerical evidence that this fixed point exists. If we assume the existence of a fixed point for now, the magnetization on locally tree-like graphs in the thermodynamic limit is given by

$$M(\beta, B) = \mathbb{E}\left[\tanh\left(\beta\Delta(B + JAD) + \sum_{i=1}^{D} \xi(\tilde{h}_i^*(\beta, B))\right)\right], \tag{5.12}$$

where $\tilde{h}_i^*(\beta, B)$ are i.i.d. copies of the fixed point of equation (5.11). Note that to get the magnetization, almost one extra step in the recursion is made. However, in this last step, the sum runs to D, instead of $D^* - 1$. This comes from the fact that in the last step, we arive at the root magnetization of a uniformly chosen root. As discussed before, the degree of the root is given by the random variable D, while the degrees of all other vertices are given by i.i.d. copies of D^*. Therefore, $D^* - 1$ is used in the recursion, because in each step only the offspring of a vertex in the tree is considered.

5.3 Application on regular trees

When the recursion formula that is discussed in the previous section is applied on regular trees, the recursion can be analysed more easily. In the paper of Dembo and Montanari [6], it is proven that the recursion formula has a unique fixed point for $B > 0$ for the case with $\sigma = \pm 1$. This result can be used for the asymmetric case when the assumption that $B > 0$ can be validated. When the degree of the tree is taken constant, we see that the recursion from equation (5.11) can be written as

$$\begin{aligned}\tilde{h}^{(t+1)} &= \beta\Delta(JAk + B) + (k-1)\,\mathrm{arctanh}\left(\tanh\left(\beta J\Delta^2\right)\tanh\left(\tilde{h}^{(t)}\right)\right) \\ &= \beta\Delta(JAk + B) + (k-1)\xi(\tilde{h}^{(t)}).\end{aligned} \tag{5.13}$$

Here k is the (fixed) degree of all vertices in the graph. As initial condition for the recursion, we use again $\tilde{h}^{(0)} = 0$, which corresponds to 'free boundary conditions'. By the regularity of the trees, we find that the recursion is reduced to a real valued recursion, instead of the stochastic recursion that we need to consider in general. This makes analyzing the recursion much easier. First of all, we can define the entire term $\beta\Delta(JAk + B)$ as some magnetic field $\hat{B}$. By doing this, the recursion can be linked directly to the symmetric recursion equation, where there is only a B term in front of the sum.

Therefore, we can use the existing results from for example Dembo and Montanari, to conclude that there is a unique fixed point to the recursion, given that we obey the conditions. The first condition is that $\hat{B} > 0$, i.e. $B > -JAk$. For the regular tree, this is the only requirement. Of course, as we have translated our problem to the symmetric Ising model, we can flip everything around and conclude that there is also one fixed point for $B < -JAk$. These fixed points can now be determined numerically for a given degree k, temperature β and magnetic field B and constants c and d (from the asymmetric Ising model). For better comparison to the results from the complete graph, we will define $J = 1/k$, which normalizes the interaction term in the Hamiltonian.

One way to determine the fixed point numerically is similar to the method used for the complete graph. We find the intersection between the left hand side and the right hand side of equation (5.13), by using that for the fixed point $\tilde{h}^{(t+1)} = \tilde{h}^{(t)} = \tilde{h}^*$, where $\tilde{h}^*$ is the fixed point. An example of this method is shown in Figure 4, where the intersections are the fixed points. Note that for a fixed magnetic field, there is only one fixed point to which the recursion will converge.

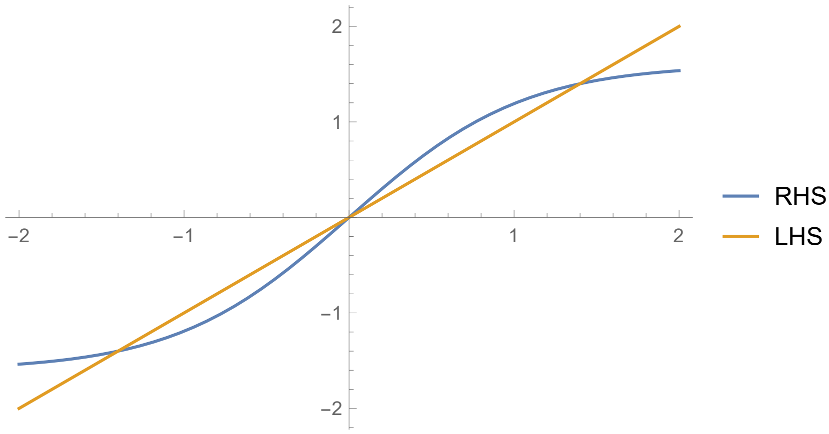


Figure 4: *Plot of the left hand side and right hand side of the fixed point equation for the regular tree in equation* (5.13)*. Here* $B = B_0(\beta)$ *and* β *is taken above the critical point.*

Using the fixed point, the magnetization is given by

$$M(\beta, B) = \tanh\Big(\beta\Delta(B + JAk) + k\xi(\tilde{h}^*)\Big). \tag{5.14}$$

From equation (5.13) and equation (5.14), we can also retrieve the critical behaviour in a similar fashion as for the complete graph. First of all, we note that by symmetry the critical curve must be equal to

$$B_0(\beta) = -AkJ = -A, \tag{5.15}$$

where the last equality holds when we fill in $J = 1/k$, which normalizes the interaction term in the Hamiltonian. Apparently, the critical curve for the regular graph is the same as for the complete graph, if we choose J in a similar way. The next step is to find the critical temperature again. From Figure 4, it becomes clear that again we need to choose β such that the slope of the right hand side of equation (5.13) equals one in the origin. Furthermore, we know that the critical point lies on the critical curve. Solving this yields the critical temperature

$$\beta_c = \frac{1}{J\Delta^2}\operatorname{arctanh}\left(\frac{1}{k-1}\right). \tag{5.16}$$

It is interesting to notice that for in the limit of $k \to \infty$, this expression tends to $1/\Delta^2$ (for $J = 1/k$), which is the same as the critical temperature for the complete graph. Lastly, we are still interested in the critical exponents. However, in the regular case, there is no reason to expect different critical exponents, as we can again rewrite the system to a symmetric form, with the proper substitutions. Therefore, the critical exponents are the same as for the symmetric model.

5.4 Random trees

The next step in the investigation of the Ising model on locally tree-like graphs is to consider the degrees to be random. In the previous section only graphs were considered with a fixed degree, while now the degree of each vertex will be independent and identically distributed. This section will consist of two parts. In the first part, the critical curve will be defined and a numerical method of finding it will be discussed. We will use the degree distribution as defined in Section 3.2. In the second part, we will try to find some bounds on the critical curve by making some mathematical estimations.

5.4.1 Critical curve

For the case where the degrees are i.i.d. random variables, the recursion equation can again be written as

$$\begin{aligned}\tilde{h}^{(t+1)} &= \beta\Delta(JAD^* + B) + \sum_{i=1}^{D^*-1} \operatorname{arctanh}\Big(\tanh\big(\beta J \Delta^2\big)\tanh\Big(\tilde{h}_i^{(t)}\Big)\Big) \\ &= \beta\Delta(JAD^* + B) + \sum_{i=1}^{D^*-1} \xi(\tilde{h}_i^{(t)}),\end{aligned} \tag{5.17}$$

where D^* is the size-biased version of the random variable D. Here $\tilde{h}^{(t)}$ is defined as before in equation (5.9). It is important to realize that now $\tilde{h}^{(t)}$ are not numbers anymore, but they represent random variables. Therefore, the fixed point of the recursion will also be a random variable. If we manage to find a fixed point $h^*(\beta, B)$ of this recursion equation, the magnetization can be written as

$$M(\beta, B) = \mathbb{E}\left[\tanh\left(\beta\Delta(JAD + B) + \sum_{i=1}^{D} \xi(h_i^*(\beta, B))\right)\right]. \tag{5.18}$$

Now there are a couple of initial observations that we can make. First of all, Dembo and Montanari showed that the recursion has a unique fixed point for a given β, B and initial condition $h^{(0)}$. Their proof can no longer be applied here, because they use the assumption that the sign of the term in front of the sum is known. However, in our case this term contains randomness from the degree distribution. One option is to choose the magnetic field such that $\beta\Delta(JAD + B) > 0$ for all possible values of D. However, it turns out that with that choice, we will not be able to get to the critical curve. Recall that we defined the critical curve as

$$C = \{(\beta, B) \mid \forall \epsilon > 0 : \ M(\beta, B + \epsilon) > 0, M(\beta, B - \epsilon) < 0\}. \tag{5.19}$$

Note that in this case, the initial condition $\tilde{h}^{(0)}$ can be very important. We know that above the critical point, there are two possible solutions for the magnetization. Depending on the magnetic field, the system will converge to one of these solutions. For now, we will assume that $\tilde{h}^{(0)} = 0$, as in the regular setting this boundary results in a magnetization of 0 on the critical curve.

For this setting, it turns out to be very hard to get an expression for the critical curve. Therefore, we will first use a numerical method, described by Mariana Olvera-Cravioto [7], in order to get an estimation of the distribution of the fixed point of the recursion. This method creates a sample pool of realizations of $\tilde{h}^{(t)}$ for each generation. For each element in this set of samples, a size-biased degree is randomly chosen according to its distribution. Then one realization of $\tilde{h}^{(t)}$ can be calculated by taking the sum over randomly chosen elements from the previous generation. It turns out that in most cases, about 10 generations is enough to get a very stable distribution. The last sample pool is therefore used as the distribution of the fixed point. The magnetization can then also be determined by making a large sample of the part inside the expectation in equation (5.18). Then taking the average gives the estimation of the magnetization. With this, we use a simple optimization algorithm to determine for which magnetic field B, the magnetization switches between positive and negative. An example of the resulting critical curve is shown in Figure 5. There we see a decrease of the critical curve for $\beta < \beta_c$ and afterwards a slight increase. The critical point for the complete graph with the same parameters is $\beta_c = 0.25$.

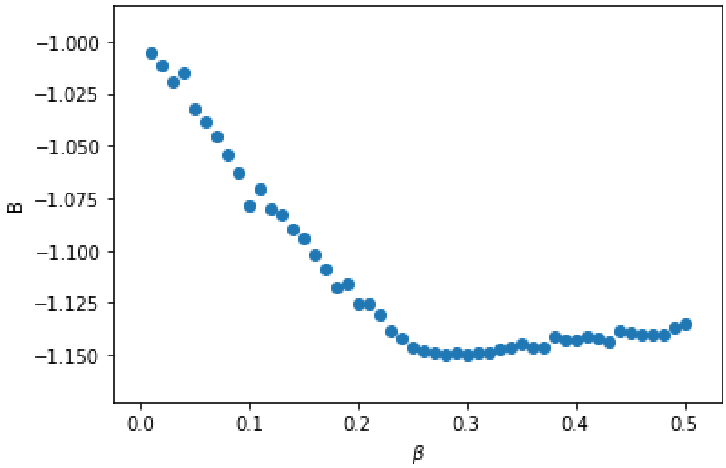


Figure 5: *Critical curve for the asymmetric Ising model with $(c, d) = (3, 1)$. The degrees are distributed with a mixed Poisson distribution with a Pareto random variable with $\alpha = 5$ an average degree of 10.*

5.4.2 Bounds on the critical curve

To get a more mathematical idea of where the critical curve lies, there are some bounds that we can find for the critical curve. First of all, we want to get rid of the hyperbolic tangent in the expectation of $M(\beta, B)$. Therefore we define $\hat{M}(\beta, B)$ by

$$\begin{aligned}\hat{M}(\beta, B) &= \mathbb{E}[\beta\Delta(B + JAD) + \sum_{i}^{D} \xi(\tilde{h}^*(\beta, B))] \\ &= \beta\Delta(B + JA\mathbb{E}[D]) + \mathbb{E}[D]\mathbb{E}[\xi(\tilde{h}^*(\beta, B))].\end{aligned} \tag{5.20}$$

To bound the critical curve, we need to make three assumptions. Proving that these assumptions hold, or perhaps only hold in certain situations, is still an open problem.

Assumption 5.1 *The sign of* $M(\beta, B)$ *equals the sign of* $\hat{M}(\beta, B)$.

Assumption 5.2 *The sign of* $\mathbb{E}[\xi(\tilde{h}^*(\beta, B))]$ *equals the sign of* $\mathbb{E}[\tilde{h}^*(\beta, B)]$.

Assumption 5.3 *The expectation of* $\xi(\tilde{h}^*(\beta, B))$ *can be bounded by*

$$\mathbb{E}[\xi(\tilde{h}^*(\beta, B)] \leq \hat{\beta}\mathbb{E}[\tilde{h}^*(\beta, B)]. \tag{5.21}$$

Note that these assumptions are quite easy to show for the symmetric Ising model. However, they are more tricky in the asymmetric case, as now the degree dependent term in front of the sum can be negative for some realizations of D. Using these assumptions, we can easily find a rough upper and lower bound for the critical curve. First, let us assume that $B = -JA\mathbb{E}[D]$. This reduces $\hat{M}(\beta, B)$ to $\hat{M}(\beta, B) = \mathbb{E}[D]\mathbb{E}[\xi(\tilde{h}^*(\beta, B))]$. Then we can take the expectation on both sides of the recursion equation from equation (5.17) to get

$$\begin{aligned}\mathbb{E}[\tilde{h}^{(t+1)}] &= \mathbb{E}[\beta\Delta(JAD^* + B) + \sum_{i=1}^{D^*-1} \xi(\tilde{h}^{(t)})] \\ &= \beta\Delta JA(\mathbb{E}[D^*] - \mathbb{E}[D]) + \mathbb{E}[D^* - 1]\mathbb{E}[\xi(\tilde{h}^{(t)})],\end{aligned} \tag{5.22}$$

where in the last step, the chosen magnetic field is filled in. Now, assuming an initial $\tilde{h}^{(0)} = 0$, we observe that $\mathbb{E}[\tilde{h}^{(t)}] > 0$ for all t, as the first term in the recursion above is a fixed positive number. Hence, we find that $\hat{M}(\beta, B) > 0$ and by assumption 5.1 that $M(\beta, B) > 0$. Therefore, the choice of $B = -JA\mathbb{E}[D]$ is an upper bound for the critical curve.

The same idea will now be used to show that $B = -JA\mathbb{E}[D^*]$ is a lower bound for the critical curve. Now, substituting this magnetic field in the recursion equation and taking the expectation gives

$$\begin{aligned}\mathbb{E}[\tilde{h}^{(t+1)}] &= \mathbb{E}[\beta\Delta(JAD^* + B) + \sum_{i=1}^{D^*-1} \xi(\tilde{h}^{(t)})] \\ &= \mathbb{E}[D^* - 1]\mathbb{E}[\xi(\tilde{h}^{(t)})].\end{aligned} \tag{5.23}$$

Now, by taking the initial condition of the recursion again equal to zero, we find that $\mathbb{E}[\tilde{h}^{(t)}] = 0$ for all t. Therefore, we get

$$\hat{M} = \beta\Delta JA(\mathbb{E}[D] - \mathbb{E}[D^*]) < 0. \tag{5.24}$$

Using assumption 5.1 again, we also have that $M(\beta, B) < 0$. Therefore, $B = -JA\mathbb{E}[D^*]$ is a lower bound for the critical curve.

Ideally, to find the critical curve, we would take $M(\beta, B) = 0$ and solve for B. As it is not possible to solve this directly, we can use the assumptions to find a tight upper bound for $\hat{M}(\beta, B)$. If this upper bound is set equal to zero and solved for B as a function of β, we know that $\hat{M}(\beta, B)$ and thus also M will be less than or equal to zero at that magnetic field. Therefore this magnetic field will be a lower bound for the critical curve. First of all, we use the inequality in assumption 5.3

and a substitution of the recursion to get

$$\begin{aligned}\mathbb{E}[\xi(\tilde{h}^*(\beta,B))] &\leq \hat{\beta}\mathbb{E}[\tilde{h}^*(\beta,B)] \\ &= \hat{\beta}(\beta\Delta(B+JA\mathbb{E}[D^*]) + \mathbb{E}[D^*-1]\mathbb{E}[\xi(\tilde{h}^*(\beta,B))]) \\ &= \hat{\beta}(\beta\Delta(B+JA(\nu+1)) + \nu\mathbb{E}[\xi(\tilde{h}^*(\beta,B))]).\end{aligned} \tag{5.25}$$

Substituting this inequality l times in equation (5.20), gives

$$\hat{M}(\beta,B) \leq \beta\Delta(B+JA\mathbb{E}[D])+\mathbb{E}[D](\beta\Delta(B+JA(\nu+1)))\hat{\beta}\frac{1-(\hat{\beta}\nu)^l}{1-\hat{\beta}\nu}+\beta J\Delta^2\mathbb{E}[D](\hat{\beta}\nu)^l. \tag{5.26}$$

Here we have made the assumption that $\hat{\beta} < 1/\nu$, where ν is defined as $\nu = \mathbb{E}[D^*-1]$. Using this assumption, a geometric series could be used. Now we take the limit of l to infinity, which gives

$$\hat{M}(\beta,B) \leq \beta\Delta(B+JA\mathbb{E}[D]) + \mathbb{E}[D](\beta\Delta(B+JA(\nu+1)))\hat{\beta}\frac{1}{1-\hat{\beta}\nu} \tag{5.27}$$

Setting this upper bound of $\hat{M}(\beta,B)$ equal to zero and solving for B as a function of β gives

$$B = \frac{-JA\mathbb{E}[D](1+\hat{\beta})}{1+\hat{\beta}(\mathbb{E}[D]-\nu)}, \tag{5.28}$$

which is now a lower bound for the critical curve. This proof of a lower bound also gave us an idea of where the critical temperature may lie. In the symmetric case, a similar proof was used to show that the critical temperature is given by $\hat{\beta} = 1/\nu$, or equivalently $\beta_c = \frac{1}{J\Delta^2}\,\mathrm{arctanh}(1/\nu)$. However, in the asymmetric case, this proof cannot be used. In Section 7, simulations will be used to find the position of the critical temperature.

To conclude this section about random trees, we will compare the bounds on the critical curve to the numerically found critical curve. In Figure 6, the same critical curve as before is shown, with the three bounds that were mathematically found. It turns out that the critical curve lies close to the bound for this set of parameters.

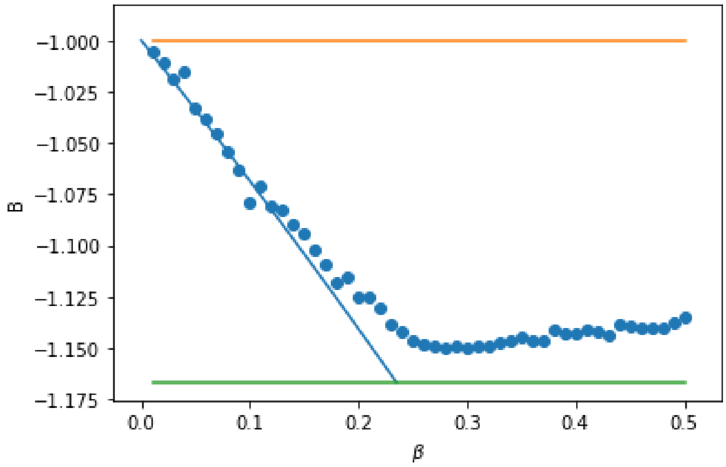


Figure 6: *Critical curve for the asymmetric Ising model with* $(c, d) = (3, 1)$*. The degrees are distributed with a mixed Poisson distribution with a Pareto random variable with* $\alpha = 5$ *and an average degree of 10. The blue dots show the numerically found critical curves and the other lines show upper and lower bounds for the critical curve.*

5.5 Proof of tree recursion

In the analysis of the Ising model on trees, the main ingredient was the recursion formula, as defined in equation (5.4). In this section, we will give a proof of the recursion for the symmetric system, followed by an explanation of how to extend this to the asymmetric variant. First of all, let us consider again the situation, as described in Section 5.2. The idea is to work from the boundary of a finite tree towards the root. Suppose we have some vertex u in the tree. Then there will be one edge that is directed towards the root, and a number of edges directing away from the root, distributed with $D^* - 1$. These last edges are the offspring of vertex u. Now we suppose that each vertex in the offspring has an effective magnetic field $h_i^{(t)}$, that is caused by all their connections, except for the connection with vertex u. This results in an expected magnetization of these vertices of $\langle \sigma_i \rangle_W = \tanh\left(\beta h_i^{(t)}\right)$, where W denotes the subtree without the edge towards the root. Furthermore, we can write $\langle \sigma_u \rangle_W = \tanh\left(\beta h^{(t+1)}\right)$, as u lies on a lower generation. To get an expression of $h^{(t+1)}$ in terms of $h_i^{(t)}$, we consider the tree consisting only of vertex u and its offspring. Using $\vec{\sigma}$ to denote the spins on these vertices, we can write the ensemble average of vertex u as

$$\langle \sigma_u \rangle_W = \frac{1}{Z} \sum_{\vec{\sigma} \in \{-1,1\}^{D^*}} \sigma_u \exp\left(\beta B \sigma_U + \sum_{i=1}^{D^*-1} \beta h_i^{(t)} \sigma_j + \beta J \sigma_u \sum_{i=1}^{D^*-1} \sigma_i \right), \tag{5.29}$$

where we have used $h_i^{(t)}$ as the magnetic fields for the offspring and the fact that the only edges are the ones connecting u to its offspring. Furthermore, Z is the normalization factor for this system, which is equal to the sum above, except for the first σ_u. This expression can now be manipulated to get rid of the sum over $\vec{\sigma}$. In the first step, the term inside the exponent is rewritten. Then the two states of

σ_u are filled in, followed by rewriting the whole expression in products. This gives

$$
\begin{aligned}
\langle\sigma_u\rangle_W =&\frac{1}{Z}\sum_{\vec{\sigma}\in\{-1,1\}^{D^*}}\sigma_u\mathrm{e}^{\beta B\sigma_u}\prod_{i=1}^{D^*-1}\exp\Big(\sigma_i\beta(J\sigma_u+h_i^{(t)})\Big)\\
=&\frac{1}{Z}\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\Big(\mathrm{e}^{\beta(J+h_i^{(t)})}+\mathrm{e}^{-\beta(J+h_i^{(t)})}\Big)-\frac{1}{Z}\mathrm{e}^{-\beta B}\prod_{i=1}^{D^*-1}(\mathrm{e}^{-\beta(-J+h_i^{(t)})}+\mathrm{e}^{\beta(-J+h_i^{(t)})})\\
=&\frac{\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\cosh\Big(\beta(J+h_i^{(t)})\Big)-\mathrm{e}^{-\beta B}\prod_{i=1}^{D^*-1}\cosh\Big(\beta(J-h_i^{(t)})\Big)}{\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\cosh\Big(\beta(J+h_i^{(t)})\Big)+\mathrm{e}^{-\beta B}\prod_{i=1}^{D^*-1}\cosh\Big(\beta(J-h_i^{(t)})\Big)}\\
=&\frac{\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\Big(1+\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)\Big)-\mathrm{e}^{-\beta B}\prod_{i=1}^{D^*-1}\Big(1-\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)\Big)}{\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\Big(1+\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)\Big)+\mathrm{e}^{-\beta B}\prod_{i=1}^{D^*-1}\Big(1-\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)\Big)}\\
=&\tanh\left(\log\left(\mathrm{e}^{\beta B}\prod_{i=1}^{D^*-1}\sqrt{\frac{1+\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)}{1-\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)}}\right)\right)\\
=&\tanh\left(\beta B+\sum_{i=1}^{D^*-1}\operatorname{arctanh}\Big(\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)\Big)\right).
\end{aligned}
\tag{5.30}
$$

If we now take the inverse hyperbolic tangent on both sides, we get the recursion relation

$$
\beta h^{(t+1)}=\beta B+\sum_{i=1}^{D^*-1}\operatorname{arctanh}(\tanh(\beta J)\tanh\Big(\beta h_i^{(t)}\Big)),\tag{5.31}
$$

which is exactly the required recursion for the symmetric Ising model. For the asymmetric model, we get a few extra terms that we need to consider. In equation (5.7), it is already shown that in that case we have $\langle\sigma_u\rangle_W=\tanh\big(\beta\Delta h^{(t+1)}\big)$. Using the Hamiltonian on the graph consisting of u and its offspring, as above, we can also write this as

$$
\begin{aligned}
\langle\sigma_u\rangle_W =&\frac{1}{Z}\sum_{\vec{\sigma}\in\{-1,+1\}^{D^*}}\sigma_u\exp\left((\beta A\Delta(D^*-1)J+\beta B\Delta)\sigma_u\right)\\
&\cdot\exp\left(\sum_{i=1}^{D^*-1}(\beta\Delta h_i+A\beta\Delta J)\sigma_i+\beta\Delta^2J\sigma_u\sum_{i=1}^{D^*-1}\sigma_i\right),
\end{aligned}
\tag{5.32}
$$

which is of the same form as equation (5.29). Therefore, the same derivation can be followed, with a proper substitution. This results in the recursion in equation (5.4) and hereby completes the proof.

6 Annealed Ising model

The situation that is considered up to now is often called the quenched setting of the Ising model. There the influence of the spin configuration is investigated on a given graph. However in order to do this, the graph needed to be fixed. Sometimes it is important to average out over the randomness in the possible graphs. One way to do this is to take the average of the quenched setting. This can be difficult to compute, as in the probability measure, there is a ratio of random variables. Therefore, in physics, sometimes the annealed setting is used to make the calculations easier. In this section, we will have a look at this annealed version of the Ising model, where the randomness of the graphs is taken into account in the definition of the Ising measure. This setting will also be analyzed using Monte Carlo simulations in Section 7.

6.1 Definition of annealed Ising model

In the annealed setting, the expectation over the graph is taken over the numerator and the denominator in the definition of the measure. This results in the measure

$$\begin{aligned} \mu_n^{\mathrm{an}}(\vec{x}) &= \frac{\mathbb{E}\left[\mathrm{e}^{-\beta H_n(\beta,B)}\right]}{\mathbb{E}\left[Z_n(\beta,B)\right]} \\ &= \frac{1}{\mathbb{E}\left[Z_n(\beta,B)\right]}\mathbb{E}\left[\exp\left(\beta J\sum_{\{i,j\}\in E_n} x_i x_j + \beta B\sum_{i\in V_n} x_i\right)\right]. \end{aligned} \tag{6.1}$$

Here the superscript 'an' denotes the fact that we talk about the annealed model. In literature, a distinction is made between two situations. The first situation is the situation with deterministic degrees and the second has i.i.d. random degrees. In the first case, the degrees of the vertices can still be drawn from a distribution, but after drawing them, they stay fixed. This means that the degree sequence is predetermined, which is the setting that can be best compared to the average of the quenched setting. If we consider the degrees to be i.i.d. in the annealed setting, this gives rise to the possibility that the graph changes drastically if that is more energy efficient for the spin configuration. In particular, the system will add many extra edges, because the cost of an extra edge is compensated by a decrease in the Hamiltonian. We will only consider the case with deterministic degrees for the configuration model, where the degree sequence is generated, but the randomness in the pairing of the half edges is still taken into account. We can now define an annealed pressure per particle as

$$\psi_n^{\mathrm{an}}(\beta,B) = \frac{1}{n}\log\mathbb{E}\left[Z_n(\beta,B)\right]. \tag{6.2}$$

In the next section, this pressure per particle will be determined for the asymmetric Ising model.

6.2 Annealed pressure per particle

In this section we will 'translate' the results from Can et al. [8] of the annealed Ising model on the configuration to the asymmetric model. We mainly focus on deriving an expression for the pressure per particle. As said before, we will only consider the case where the degrees of the configuration model are fixed. Therefore define a degree sequence $(d_i)_{i\in V_n}$, where we assume that $d_i > 0$ and that the sum of the degrees ℓ_n is even. Again the set of all vertices will be defined as $V_n = \{1, 2, ..., n\}$. Then, for a subset $A \subset V_n$, the sum of the degrees of the vertices in that set is denoted by

$$\ell_S = \sum_{i\in S} d_i. \tag{6.3}$$

Now, in line with the notation defined before, ℓ_n will be used to denote ℓ_{V_n}. Using this degree sequence, the random variable D_n, denoting the degree of a uniform vertex, can be defined by the empirical distribution F_n, given by

$$F_n(x) = \frac{1}{n}\sum_{i\in V_n} \mathbb{1}_{\{d_i\leq x\}}. \tag{6.4}$$

Furthermore, we define the number of vertices with degree k by n_k and the probability of a uniform vertex to have degree k by $p_k^{(n)}$. These values are given by

$$n_k = |\{i \leq n \mid d_i = k\}| \tag{6.5}$$

and

$$p_k^{(n)} = \mathbb{P}(D_n = k) = \frac{n_k}{n}. \tag{6.6}$$

Here the absolute value signs denote the number of elements in the set, so that n_k is the number of vertices with degree k. Note that by construction we now have $\mathbb{E}[D_n] = \ell_n/n$. In order to work with this degree sequence, some conditions must be made for the random variable D_n.

Condition 6.1 *There exists a distribution function F such that for $n \to \infty$,*

$$D_n \xrightarrow{d} D, \tag{6.7}$$

where D_n and D have distributions F_n and F respectively and $\xrightarrow{d}$ denotes convergence in distribution. Furthermore $F(0) = 0$, or equivalently $\mathbb{P}(D \geq 1) = 1$.

Condition 6.2 *For $n \to \infty$, we have*

$$\mathbb{E}[D_n] \to \mathbb{E}[D]. \tag{6.8}$$

Condition 6.3 *Let $d_{\max}^{(n)} = \max\{d_i \mid i \in V_n\}$ be the maximum degree. Then for $n \to \infty$,*

$$d_{\max}^{(n)} = o(n/\log n). \tag{6.9}$$

In a paper from H. Can et al. [8], the partition function for the deterministic case is determined for the symmetric Ising model. Their method will now be extended to the asymmetric case. First of all, for a given spin configuration, the two sums in the Hamiltonian can be written as

$$\begin{aligned}\sum_{i\in V_n} x_i = \sum_{i\in V_n}(A + \Delta\sigma_i) &= nA + \Delta(2|\sigma_+| - n)\\ &= 2\Delta|\sigma_+| - n(\Delta - A)\end{aligned} \tag{6.10}$$

and

$$\sum_{\{i,j\}\in E_n} x_i x_j = \sum_{\{i,j\}\in E_n} (A^2 + A\Delta(\sigma_i + \sigma_j) + \Delta^2\sigma_i\sigma_j) \\ = \frac{\ell_n}{2}A^2 + A\Delta \sum_{i\in V_n} d_i\sigma_i + \Delta^2(\frac{\ell_n}{2} - 2\mathrm{e}(\sigma_+, \sigma_-)). \tag{6.11}$$

Here $\sigma_\pm$ are the sets of all vertices in V_n with $\sigma_i = \pm 1$ and $\mathrm{e}(S_1, S_2)$ denotes the number of edges between the sets S_1 and S_2. Furthermore the sum over $d_i\sigma_i$ can be written as

$$\sum_{i\in V_n} d_i\sigma_i = \ell_{\sigma_+} - \ell_{\sigma_-} = 2\ell_{\sigma_+} - \ell_n. \tag{6.12}$$

This gives

$$H_n = -J\left(\frac{\ell_n(A^2 + \Delta^2 - 2A\Delta)}{2} + 2A\Delta\ell_{\sigma_+} - 2\Delta^2\mathrm{e}(\sigma_+, \sigma_-)\right) - B\big(2\Delta|\sigma_+| - n(\Delta - A)\big). \tag{6.13}$$

Now to get $\exp(-\beta H_n)$, the expectation can be taken over the graph for a given spin configuration. Here we also fill in $\mathbb{E}[D_n] = \ell_n/n$ to obtain

$$\mathbb{E}[\exp(-\beta H_n)] = \exp\left(\beta n\left(\frac{1}{2}J\mathbb{E}[D_n](A-\Delta)^2 - B(\Delta - A)\right)\right) \\ \cdot \mathrm{e}^{2\beta B\Delta|\sigma_+|} \cdot \mathbb{E}\Big[\exp\big(2\beta JA\Delta\ell_{\sigma_+} - 2\beta J\Delta^2\mathrm{e}(\sigma_+, \sigma_-)\big)\Big]. \tag{6.14}$$

Note that in the expectation, only the terms remain that are dependent on the graph, while everything outside is only dependent on the spins. Now the annealed partition function can be expressed as

$$\mathbb{E}[Z_n(\beta, B)] = \mathbb{E}\left[\sum_{\vec{x}\in\{-d,c\}} \exp(-\beta H_n)\right] = \sum_{\vec{x}\in\{-d,c\}} \mathbb{E}\left[\exp(-\beta H_n)\right] \\ = \mathrm{e}^{\beta n(J\mathbb{E}[D_n](A-\Delta)^2/2 - B(\Delta - A))} \\ \cdot \sum_{S\subset V_n} \mathrm{e}^{2\beta B\Delta|S| + 2\beta JA\Delta\ell_S}\mathbb{E}[\exp\big(-2\beta J\Delta^2\mathrm{e}(S, V_n\setminus S)\big)]. \tag{6.15}$$

In the last step, the sum over all spin configurations is written as a sum over all possible subsets S of V_n with positive spins. The exponent with ℓ_S could be taken out of the expectation because we have a fixed degree sequence. Hence, for a given subset S, the value of ℓ_S is known.

Now in a paper from Can [9, equation (2.2)], he concludes that the function denoting the number of edges between two complementary sets S and $V_n \setminus S$, can be written as

$$\mathrm{e}(S, V_n\setminus S) = X(\ell_S, \ell_n), \tag{6.16}$$

in distribution. Here $X(\ell_S, \ell_n)$ denotes the number of edges between the sets ℓ_S and $\ell_n \setminus \ell_S$ in a configuration model where every vertex has degree 1. This can be understood by considering that the configuration model randomly pairs the half-edges. Hence, a configuration model with ℓ_n vertices with degree 1 has the same configuration as a configuration model with n vertices with degrees summing up to ℓ_n. Using this, we can write the partition function as

$$\mathbb{E}[Z_n(\beta,B)] = \mathrm{e}^{R(\beta,B)} \sum_{S\subset V_n} \mathrm{e}^{2\beta B\Delta|S|+2\beta JA\Delta\ell_S}\hat{g}_\beta(\ell_S,\ell_n), \tag{6.17}$$

where we have made the two substitutions

$$R(\beta,B) = \beta n(J\mathbb{E}[D_n](A-\Delta)^2/2 - B(\Delta-A)) \tag{6.18}$$

and

$$\hat{g}_\beta(\ell_S,\ell_n) = \mathbb{E}[\exp\bigl(-2\beta J\Delta^2 X(\ell_S,\ell_n)\bigr)]. \tag{6.19}$$

Now equation (6.17) can be rewritten as

$$\mathbb{E}[Z_n(\beta,B)] = \mathrm{e}^{R(\beta,B)} \sum_{(j_k)\le(n_k)} \left(\prod_{k\ge1}\binom{n_k}{j_k}\right)\exp\left(2\beta\Delta\left(B\sum_{k\ge1}j_k + JA\sum_{k\ge1}kj_k\right)\right) \cdot \hat{g}_\beta\left(\sum_{k\ge1}kj_k,\ell_n\right). \tag{6.20}$$

Here the product of combinations denotes the number of ways to choose a sequence (j_k) with $j_k < n_k$ for all k. The numbers j_k in the sequence (j_k) denote the number of vertices of degree k in the set of vertices with a positive spin. The sequence (n_k) is the fixed sequence denoting the total number of vertices with each degree, as defined in equation (6.5). Using this expression for the partition function, the annealed pressure per particle can be determined using equation (6.2). This results in

$$\psi_n^{\mathrm{an}}(\beta,B) = \frac{\beta\mathbb{E}[D_n](A-\Delta)^2}{2} - \beta B(\Delta-A) + \frac{1}{n}\log\sum_{(j_k)\le(n_k)}\exp\left[\sum_{k\ge1}\log\binom{n_k}{j_k} + 2\beta\Delta\sum_{k\ge1}j_k(B+JAk) + \log\hat{g}_\beta\left(\sum_{k\ge1}kj_k,\ell_n\right)\right]. \tag{6.21}$$

Here an exponent of a logarithm is taken inside the outer sum to get rid of the product over k. Now an approximation for this expression can be made in a similar way as done by Can et al. [8]. First of all the sum over (j_k) can be bounded from below by taking the maximum of the expression inside the sum. It can be bounded from above by multiplying this maximum by the number of sequences (j_k) with $(j_k) \le (n_k)$. Now we note that this number of sequences can be bounded from above by $n^{d_{\max}^{(n)}} = \mathrm{e}^{d_{\max}^{(n)}\log n} = \mathrm{e}^{o(n)}$. Here we have used condition 6.3 to bound the maximal degree. The next step is to use the Stirling approximation to write

$$\binom{n}{j} = \frac{n^n}{j^j(n-j)^{n-j}}\mathrm{e}^{O(\log n)} = \mathrm{e}^{nI(j/n)+O(\log n)}, \tag{6.22}$$

where $I(t)$ is defined by $I(0) = I(1) = 0$ and, for $t \in (0,1)$,

$$I(t) = -t\log t - (1-t)\log(1-t). \tag{6.23}$$

With all these definitions the pressure per particle can be rewritten. If the sum in equation (6.21) is replaced by the maximum of its argument, the logarithm in front of the sum cancels with the exponent on the inside. Then taking the $1/n$ inside of the brackets, all terms are rewritten using $s_k = j_k/n_k$ and $p_k = n_k/n$. This results in

$$\psi_n^{\mathrm{an}}(\beta, B) = \frac{\beta\mathbb{E}[D_n](A-\Delta)^2}{2} + \max_{0\le s_k\le 1}\left[\sum_{k\ge1} p_k^{(n)} I\left(s_k\right) + 2\beta\Delta\sum_{k\ge1}(B+JAk)s_k p_k^{(n)} \right.$$
$$\left. -\beta B(\Delta - A) + \mathbb{E}[D_n]\hat{F}_\beta\left(\frac{\sum_{k\ge1} k p_k^{(n)} s_k}{\mathbb{E}\left[D_n\right]}\right)\right] + O\left(\frac{d_{\max}^{(n)}\log n}{n}\right). \tag{6.24}$$

Here the following bound for $\hat{g}_\beta$ is used, which is very similar to the bound used by Can,

$$\max_{0\le k\le m}\left|\frac{\log \hat{g}_\beta(k,m)}{m} - \hat{F}_\beta(k/m)\right| = \frac{O(1)}{m}, \tag{6.25}$$

where, for $t\in[0,1]$,

$$\hat{F}_\beta(t) = \int_0^{\min\{t,1-t\}} \log \hat{f}_\beta(u)\mathrm{d}u, \tag{6.26}$$

and, for $u\in[0,1]$,

$$\hat{f}_\beta(u) = \frac{\mathrm{e}^{-2\beta J\Delta^2}(1-2u) + \sqrt{1+\left(\mathrm{e}^{-4\beta J\Delta^2}-1\right)(1-2u)^2}}{2(1-u)}. \tag{6.27}$$

Now following the proof of Can, it can be shown that the maximum in equation (6.24) can be replaced by a supremum over the open interval $(0,1)$, as the the maximum is taken at s_k in this open interval for all k. Using condition 6.1 and 6.2, $p_k^{(n)}$ and D_n can be replaced by p_k and D respectively. Furthermore, it turns out that all other terms are $o(1)$. Therefore, the annealed pressure per particle can be written as

$$\psi_n^{\mathrm{an}}(\beta, B) = \frac{\beta\mathbb{E}[D_n](A-\Delta)^2}{2} + \sup_{s_k\in(0,1)} \hat{G}\left((s_k)_{k\ge1}\right) + o(1), \tag{6.28}$$

where $\hat{G}(s_k)_{k\ge1}$ is given by

$$\hat{G}\left((s_k)_{k\ge1}\right) = \sum_{k\ge1} p_k I\left(s_k\right) + 2\beta\Delta\sum_{k\ge1}(B+JAk)s_k p_k$$
$$-\beta B(\Delta - A) + \mathbb{E}[D]\hat{F}_\beta\left(\frac{\sum_{k\ge1} k p_k s_k}{\mathbb{E}\left[D\right]}\right). \tag{6.29}$$

From this analysis, we have found the pressure per particle for the annealed case on the asymmetric Ising model. Using this pressure, one could numerically find the (spontaneous) magnetization by finding the optimal sequence (s_k) and then taking the derivative of the pressure per particle to B. For the symmetric model, Can et al. found an exact expression from which the optimal sequence (s_k) can be found. For the asymmetric model, their proof could not be followed without encountering some new problems. For a future research, it would be interesting to see if this sequence can be found mathematically for the asymmetric model. In this thesis, we will further analyze the annealed setting with a Monte Carlo simulation in Section 7.5.

7 Monte Carlo simulations

At this point, many results have been obtained by mathematically looking at the Ising model. At some points it was not possible to obtain exact results. However, sometimes it was possible to obtain numerical results. For example, it was possible to numerically find the fixed points for regular trees. In these calculations, the only errors came from the precision of the software that was used. In the calculations for the fixed points of the i.i.d. trees, the results had larger errors, as they were found by taking samples from random distributions. Therefore the precision was dependent on the amount of samples that could be taken.

When we considered the configuration model, it turned out that it was not always possible to find the critical values and critical exponents mathematically. This happened when the degrees of the graphs became random. To get an idea of the values of these critical values and exponents, we will use a Monte Carlo simulation. This is a simulation method that is often used in molecular physics and statistical physics. First we will explain the idea behind Monte Carlo simulations and then it will be applied to the quenched and annealed Ising model on the configuration model.

7.1 Monte Carlo Metropolis method

The goal of a Monte Carlo simulation is to efficiently compute thermodynamic quantities. Suppose you want to know the expectation of the average magnetization of a given graph. To calculate this exactly, you could use the following equation:

$$\langle \frac{1}{n} \sum_{i \in V_n} x_i \rangle_{\mu_n} = \sum_{\vec{x} \in \{-d,c\}^n} \frac{\frac{1}{n} \sum_{i \in V_n} x_i}{Z_n(\beta, B)} e^{-H(\vec{x})}. \tag{7.1}$$

For a graph consisting of 1000 vertices, the configuration space would be $\{c, -d\}^{1000}$, which has 2^{1000} different elements. To calculate the magnetization, you would have to sum over all of these possible configurations. This is of course impossible, even with a very fast computer. Therefore, one method to approximate the magnetization would be to take a number of random samples from the configuration space and take a weighted average of the resulting magnetizations. The weight of each sample would be the probability of this configuration according to the Ising measure. There is however one catch to this method. If the distribution of the probability measure has one very small peak, the random sampling will most likely give bad results. For most of the samples, the weighted magnetization will be negligible.

In the book Understanding Molecular Simulations from Daan Frenkel and Berend Smit [10], a comparison is made to measuring the depth of the Nile in Africa. The random sampling method corresponds to a situation where you take a random point in Africa and if you happen to be in the Nile, you measure the depth. Of course, most of the time you will not be in the Nile and you are wasting a lot of time.

The Metropolis method suggests a more efficient method. Here a random walk is created through the region of the configuration space where the weights are not negligible. Now you repeatedly take a random step (trial move) from your current position. This trial move is rejected if it takes you out of the Nile, while the move is accepted if it keeps you inside the Nile.

Now we apply this method to the (quenched) Ising model. In that case, the trial move consists of flipping the spin of one random vertex between $+1$ and -1 or between c and $-d$ in the asymmetric case. Then the probability that this move is accepted is 1 if the energy of the system decreases by accepting the move. If the energy of the system increases, the step can still be accepted, but with a smaller probability. This makes sure that the system will not get stuck in a local minimum. In particular, the probability of accepting a flip of one spin is given by

$$\begin{aligned}\mathbb{P}(\text{accept move}) &= \min\{1, \frac{\mu(\vec{x}_{\text{new}})}{\mu(\vec{x}_{\text{old}})}\} \\ &= \min\{1, \frac{\exp(-\beta H(\vec{x}_{\text{new}}))}{\exp(-\beta H(\vec{x}_{\text{old}}))}\} \\ &= \min\{1, \exp(-\beta(H(\vec{x}_{\text{new}}) - H(\vec{x}_{\text{old}})))\}.\end{aligned} \tag{7.2}$$

Here $\vec{x}_{\text{old}}$ and $\vec{x}_{\text{new}}$ stand for the spin configuration before and after the proposed trial move respectively. To get an estimation of the desired thermodynamic quantity, the average is taken over the quantity after each trial move in the Metropolis scheme. There are two important things to note here. First of all, the average that is taken is not weighted. Secondly, the average is taken after each trial move, so also if the move is not accepted. In that case, for example the magnetization of the old state contributes multiple times in the average. If the magnetizations are averaged in this manner, it can be shown that this average converges to the actual magnetization according to the Ising measure, when the number of cycles is increased [10]. This is because the Ising measure is the stationary distribution of the Metropolis method.

In a Monte Carlo simulation, many cycles have to be performed to get accurate results. Therefore it is important to minimize the amount of calculations in each cycle. One thing that can be simplified is the difference between the two Hamiltonians in equation (7.2). This can be done by noticing that there is only one spin that is changed in one step. This idea gives the following simplification, where x_f and $\bar{x}_f$ are the spin of the vertex that will be flipped and the spin after it is flipped. Then using $x_i = A + \Delta\sigma_i$, we have $\bar{\sigma}_f = -\sigma_f$.

$$\begin{aligned}H(\vec{x}_{\text{new}}) &= -J \sum_{\{i,j\}\in E_n} x_i x_j + J x_f \sum_{(f,j)\in E_n} x_j - J\bar{x}_f \sum_{(f,j)\in E_n} x_j + Bx_f - B\bar{x}_f \\ &= H(\vec{x}_{\text{old}}) + J(x_f - \bar{x}_f) \sum_{(f,j)\in E_n} x_j + B(x_f - \bar{x}_f) \\ &= H(\vec{x}_{\text{old}}) + J\Delta(\sigma_f - \bar{\sigma}_f) \sum_{(f,j)\in E_n} (A + \Delta\sigma_j) + B\Delta(\sigma_f - \bar{\sigma}_f) \\ &= H(\vec{x}_{\text{old}}) + 2J\Delta\sigma_f \sum_{(f,j)\in E_n} (A + \Delta\sigma_j) + 2B\Delta\sigma_f.\end{aligned} \tag{7.3}$$

From this we see that the only things we need to know to calculate the difference in the Hamiltonian is the current spin σ_f and the spins of the vertices that are connected to vertex f. Using this method, the calculations needed in each Monte Carlo step are minimized. For example, the partition function never has to be calculated.

7.2 Application to the Ising model

There are a couple of things that we need to consider before applying the Metropolis method to the Ising model. First of all, the simulation obviously has to use a finite system to work with. Therefore we will never really get to the thermodynamic limit. This has a few implications for which a solution needs to be found.

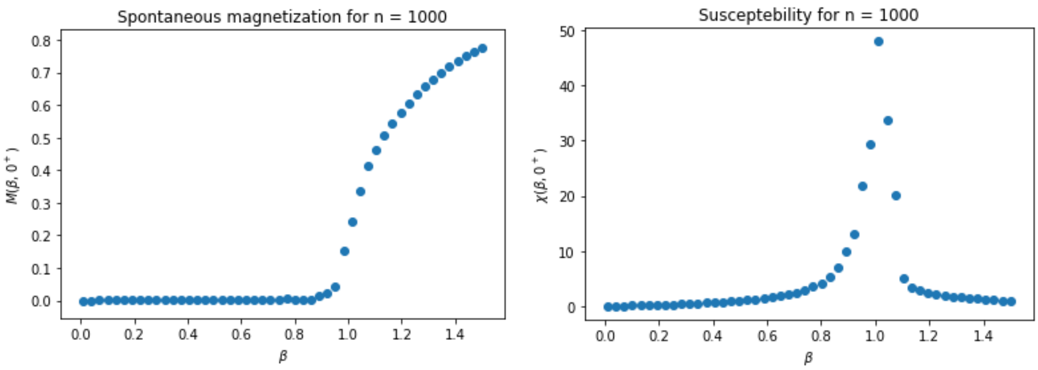
For the Ising model, we are often interested in the behaviour around the critical curve and around the critical point. We also know that for a fixed β and B, there will be exactly one solution for the magnetization. However that is only true in the thermodynamic limit. For a finite system, there will be two stable solutions for $\beta > \beta_c$, one positive and one negative. Therefore, if the simulation is run long enough, the average of the magnetizations will be somewhere in between the two stable solutions.

There are multiple ways of fixing this, but there is no known way to completely avoid this problem without getting additional errors. The method that will be used, is to add a small magnetic field to the critical curve. This increase in the field makes sure that the system is pushed into one of the two branches, depending on the sign of the added magnetic field. In this way, also the behaviour in the positive and negative branch can be compared.

Furthermore, there are two other considerations about applying the Metropolis method on the Ising model. One problem of this kind of simulation is that around the critical point it takes a lot of time to reach an equilibrium situation, while for very small β, the equilibrium is reached very quickly. This last observation is used to organize the simulations. When a plot of the spontaneous magnetization is made, the measurements are performed for increasing β, while taking the last spin configuration of a β-value as initial position for the next value. This also directly solves the question of what initial spin configuration needs to be used. For the first β, the spin configuration can just be taken randomly, as this simulation will converge very fast.

Another consequence of considering finite systems is that the phase transition at the critical point is less sharp. Therefore, from one simulation it is very difficult to retrieve the critical temperature and the critical exponents. To solve this problem, the method of finite size scaling is used, which will be explained later.

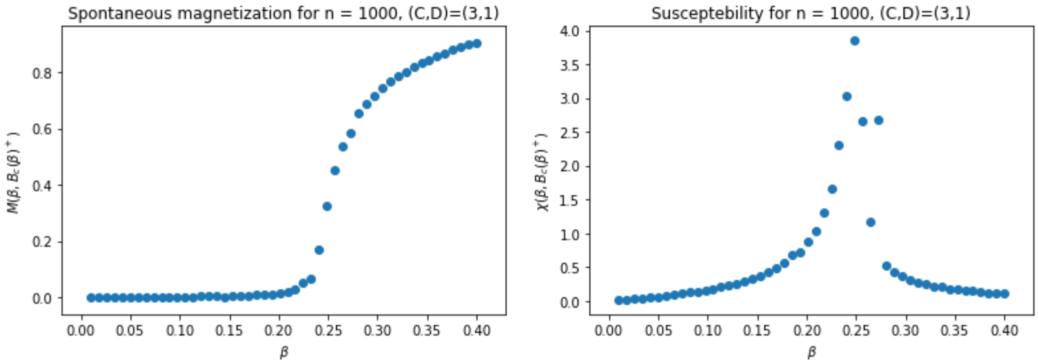
With all of this in mind, it is possible to get a plot of the spontaneous magnetization. In Figure 7a, the spontaneous magnetization is shown, as determined by a Monte Carlo simulation. Here the symmetric Ising model is used ($c = d = 1$). To perform this simulation, first a graph is created with the configuration model. The system size is set to $n = 1000$, with $\alpha = 5$ and $\overline{D} = 10$ as parameters to create the graph. Recall that α is the strength of the power law and $\overline{D}$ is the average degree. For the simulation, $5 \cdot 10^5$ Monte Carlo cycles are performed, where the magnetization is only used in the average after each n cycles. This is done in order to get data points that are less dependent of each other. It also prevents us from having to work with a million data points for each measurement, which takes relatively much time for the computer. In this way, 500 independent data points are created. This will be defined as 500 sweeps, where one sweep denotes n cycles. Then, to reduce the errors due to randomness in the graphs, the average is taken over 20 different graphs. Lastly, for each β value, the first 20 sweeps are not used, to make sure that the system has some time to reach the right region of the configuration space.



(a) *Spontaneous magnetization* ***(b)*** *Susceptibility*

Figure 7: *Spontaneous magnetization (a) and susceptibility (b) for the symmetric Ising model on a configuration model with $n = 1000$, $\alpha = 5$ and $\overline{D} = 10$. The results are the average of the Monte Carlo simulation of 10 graphs with 500 sweeps. The 0^+ denotes the limit from B to zero from above.*

During the same Monte Carlo simulation, the susceptibility is determined as well. The resulting data are shown in Figure 7b. One thing to note here is that the determination of the susceptibility is less accurate than the magnetization, as it corresponds to a second derivative of the pressure per particle, instead of a first derivative. If we now want to compare these results to the asymmetric Ising model, we have to perform the simulation along the critical curve. For the quenched system, the critical curve can be determined by the method described in Section 5.4. It is also possible to calculate this critical curve with Monte Carlo simulations, by finding the magnetic field at which the average magnetization changes from positive to negative. However, this method is less accurate and will therefore only be used if there is no better method. In Figure 8, the spontaneous magnetization and susceptibility are shown for an asymmetric system with $(c, d) = (3, 1)$, as determined by a Monte Carlo simulation.



(a) *Spontaneous magnetization* ***(b)*** *Susceptibility*

Figure 8: *Spontaneous magnetization (a) and susceptibility (b) for the asymmetric Ising model on a configuration model with $\alpha = 5$ and $\overline{D} = 10$. The results are the average of the Monte Carlo simulation of 10 graphs with 500 sweeps. The $B_0(\beta)^+$ denotes the limit from B to $B_0(\beta)$ from above.*

7.3 Finite size scaling

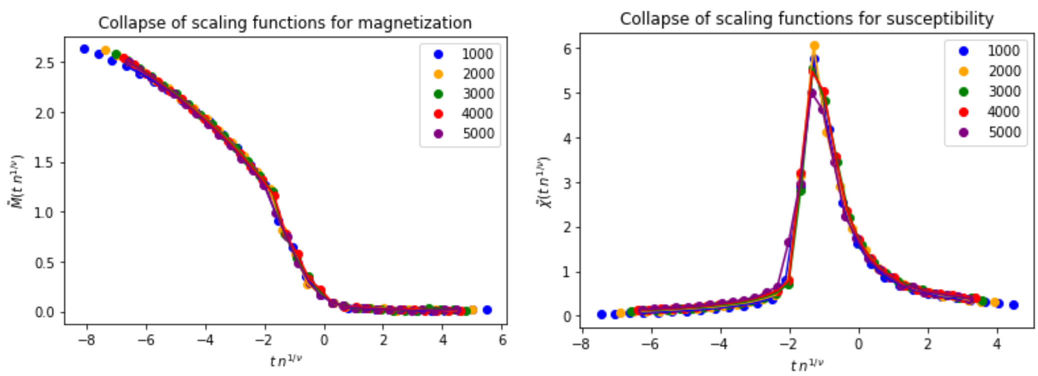
The reason that we started with the Monte Carlo simulations was to be able to find the critical values and critical exponents from the asymmetric Ising model. To find an estimation of these values in the thermodynamic limit, while using finite systems, we will use the method of finite size scaling. This method also works if we do not know exactly where the critical temperature lies. Using the method described in chapter 8 of Monte Carlo Methods in Statistical Physics [11], the magnetization and the susceptibility can be written as respectively

$$M(t) = n^{-\boldsymbol{\beta}/\nu}\tilde{M}(n^{1/\nu}t) \tag{7.4}$$

and

$$\chi(t) = n^{\boldsymbol{\gamma}/\nu}\tilde{\chi}(n^{1/\nu}t). \tag{7.5}$$

Here n denotes the system size and $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are the critical exponents as described in Section 2. Furthermore a new critical exponent ν is introduced, which tells something about the correlation length. It is believed that this exponent is intrinsic to the Ising model. In this way $M(t)$ and $\chi(t)$ are written in terms of the so-called scaling functions $\tilde{M}(n^{1/\nu}t)$ and $\tilde{\chi}(n^{1/\nu}t)$. Here t denotes the reduced temperature $T - T_c = 1/\beta - 1/\beta_c$. It turns out that these scaling functions are independent of the system size if and only if the right critical exponents and critical temperature are used. This gives us a way to determine the critical exponents, as well as the critical temperature, by collapsing the scaling functions for different system sizes. In Figure 9 such a collapse of the scaling functions is shown for the symmetric Ising model for the spontaneous magnetization and the susceptibility.



(a) *Spontaneous magnetization* ***(b)*** *Susceptibility*

Figure 9: *Collapse of the scaling functions for the magnetization and the susceptibility. The critical values for which the collapse of the spontaneous magnetization was optimal, are* $\beta_c = 0.9460$*,* $\boldsymbol{\beta} = 0.5054$ *and* $\nu = 2.6798$*. For the susceptibility, the critical values are* $\beta_c = 0.9197$*,* $\boldsymbol{\gamma} = 0.6399$ *and* $\nu = 2.8199$

The critical values for which the collapse is optimal are determined by minimizing the distance between the graphs for the different n-values, using

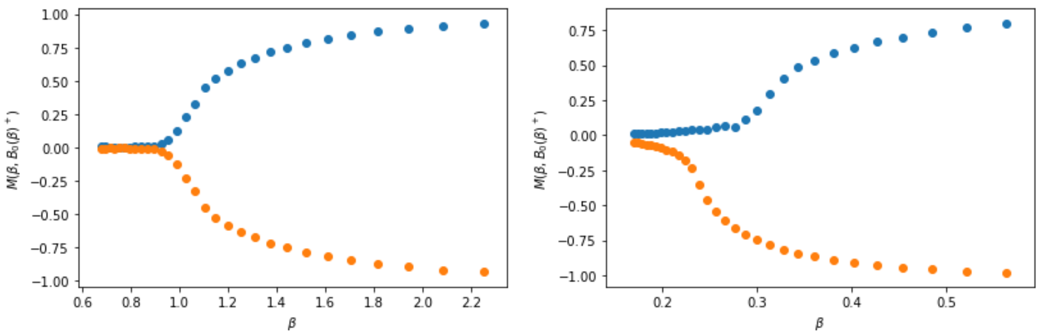
$$V = \frac{1}{x_{\max} - x_{\min}} \int_{x_{\min}}^{x_{\max}} \left(k \sum_{i=1}^{k} \tilde{M}_i^2(x) - \left[\sum_{i=1}^{k} \tilde{M}_i(x) \right]^2 \right) \mathrm{d}x. \tag{7.6}$$

Here $x_{\max}$ and $x_{\min}$ are a chosen upper and lower bound on $n^{1/\nu}t$. Furthermore, k is the number of different system sizes and the functions $\tilde{M}_i(x)$ are determined by an interpolation of the data. To make sure that V function is normalized, V is divided by the average of all data points and than minimized by changing the critical exponents and the critical temperature. The same equation can also be used for $\tilde{\chi}(x)$.

From the results in Figure 9 and other performed measurements, we observe that the determination of the critical exponents for the susceptibility is not very precise. This might be caused by the fact that the susceptibility represents a second derivative, while the magnetization only represents a first derivative. The theoretical critical exponents for the symmetric Ising model on locally tree-like graphs for the considered situation, are $\boldsymbol{\beta} = 0.5$ and $\boldsymbol{\gamma} = 1$ [2]. The magnetization does in general give results close to the theoretically expected values. In the simulation shown above, the critical exponent found was $\boldsymbol{\beta} = 0.5054$, which is very close to 0.5. Furthermore, it turns out that the determination of the critical temperature is also quite good. The theoretical critical temperature is given by $\beta_c = \operatorname{arctanh}(1/\nu)/J \approx 0.94027$ and the critical value found by the magnetization is $\beta_c = 0.9460$. Therefore, from now on, we will mainly discuss the results of the spontaneous magnetization.

7.4 Results for asymmetric model

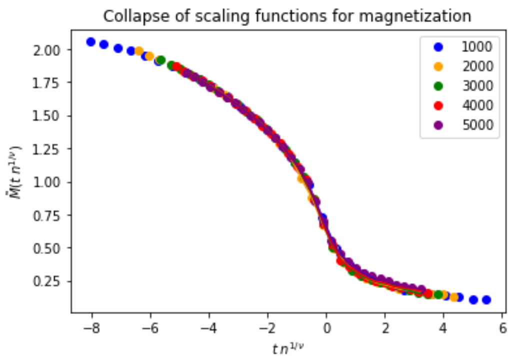
For the asymmetric Ising model, the collapse functions need to be made for the positive magnetizations, as well as for the negative magnetizations. This is because the definition of the critical curve that is used throughout this paper results in an asymmetrical behaviour around the critical curve. In Figure 10, the spontaneous magnetization is plotted for the positive and negative part and for both the symmetric and the asymmetric model. To get the data for these plots, a small magnetic field was added to push the system in the required branch. This is the reason that before the critical point, the results do not give a straight line at zero magnetization.



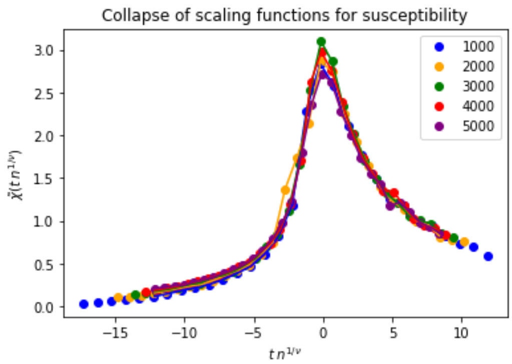
(a) Symmetric *(b) Asymmetric*

Figure 10: *Spontaneous magnetization for the symmetric and asymmetric model for $n = 1000$, $\alpha = 5$ and $\overline{D} = 10$.*

From these plots and the results of other measurements, the conclusion can be made that above the critical point, the asymmetric model does not have two branches with equal behaviour, even if we take the magnetization at the critical curve. In the discussion in Section 7.6, some possible problems with the critical curve will be discussed. For now, we will just consider the positive and the negative branch separately. It turned out that the simulation of the positive branch gave very inaccurate results, as the system often switched to the negative branch. This resulted in bad collapses. The negative branch gave much better results. In Figure 11, the resulting collapses are shown for this negative branch.



(a) Spontaneous magnetization



(b) Susceptibility

Figure 11: *Collapse of the scaling functions for the absolute value of the magnetization and the susceptibility for the negative branch of the asymmetric Ising model with $(c, d) = (3, 1)$. The critical values for which the collapse of the spontaneous magnetization was optimal, are $\beta_c = 0.2363$, $\boldsymbol{\beta} = 0.3114$ and $\nu = 5.8024$. For the susceptibility, the critical values are $\beta_c = 0.2369$, $\boldsymbol{\gamma} = 0.7128$ and $\nu = 3.5326$.*

First of all, we can draw some conclusions about the critical temperature. The Monte Carlo simulations give more evidence that also in the asymmetric Ising model with i.i.d. random degrees, the critical value is given by

$$\beta_c = \frac{1}{J\Delta^2} \operatorname{arctanh}(1/\nu). \tag{7.7}$$

For the asymmetric model with $(c, d) = (3, 1)$, this gives a theoretical critical temperature of $\beta_c \approx 0.235067$, which is very close to the found critical temperature of $\beta_c = 0.2363$ from the simulations.

Furthermore, it is interesting to note that the critical exponent found for the magnetization is $\boldsymbol{\beta} = 0.3114$. This is significantly lower that the critical exponent that is found for the symmetric Ising model, which is $\boldsymbol{\beta} = 0.5$. In Section 7.6, the results will be discussed and some problems with the critical curve will be discussed.

7.5 Annealed Monte Carlo simulation

In Section 6, the asymmetric Ising model was considered for the annealed setting. It turned out that it was possible to find an expression for the pressure per particle in the thermodynamic limit. One problem that we encountered, is the fact that the optimum needed to be found of the function $\hat{G}\left((s_k)_{k\geq 1}\right)$. In this thesis, no sequence $(s_k)_{k\geq 1}$ has been found for which this function is maximal. Can et al. did find a solution for the symmetric Ising model, so it might still be possible to extend their solution to the asymmetric model. However, in this section we will use a Monte Carlo simulation to find the spontaneous magnetization and the critical behaviour of the annealed Ising model on the configuration model. This simulation will be based on equations that are exact for finite system sizes.

Recall that in the Metropolis method for Monte Carlo simulations, the probability to accept a trial move depends on the ratio of the measure of the two states. In the quenched setting, the measure depended on the specific graph, while in the annealed setting, the measure is defined by a ratio of the expectations. Again, the denominator is independent of the spin configuration, hence we only need to consider the numerator in the annealed measure. By combining equation (6.14), equation (6.16) and equation (6.19), we find that this numerator is given by

$$\begin{aligned}\mathbb{E}[\exp(-\beta H_n)] =& \exp\big(\beta n(J\mathbb{E}[D_n](A-\Delta)^2/2 - B(\Delta - A))\big) \\ &\cdot \exp\big(2\beta\Delta(B|\sigma_+| + JA\ell_{\sigma_+})\big)\hat{g}_\beta(\ell_{\sigma_+}, \ell_n).\end{aligned} \tag{7.8}$$

Note that in this equation, the function $\hat{g}_\beta$ is the only place where the structure of the configuration model can be found. All other terms only depend on the number of vertices with a positive spin and the total degree of these vertices. If we now propose to flip the spin σ_k of vertex k to its inverse $\sigma'_k = -\sigma_k$ in a Monte Carlo step, the probability to accept this step is given by

$$\begin{aligned}\mathbb{P}(\text{accept move}) &= \min\left\{1, \exp\Big(2\beta\Delta B(|\sigma'_+| - |\sigma_+|) + 2\beta\Delta JA(\ell_{\sigma'_+} - \ell_{\sigma_+})\Big)\frac{\hat{g}_\beta(\ell_{\sigma'_+}, \ell_n)}{\hat{g}_\beta(\ell_{\sigma_+}, \ell_n)}\right\} \\ &= \min\left\{1, \exp(-2\beta\Delta\sigma_k(B + JAd_k))\frac{\hat{g}_\beta(\ell_{\sigma_+} - \sigma_k d_k, \ell_n)}{\hat{g}_\beta(\ell_{\sigma_+}, \ell_n)}\right\},\end{aligned} \tag{7.9}$$

where d_k is the degree of vertex k and $\sigma_k = \pm 1$ is the spin of the vertex before the proposed flip. Now we still need to be able to calculate the ratio of $\hat{g}_\beta$ functions. In his paper about regular graphs [9], Can shows that, for $k \leq m/2$, this function satisfies

$$\hat{g}_\beta(k, m) = \frac{(m - 2k + 2)\mathrm{e}^{-2\beta J\Delta^2}}{m - k + 1}\hat{g}_\beta(k-1, m) + \frac{k-1}{m-k+1}\hat{g}_\beta(k-2, m), \tag{7.10}$$

with $\hat{g}_\beta(0, m) = 1$ and $\hat{g}_\beta(1, m) = \mathrm{e}^{-2\beta J\Delta^2}$. Using the fact that $\hat{g}_\beta(k, m) = \hat{g}_\beta(m - k, m)$, we can determine the value of this function for all $k \leq m$. One disadvantage is that the values of these functions can get very close to zero, which gives inaccuracies when calculation the fraction of two very small numbers. Therefore, we can define for $1 \leq k \leq m$,

$$\hat{h}_\beta(k, m) = \frac{\hat{g}_\beta(k, m)}{\hat{g}_\beta(k-1, m},\tag{7.11}$$

which satisfies, for $k \leq m/2$, the recursion relation

$$\hat{h}_\beta(k+1, m) = \frac{\mathrm{e}^{-2\beta J\Delta^2}(m - 2k)}{m - k} + \frac{k}{(m-k)\hat{h}_\beta(k, m)}. \tag{7.12}$$

For $k > m/2$, we can calculate the value of $\hat{h}_\beta(k, m)$ by taking the reciprocals of $\hat{h}_\beta(m - k, m)$. Using this recursion, we can calculate the required fraction in the probability in equation (7.9) by taking the product of several evaluations of $\hat{h}_\beta(k, m)$.

Now, using this probability to accept a move, the same Monte Carlo procedure can be used as described before. However, in the annealed case, the critical curve is still unknown. It might be possible that there exists a more precise method to determine this critical curve, but by lack of such a method, the critical curve is determined using Monte Carlo simulations. Here, the initial spin configuration is taken with an average magnetization of zero. Then for different values of B, the simulation is run. If the resulting magnetization is on average more positive, the magnetic field is reduce and vice versa. In this way, the critical curve in Figure 12 is obtained.

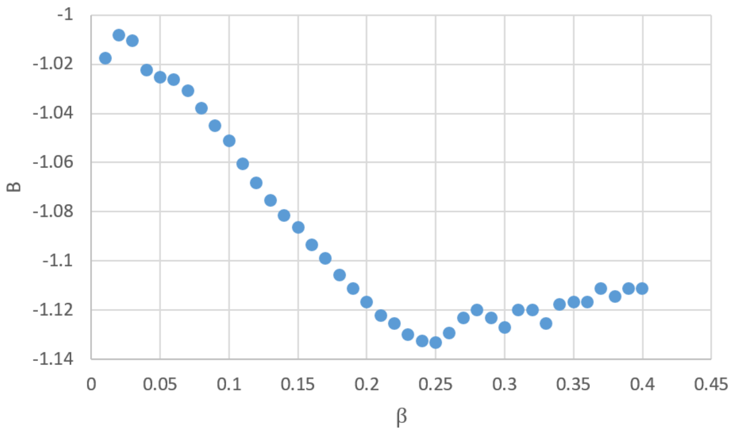
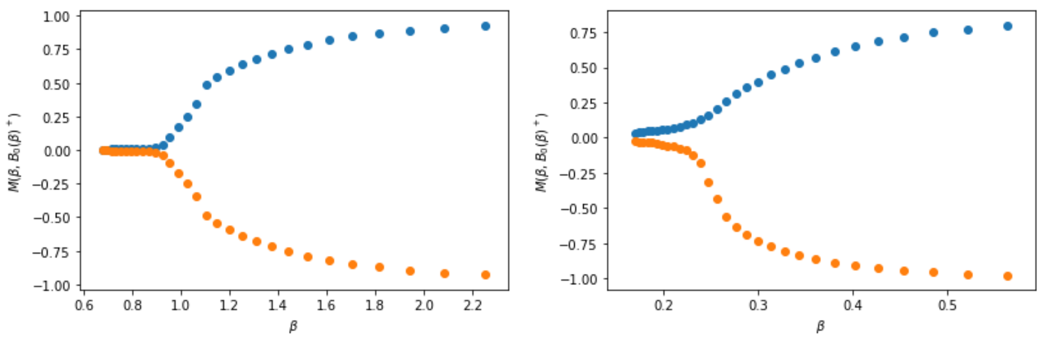


Figure 12: *Critical curve for the annealed Ising model with* $(c, d) = (3, 1)$*,* $\alpha = 5$*,* $\overline{D} = 10$*, obtained with a Monte Carlo simulation.*

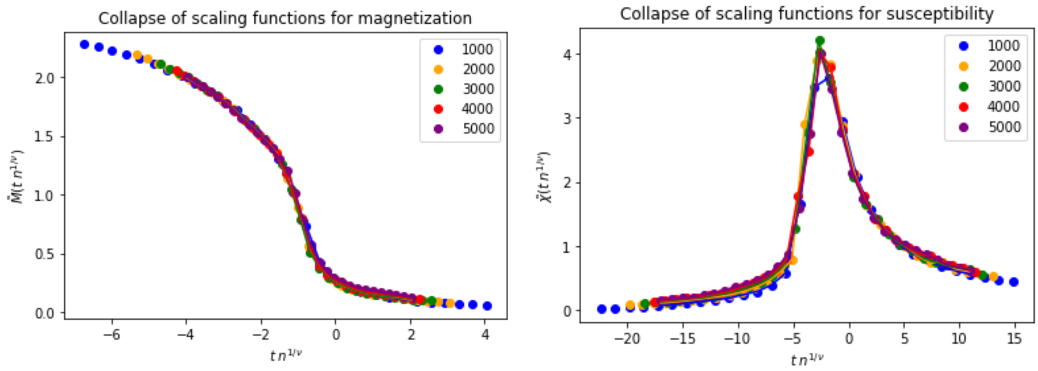
Using this critical curve, the behaviour of the spontaneous magnetization can be investigated with a Monte Carlo simulation. In Figure 13, the difference between the positive and negative branch is shown at the critical curve. Again, it is found that there is an asymmetry in the magnetization around the critical curve. The negative branch is also more attractive for the system. However, this effect is less noticeable in the annealed setting, than in the quenched setting.



(a) *Symmetric* ***(b)*** *Asymmetric*

Figure 13: *Spontaneous magnetization for the annealed Ising model for the symmetric and asymmetric model, for* $n = 1000$*,* $\alpha = 5$ *and* $\overline{D} = 10$*.*

Using the results from the Monte Carlo simulations for the annealed Ising model, again collapse functions can be made. In Figure 14, the collapses are shown for the asymmetric model on the negative branch.



(a) *Spontaneous magnetization* ***(b)*** *Susceptibility*

Figure 14: *Collapse of the scaling functions for the absolute value of the magnetization and the susceptibility for the negative branch of the asymmetric annealed Ising model with* $(c, d) = (3, 1)$*. The critical values for which the collapse of the spontaneous magnetization was optimal, are* $\beta_c = 0.2300$*,* $\boldsymbol{\beta} = 0.3965$ *and* $\nu = 7.1837$*. For the susceptibility, the critical values are* $\beta_c = 0.2351$*,* $\boldsymbol{\gamma} = 0.5390$ *and* $\nu = 3.1375$*.*

From this collapse, again we can find a reasonable estimate for the critical temperature, which turned out to be $\beta_c = 0.2300$ for this situation. This is very close to the expected temperature, as in the symmetric Ising model, the critical temperatures are the same for the quenched setting and the annealed setting with deterministic degrees. Here, we also see that $\boldsymbol{\beta} = 0.3965$. If the same simulation is run for the symmetric Ising model, we find $\boldsymbol{\beta} = 0.4899$, which is very close to the theoretical exponent for the quenched setting. There is no literature in which the critical exponents are determined for the annealed setting of the configuration model. As the quenched simulations resulted in large errors in the critical values of the susceptibility, we can not draw many conclusions about the susceptibility in the annealed setting.

7.6 Discussion of critical curve

In the previous two sections, we showed the results of the Monte Carlo simulations in the quenched and in the annealed setting. The first observation that we could make, is that the results of the magnetization were much better than of the susceptibility. One possible reason for this is that the susceptibility is determined by taking squares of the magnetizations in each Monte Carlo step. This can cause larger uncertainties in the retrieved susceptibilities.

The next observation that we made, was a large difference between the positive and the negative branch of the magnetization. This could be caused by a real intrinsic difference in these two branches in the defined system. However, after performing more simulations with other values for α, $\overline{D}$ and c and d, it turned out that there were much larger deviations in the results. For example, the negative branch started to deviate from $M = 0$, even for $\beta < \beta_c$. This also happened if there was no offset of the external magnetic field. Therefore, the most logical explanation would be that there is an error in the determined critical curve.

In the quenched setting, the critical curve has been defined using the numerically found distribution of the fixed point, as explained in Section 5.4.1. To test whether the determined critical curves were correct, we also determined the position of the critical curve with a Monte Carlo simulation. As explained before, this way of determining the critical curve is very time consuming and is also less accurate. However, this did result in critical curves that were (at least in some cases) located higher than the earlier found critical curves. A possible explanation of this difference is that the first method determines the critical curves in the thermodynamic limit and the second method uses a finite system. However, even with the second method, the errors in the Monte Carlo simulations were still not completely removed.

8 Conclusion and discussion

In this project, many different steps have been taken to investigate the asymmetric Ising model on the configuration model. The goal was to compare the critical behaviour of the asymmetric model to the known behaviour on the symmetric Ising model. In the first part of the project, the critical temperature has been determined for the complete graph and for regular trees. It turned out that in these cases, the asymmetry only caused a translation of the model, which could be resolved by a substitution of a slightly different magnetic field and inverse temperature.

In the next section, trees were considered with a random degree. This immediately resulted in a much more complex situation. Now it was no longer clear at which magnetic field, the critical temperature would be found. At this point the definition of the critical curve became very important. The way that the critical curve has been defined in this project, was based on the idea that at the critical curve the magnetization should be zero. Using numerical methods, it was possible to find this critical curve for a given degree distribution. Using mathematical tricks, also some bounds on this critical curve could be found.

After considering trees, the results could directly be applied on the configuration model. Then the annealed Ising model has been explained. In the annealed Ising model, the expectation is taken over all possible graphs in the denominator and the numerator of the Ising measure. This means that the graphs are allowed to choose the optimal configuration to minimize the Hamiltonian of the Ising model. For the annealed model, the pressure per particle has been calculated for the asymmetric model in the thermodynamic limit. However, the expression for this pressure still contained an optimization, for which the optimal value has not been found. It would be interesting for a future research to see if this optimum can be found mathematically.

Then, for both the quenched and the annealed setting, Monte Carlo simulations have been performed to determine the critical temperature and the critical exponents. This is done in addition to the mathematical analysis, as it was not always possible to find exact results for these critical values. From these simulations, the critical temperature could be determined for the case with random degrees. It became clear that the critical temperature was very close to the critical temperature that one would naively assume from the symmetrical case. For the annealed setting, it also became clear that the critical temperature was at least very close to the quenched critical temperature, as expected from the symmetrical model. It turned out that it was a lot harder to determine the critical exponents with high precision. This was mainly caused by a problem in the simulation with the critical curves. The critical curves that were used in the simulation, resulted in non-zero magnetization, even before reaching the critical temperature. One possible explanation for this behaviour is that the critical curve could be different in the thermodynamic limit than in the finite systems that were used in the simulations.

Even if the simulations did not always give the wanted results, the techniques that were used are still very interesting. Mainly the annealed Monte Carlo simulation for the configuration model is something that has not been done before. It would be interesting to use this in future projects.

9 Future research

In this project, there were still some problems that have not be solved yet. It would be interesting for a future project to investigate the critical curve in more detail. Now, only one definition is used throughout the project. However, during the last simulations, a difference was found in the positive and negative branch of the solutions. This could be caused by a difference in the energy of these two branches. Therefore, it might be better to define the critical curve as the curve where the energy in the two branches is equal. This is only important in the region with $\beta > \beta_c$.

Another aspect that could be explored more, is the optimization problem for the pressure per particle in the annealed setting. It might well be possible to 'translate' the symmetric solution to the asymmetric model. If this optimization problem is solved, this could really help to find the critical curve for the annealed setting with much more precision. It would also be interesting to consider the annealed situation where the degrees are also allowed to change. This could also be added in a Monte Carlo simulation.

And thirdly, the simulations that were run in this project, did not always give very accurate results. Mainly, the susceptibility had large errors. It would be interesting to try other methods for the Monte Carlo simulations. One underlying problem in the simulations is the fact that both branches can be reached, while we want to determine the magnetization in each branch separately. By correcting this problem in the simulation, other errors are created. Perhaps there are other methods that do not cause these problems.

Furthermore, in this project, mainly the mathematical and physical interpretation of the Ising model was considered. For a future project, it would be interesting to look even more into the sociological implications of the asymmetrical model. An interesting topic would be to investigate how difficult it is to switch between different branches. One could also look more at the effects of finite size systems and at what happens with really small graphs.

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