BACHELOR

The ruin probability of insurance models with linear and stochastic premium

Geurts, Sabine A.F.

Award date:
2019

5612 AZ Eindhoven
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
www.tue.nl

Author

S.A.F. Geurts (0961888)

Supervisor

J.A.C. Resing

Date

November 13, 2019

The ruin probability of insurance models with linear and stochastic premium

Bachelor Final project

Abstract

The primary purpose of this study is to determine the ruin probability of insurance models with linear and stochastic premium, and to investigate the difference with insurance models without linear or stochastic premium.
In the first part, three insurance models are mathematically defined and the ruin probability of those models is analyzed theoretically. Results of the analysis are lower-bounds for the non-ruin probability, integral equations the non-ruin probability needs to satisfy and in the case of exponentially distributed claims an exact formula for the non-ruin probability is found.
Then, a simulation of the insurance models is written to visualize the results found in the theoretical analysis. This section of the study illustrates the influence of the parameters of the models and the difference between the models.
The results of the simulation show that the ruin probability is higher in the case of an insurance model with only stochastic premium and that the lowest ruin probability is obtained in the case of an insurance model with only linear premium.

Contents

1 Introduction

Risk theory is variant of mathematics which is still developing today. The origin takes back to the early 1900 in Scandinavia where life insurance companies where starting to develop. A lot of those startups went bankrupt, or in other words were ruined, therefore the question rose on how this could be prevented. This called for a new way of mathematics, which was not developed yet. One of the first mathematicians that studied risk and collective risks was Filip Lundberg, a Swedish mathematician that lived from 1876 up and until 1965. Lundberg's notation was new and strangely written. One of the few that could decipher his papers was Harald Cramér (1893-1985), also a Swedish mathematician. He wrote out Lundberg's notes in a more modern way and used it for insurance modelling. It is logical that one of the first models to determine the probability of a insurance company to become bankrupt, is named after them. The model in question is the Cramér-Lundberg model, or what is nowadays (sometimes) called the classical Poisson risk model. [3] This model is still one of the main modelsin risk theory that is investigated in the 21th century. Multiple researches start with the classical Poisson risk model and then add some extra features to the model, hoping to still find results with those extra features.
As the title states, this research will focus on the ruin probability of insurance models with linear and stochastic premium. This will be done by comparing multiple models, found in literature or presented in this report. The Cramér-Lundberg model will be the first model to be investigated. This model only uses linear premium and can be found in the book "Modern Actuarial Risk Theory" [4]. For further literature research about stochastic premium, an insurance model which only uses stochastic premium is investigated. This model can be found in the paper "The Cramér-Lundberg model with stochastic premium process" by Boikov [1]. To fully research the ruin probability of models with linear and stochastic premium, a new model is developed which will use both linear and stochastic premium.
One can ask what the difference is between the three models and what influence the extra stochastic factor has on the ruin probability. Not only would one like to model the ruin probability and make approximations, one would also like to find exact equations the ruin probability needs to satisfy. If it is possible, in some special cases of the models, one might also like to know the exact formula for the ruin probability. The report tries to find the answers to these questions.

1.1 Outline of the report

To reach the goal of the project, three models for the capital of an insurance company are investigated. First, in Chapter 2, the classical Cramér-Lundberg will be formulated in mathematical terms. This chapter will also contain some analysis for the ruin probability with exact results. The second model, a model with only stochastic premium, will be discussed in Chapter 3. This chapter will state the model and an analysis of the ruin probability is discussed. After that, Chapter 4 contains the third model. This model will consist of both linear and stochastic premium. In this chapter, the model description is given, some analysis for the non-ruin probability will be discussed and the model will be compared to the first two models. Furthermore, to obtain numerical results, a simulation is written and will be stated in Chapter 5. This chapter contains the written simulation and numerical results from that simulation for all models, which will be compared and discussed. After the numerical results, a conclusion and discussion are given to discuss the goals and results of the report.

2 Cramér-Lundberg model

To understand the classical Cramér-Lundberg model and ruin probability, the book "Modern Actuarial Risk Theory" [4] is used. Particular, some results from Chapter 4 are discussed.

2.1 The classical ruin process

The model is defined as the surplus process or risk process as the following:

$$X(t) = x + c \cdot t - S(t), t \geq 0 \tag{2.1}$$

where $X(t) =$ the insurer's capital at time t, $x = X(0) =$ the initial capital, $c =$ the (constant) premium income per unit of time and $S(t) = \sum_{i=1}^{N(t)} Y_i$. Here, $N(t)$ is the number of claims up to time t and Y_i is the size of the ith claim, assumed non-negative.
The time that we are interested in, is the moment where the capital is less than 0. This state of the process is called *ruin*, and the point at which this occurs for the first time is denoted by T. We have that $T = min\{t : t \geq 0 \ \&X(t) < 0\}$ or $T = \infty$ if $X(t) \geq 0$ for all t. Then, the probability that ruin occurs is the probability that T is finite, called $\psi(x) = \mathbb{P}(T < \infty | X(0) = x)$.
The first assumption in the book is that the process $N(t)$ follows a Poisson process with parameter λ. Therefore, we get that $S(t)$ is a compound Poisson process. For further notation, the cumulative distribution function (cdf) and the moments of the individual claims Y_i are denoted by $P(y) = \mathbb{P}(Y_i \leq y)$ and $\mu_j = \mathbb{E}[Y_i^j]$. The *safety loading factor* θ is defined by the relation $c = (1 + \theta)\lambda\mu_1$. To make sure a company receives more premium than that they need to pay for claims, they make sure the *safety loading condition* is maintained. The safety loading condition in this model is equal to $c > \lambda\mathbb{E}[Y_i]$, or in other words $c > \lambda\mu_1$.

2.2 Results on ruin probability

A couple of results from the book "Modern Actuarial Risk Theory" [4] are stated below. Those results include an upper-bound for the ruin-probability and an explicit expression for the probability in case of exponential claims. This section also contains an integral equation for the ruin-probability, derived from other literature and own knowledge. The theorems will be explained and a proof is given directly after the theorem.

Theorem 1. *(Lundberg's exponential bound for the ruin probability) If R is a positive root of the adjustment equation*

$$1 + (1 + \theta)\mu_1 R = \mathbb{E}[e^{RY_i}], \text{ or equivalently } \lambda + cR = \lambda\mathbb{E}[e^{RY_i}],$$

then the upper-bound for the ruin probability equals $\psi(x) \leq e^{-Rx}$.

Proof. Define $\psi_k(x)$, with $-\infty < x < \infty$ and $k = 0, 1, 2, \ldots$ as the probability that ruin occurs at or before the k^{th} claim. For $k \to \infty$, $\psi_k(x)$ increases to its limit $\psi(x)$, $\forall x$, thus it suffices to prove the inequality for each k.

If $k = 0$, then the inequality holds because for $x < 0$ we have that $\psi_0(x) = 1$, and $\psi_0(x) = 0$ for $x \geq 0$. For general k, the probability will be calculated by conditioning on time and size of the first claim. The probability will be split up as regards of the time and the size of the first claim. It will be assumed that the first claim occurs between time t and $t + dt$, this event has a probability of $\lambda e^{-\lambda t}dt$ to occur. The claim has a probability of $dP(y)$ to be of size between y and $y+dy$. Combining this, $\psi_k(x)$ can be written in terms of $\psi_{k-1}(x)$ in the following relation:

$$\psi_k(x) = \int_{t=0}^{\infty}\int_{y=0}^{\infty} \psi_{k-1}(x + ct - y)dP(y)\lambda e^{-\lambda t}dt.$$

Assuming the inequality holds for $\psi_{k-1}(x)$, the equation can be written as the following

$$\begin{aligned}
\psi_k(x) &\leq \int_{t=0}^{\infty}\int_{y=0}^{\infty} e^{-R(x+ct-y)}dP(y)\lambda e^{-\lambda t}dt \\
&= e^{-Rx}\int_{t=0}^{\infty} \lambda e^{-Rct-\lambda t}\int_{y=0}^{\infty} e^{Ry}dP(y)dt \\
&= e^{-Rx}\int_{t=0}^{\infty} \lambda e^{-t(Rc+\lambda)}\mathbb{E}[e^{RY_i}]dt \\
&= e^{-Rx}\lambda\mathbb{E}[e^{RY_i}]\Big[\frac{-1}{\lambda + cR}e^{-t(Rc+\lambda)}\Big]_0^{\infty} \\
&= e^{-Rx}\frac{\lambda\mathbb{E}[e^{RY_i}]}{\lambda + cR}.
\end{aligned}$$

Using the adjustment equation stated before, the fraction behind e^{-Rx} becomes one. Therefore, we have that $\psi_k(x) \leq e^{-Rx}$. Through induction, it is shown that for all $k \in \mathbb{N}$ the upper-bound holds. With this result, the upper-bound holds also for $\psi(x)$. $\square$

An upper-bound for the ruin-probability is a nice result for insurance companies, however, in some cases it is nice to have an exact function for the ruin-probability. This can be found, with the use of an integral equation. This integral equation stated below, holds for all cases and is sometimes not solvable because of the difficult probability density functions of the claims.

Theorem 2. *The non-ruin probability $\phi(x)$ satisfies the following integral equation*

$$\phi(x) = \int_{t=0}^{\infty}\int_{u=0}^{x+ct} \phi(x + ct - u)p(u)du\ \lambda e^{-\lambda t}dt.$$

Proof. The probability that there is no ruin at a company, is equal to the probability that there is no ruin after the first claim of size u. The claim has a probability of $p(u)du$ to be of the size between u and $u + du$. This claim comes in at some time t. The probability of this claim coming in around time t and $t + dt$ equals to $\lambda e^{-\lambda t}dt$. If the first claim is larger then the capital at time t, the company will be ruined. Therefore, the claim size of u should be smaller or equal to $x + ct$. The capital after the first claim is equal to $x + ct - u$. When there is no ruin after the first claim, one can say that the company starts again with the same process. Therefore, we have that we can look at the same ruin probability, only now for $\phi(x + ct - u)$. Combining this altogether, we get

$$\phi(x) = \int_{t=0}^{\infty}\int_{u=0}^{x+ct} \phi(x + ct - u)p(u)du\ \lambda e^{-\lambda t}dt.$$

$\square$

In the case that the claims are exponentially distributed and independent of each other, an exact formula for $\psi(x)$ can be derived. First, following the method of the book "Modern Actuatial

Risk Theory" [4], an exact formula for $\psi(x)$ is given when both inter-arrival times and claims sizes are exponential(1). After that, an alternative derivation is given for an exact formula for $\phi(x)$. This proof follows the same method as in the paper of Boikov [1]. Here, the inter-arrival times and claims sizes are exponential(λ) and exponential(μ) respectively.

Theorem 3. *If $P(y) = 1 - e^{-\mu y}$, the intensity of the Poisson process equals $\lambda = 1$, the mean claim size $\mu_1 = 1$ and the premium rate $c = (1+\theta)\lambda\mu_1$, then $\psi(x)$ equals $\psi(x) = \psi(0)e^{-Rx}$ where $R = \frac{\theta}{(1+\theta)\mu_1}$ and $\psi(0) = \frac{1}{1+\theta}$.*

Proof. The proof contains two steps. The first step will explain how we derive a differential equation that the ruin probability must satisfy. From this step, an expression is obtained where only $\psi(0)$ is still unknown. In the second step, $\psi(0)$ will be found to fill in the first step.
To derive a differential equation for the ruin probability, there will be looked at the ruin probability when started at capital x and when started at capital $x + dx$. If a company gets ruined at the time claim X comes in, where the capital right before the claim was ν, there are two cases that can happen. Case one is that the claim is bigger than $\nu + dx$, so that both starting with capital x or $x + dx$ the company is ruined. The second case is that the remaining claim size was just smaller than dx, so that ruin did not occur yet when starting with capital $x + dx$, but it did occur if started with capital x. In the first case, the probability that a claim is larger than dx, equals e^{-dx}. This is because the claims are exponentially distributed with mean one. When this is approximated with a Taylor-expansion, we get $\mathbb{P}(X > dx) \approx 1 - dx$. For the second case, the probability that claim X is smaller than dx is the complement of the first case. So, that probability equals dx. But, the company might get ruined later on in the process. Because of the memory-less property of the Poisson process, the probability of ruin is in this second case $\psi(0)$. Combining the first case and second case, we get the differential equation:

$$\psi(x + dx) = \psi(x)(1 - dx + \psi(0)dx).$$

Rewritten this equation gives $\frac{\psi'(x)}{\psi(x)} = \psi(0) - 1$. The left hand side of the equation is the derivative of the logarithm of $\psi(x)$, therefore we obtain the following equation:

$$log(\psi(x)) = x(\psi(0) - 1) + log(\psi(0)).$$

Taking the logarithm out of the equation on both sides gives us the last equation:

$$\psi(x) = \psi(0)e^{-(1-\psi(0))x} \text{ for } x \geq 0 \text{ and } \psi(x) = 1 \text{ for } x < 0. \tag{2.2}$$

The second step in the proof is to find $\psi(0)$. For this, it is assumed the first claim is of size between y and $y + dy$ and the claim occurs around time t and $t + dt$. The probability of this event equals $e^{-y}dye^{-t}dt$. After the claim, the capital equals $x + ct - y$, so the ruin probability at x can be written in the following equation:

$$\psi(x) = \int_{y=0}^{\infty}\int_{t=0}^{\infty}\psi(x + ct - y)e^{-y}e^{-t}dtdy.$$

For the case of $x = 0$, it can be written using the result of the first step as the following:

$$\psi(0) = \int_{y=0}^{\infty}\int_{t=y/c}^{\infty}\psi(0)e^{-(1-\psi(0))(ct-y)}e^{-t}e^{-y}dtdy + \int_{y=0}^{\infty}\int_{t=0}^{y/c}e^{-y}e^{-t}dtdy.$$

Using the integration rule of $\int_0^\infty e^{-\alpha x}dx = \frac{1}{\alpha}$ and the substitution of $x = ct - y$, this results in:

$$\psi(0) = \frac{\psi(0)}{c}\int_{y=0}^{\infty}e^{-y/c}\int_{x=0}^{\infty}e^{-(1-\psi(0)+1/c)x}dxe^{-y}dy + \int_0^\infty(1 - e^{-y/c})e^{-y}dy.$$

Therefore, we get

$$\psi(0) = \frac{\psi(0)}{c} \frac{1}{1-\psi(0)+1/c} \frac{1}{1+1/c} + \frac{1}{c+1}.$$

This can be simplified to the equation $\psi(0) = \frac{1}{c}\frac{1}{1-\psi(0)+1/c}$. For this equation, there are two solutions. One is that $\psi(0) = 1$, which will be excluded as wanted solution. The other solution for this equation is $\psi(0) = \frac{1}{c}$. Using that $\mu_1 = \lambda = 1$ and $c = (1+\theta)\lambda\mu_1$, we obtain the result from the theorem. □

Theorem 4. *If $P(y) = 1 - e^{-\mu y}$ and the intensity of the Poisson process equals λ, then $\phi(x)$ equals $\phi(x) = 1 - \frac{\lambda}{\mu c} e^{-\mu(1-\frac{\lambda}{\mu c})x}$.*

Proof. Using the integral equation from Theorem 2 and $p(u) = P'(u) = \mu e^{-\mu u}$ we get that $\phi(x) = \int_{t=0}^{\infty}\int_{u=0}^{x+ct} \phi(x+ct-u)\mu e^{-\mu u}du\ \lambda e^{-\lambda t}dt$.
To solve this integral equation, first we take the derivative on both sides twice to obtain a differential equation. After this step in the proof, we obtain a formula for $\phi(x)$ with constants. To find the constants, some properties of the ruin probability are used. Using change of variables $v = x + ct - u$ the integral equation becomes

$$\phi(x) = -\int_{t=0}^{\infty}\int_{v=x+ct}^{0} \phi(v)\mu e^{-\mu(x+ct-v)}dv\ \lambda e^{-\lambda t}dt \tag{2.3}$$

$$= \int_{t=0}^{\infty} \lambda\mu e^{-\lambda t} \int_{v=0}^{x+ct} \phi(v)e^{-\mu(x+ct-v)}dvdt. \tag{2.4}$$

Now, differentiating both sides using Leibniz integral rule for differentiation gives us:

$$\begin{aligned}\frac{d}{dx}\phi(x) &= \int_{t=0}^{\infty} \lambda\mu e^{-\lambda t}\frac{d}{dx}\Big(\int_{v=0}^{x+ct}\phi(v)e^{-\mu(x+ct-v)}dv\Big)dt\\ &= \int_{t=0}^{\infty} \lambda\mu e^{-\lambda t}\Big(\phi(x+ct) - \int_{v=0}^{x+ct}\mu\phi(v)e^{-\mu(x+ct-v)}dv\Big)dt\\ &= \int_{t=0}^{\infty} \lambda\mu e^{-\lambda t}\phi(x+ct)dt - \mu\int_{t=0}^{\infty}\lambda\mu e^{-\lambda t}\int_{v=0}^{x+ct}\phi(v)e^{-\mu(x+ct-v)}dvdt.\end{aligned}$$

Using that the integral obtained is the same as the integral in Equation (2.3) we get $\phi'(x) = -\mu\phi(x) + \int_{t=0}^{\infty}\lambda\mu e^{-\lambda t}\phi(x+ct)dt$. Again, using change of variables $w = x + ct$ we get

$$\phi'(x) = -\mu\phi(x) + \int_{w=x}^{\infty}\lambda\mu\phi(w)e^{-\lambda\frac{w-x}{c}}\frac{1}{c}dw. \tag{2.5}$$

Then, differentiating again on both sides we obtain

$$\begin{aligned}\frac{d}{dx}\phi'(x) = \phi''(x) &= -\mu\phi'(x) + \frac{d}{dx}\Big(\int_{w=x}^{\infty}\lambda\mu\phi(w)e^{-\lambda\frac{w-x}{c}}\frac{1}{c}dw\Big)\\ &= -\mu\phi'(x) - \frac{\lambda\mu}{c}\phi(x) + \frac{\lambda}{c}\int_{w=x}^{\infty}\lambda\mu\phi(w)e^{-\lambda\frac{w-x}{c}}\frac{1}{c}dw.\end{aligned}$$

The integral we have obtained by this step can also be found in Equation (2.5) and hence the integral equals $\phi'(x) + \mu\phi(x)$. Filling this in the equation with the second derivative we get $\phi''(x) = -\mu\phi'(x) - \frac{\lambda\mu}{c}\phi(x) + \frac{\lambda}{c}(\phi'(x) + \mu\phi(x))$. This becomes $\phi''(x) = (\frac{\lambda}{c} - \mu)\phi'(x)$. Hence, we have obtained a second order differential equation in the form of $\phi''(x) = H\phi'(x)$ with

$H = \frac{\lambda}{c} - \mu$. Solving the differential equation gives us $\phi(x) = C_1 + C_2 e^{(\frac{\lambda}{c}-\mu)x}$.
Now, we need to solve for the constants C_1 and C_2 with the conditions of

$$\lim_{x\to\infty} \phi(x) = 1 \tag{2.6}$$

and

$$\phi(0) = \int_{t=0}^{\infty}\int_{u=0}^{ct} \phi(ct-u)\mu e^{-\mu u} du\, \lambda e^{-\lambda t} dt. \tag{2.7}$$

The first condition of the limit to infinity gives us $\lim_{x\to\infty} C_1 + C_2 e^{-\mu(1-\frac{\lambda}{\mu c})x} = C_1 = 1$. Hence, we have $\phi(x) = 1+C_2 e^{-\mu(1-\frac{\lambda}{\mu c})x}$. Using the formula obtained for $\phi(x)$ we get that $\phi(0) = 1+C_2$ and $\phi(ct-u) = 1+C_2 e^{-\mu(1-\frac{\lambda}{\mu c})(ct-u)} = 1+C_2 e^{\lambda t - \frac{\lambda u}{c} - \mu ct + \mu u}$. Filling this in for the inner integral of Equation (2.7), we get

$$\begin{aligned}\int_{u=0}^{ct} \left(1 + C_2 e^{\lambda t - \frac{\lambda u}{c} - \mu ct + \mu u}\right)\mu e^{-\mu u} du &= \int_{u=0}^{ct} \mu e^{-\mu u} du + \mu C_2 e^{\lambda t - \mu ct} \int_{u=0}^{ct} e^{-\frac{\lambda}{c}u} du \\ &= 1 - e^{-\mu ct} + \mu C_2 e^{\lambda t - \mu ct}\left(\frac{c}{\lambda} - \frac{c}{\lambda} e^{-\lambda t}\right).\end{aligned}$$

Filling this result into the integral of Equation (2.7) gives

$$\begin{aligned}\phi(0) &= \int_{t=0}^{\infty} \left(1 - e^{-\mu ct} + \mu C_2 e^{\lambda t - \mu ct}\left(\frac{c}{\lambda} - \frac{c}{\lambda} e^{-\lambda t}\right)\right)\lambda e^{-\lambda t} dt \\ &= \int_{t=0}^{\infty} \lambda e^{-\lambda t} - \lambda e^{-(\lambda+\mu c)t} + C_2 \mu c e^{-\mu ct} - C_2 \mu c e^{-(\lambda+\mu c)t} dt \\ &= 1 - \frac{\lambda}{\lambda+\mu c} + C_2 - \frac{C_2 \mu c}{\lambda + \mu c}.\end{aligned}$$

Hence, we obtain the following equation to solve

$$1 + C_2 = 1 - \frac{\lambda}{\lambda+\mu c} + C_2 - \frac{C_2 \mu c}{\lambda + \mu c}$$

Solving this, we get that $C_2 = \frac{-\lambda}{\mu c}$. Therefore, we have that $\phi(x) = 1 - \frac{\lambda}{\mu c} e^{-\mu(1-\frac{\lambda}{\mu c})x}$. □

To check the result of Theorem 4 we need to fill in the correct parameters μ_1, λ and c in the formula. If the Theorem is correct, we should obtain the same result as in Theorem 3. When $\lambda = \mu_1 = \frac{1}{\mu} = 1$ and $c = (1+\theta)\lambda\mu_1$ we should get that $\phi(x) = 1 - \frac{1}{1+\theta} e^{-\frac{\theta}{1+\theta}x}$.
So, filling in the parameters in Theorem 4 we get

$$\begin{aligned}\phi(x) &= 1 - \frac{1}{1\cdot(1+\theta)\cdot 1\cdot 1} e^{-1\cdot(1-\frac{1}{1\cdot(1+\theta)\cdot 1\cdot 1})x} \\ &= 1 - \frac{1}{1+\theta} e^{-(1-\frac{1}{1+\theta})x} \\ &= 1 - \frac{1}{1+\theta} e^{-\frac{\theta}{1+\theta}x}.\end{aligned}$$

From which we can see that $\phi(x) = 1 - \psi(x)$ from Theorem 3. Therefore, we have obtained the same result as before.

3 Model with stochastic premium

One of the papers that is used for this report is "The Cramér-Lundberg model with stochastic premium process" by A.V. Boikov [1]. This paper describes the mathematical model used and derives some analytical results thereafter. An analysis of this paper is described below, with first explaining the model, then stating some theorems and proving those theorems.

3.1 Mathematical Model

The model used in this paper is a variant of the classical Cramér-Lundberg model. The classical model describes $X(t)$, the capital of an insurance company at time t, with a linear function for the premium and a compound Poisson process for the claims.
The paper uses a model with stochastic premium process which is independent of the claim process. The Poisson process, $N^-(t)$ with intensity λ^- describes the claims coming in at the company during time $(0, t]$. The claim sizes Y_i, $i \in \{1, 2, ...\}$ are non-negative, independent, identically distributed with the distribution function $F(u)$ and probability density function $f(u)$, which are independent of $N^-(t)$. Then the aggregate claim amount at time t is equal to $R(t) = \sum_{i=1}^{N^-(t)} Y_i$.
The same technique is used to describe the premium process. The Poisson process, $N^+(t)$ with intensity λ^+ describes the premium coming in during time $(0, t]$, the premium sizes C_i, $i \in \{1, 2, ...\}$ are non-negative, i.i.d random variables with distribution function $G(v)$ and probability density function $g(v)$. The premium sizes are also independent of $N^+(t)$. The aggregate premium amount at time t equals $\Pi(t) = \sum_{i=1}^{N^+(t)} C_i$. If started with initial capital x, then $X(t)$ is equal to the following relation

$$X(t) = x + \sum_{i=1}^{N^+(t)} C_i - \sum_{i=1}^{N^-(t)} Y_i. \tag{3.1}$$

The company intends to increase the capital, which provides the *safety loading condition* $\mathbb{E}[\Pi(t)] > \mathbb{E}[R(t)]$, or equivalently, $\lambda^+\mathbb{E}[C_i] > \lambda^-\mathbb{E}[Y_i]$. The paper will investigate the non-ruin probabilities on infinite and finite intervals, respectively $\phi(x)$ and $\phi(x, t)$:

$$\phi(x) = \mathbb{P}(X(t) \geq 0 \; \forall t \in \mathbb{R}_+) \text{ and } \phi(x, t) = \mathbb{P}(X(s) \geq 0, s \in [0, t]).$$

3.2 Non-ruin probability

A couple of theorems are derived for the non-ruin probabilities. The most theorems state something about the non-ruin probability on an infinite interval. Only the last theorem states a relation about $\phi(x, t)$. In this section, the theorems and their corresponding proofs are stated.

Theorem 5. *The non-ruin probability $\phi(x)$ satisfies the following integral equation*

$$(\lambda^- + \lambda^+)\phi(x) = \lambda^- \int_0^x \phi(x - u)f(u)du + \lambda^+ \int_0^\infty \phi(x + v)g(v)dv.$$

Proof. The integral equation follows from looking at the events that are possible during an infinitesimal time Δt. There are four possible, disjoint events that can happen during this time:

1. No claims and no premium come in. So, no jumps of $N^+(t)$ and $N^-(t)$.
2. One claim and no premium come in. So, no jumps of $N^+(t)$ and one jump of $N^-(t)$.
3. One premium and no claims come in. So, one jump of $N^+(t)$ and no jumps of $N^-(t)$.
4. A claim and premium come in simultaneous, or more than one claim and/or premium comes in.

These events can happen with a certain probability. The Poisson processes are independent of each other, therefore the probability that a claim comes in can be multiplied with the probability no premium comes in to find the probability for event 2 and so forth. Combined with the knowledge that $\mathbb{P}(N(\Delta t) = 1) = \lambda \Delta t + o(\Delta t)$ for a Poisson process $N(t)$ with rate λ, the following probabilities are obtained for the events.

1. $(1 - \lambda^- \Delta t)(1 - \lambda^+ \Delta t) + o(\Delta t)$,
2. $\lambda^- \Delta t(1 - \lambda^+ \Delta t) + o(\Delta t)$,
3. $\lambda^+ \Delta t(1 - \lambda^- \Delta t) + o(\Delta t)$,
4. $o(\Delta t)$.

A claim that comes in, has a probability of $f(u)du$ to be of a size between u and $u + du$ ($\mathbb{P}(u \leq Y_i \leq u + du) \approx f(u)du$). This also holds for premium, where it has a probability of $g(v)dv$ to be of a size between v and $v + dv$. When looking at $\phi(x)$ at the small time interval, the following equation holds:

$$\begin{aligned}\phi(x) &= (1 - \lambda^- \Delta t)(1 - \lambda^+ \Delta t)\phi(x) \\ &\quad + \lambda^- \Delta t(1 - \lambda^+ \Delta t) \int_{u=0}^{x} \phi(x - u) f(u) du \\ &\quad + \lambda^+ \Delta t(1 - \lambda^- \Delta t) \int_{v=0}^{\infty} \phi(x + v) g(v) dv + o(\Delta t).\end{aligned}$$

First, this is divided by Δt, which gives the following equation:

$$((\lambda^- + \lambda^+) - \lambda^- \lambda^+ \Delta t)\phi(x) = \lambda^-(1 - \lambda^+ \Delta t) \int_{u=0}^{x} \phi(x-u) f(u) du + \lambda^+(1 - \lambda^- \Delta t) \int_{v=0}^{\infty} \phi(x+v) g(v) dv + \frac{o(\Delta t)}{\Delta t}.$$

When applying the limit of $\Delta t \to 0$, the equation of Theorem 5 is obtained. □

An alternative derivation for the integral equation would be to look at the next event that can take place.

Proof. Alternative
If started with $\phi(x)$, there are two events that can happen. A claim can come in with probability $\frac{\lambda^-}{\lambda^- + \lambda^+}$ and a premium can come in with probability $\frac{\lambda^+}{\lambda^- + \lambda^+}$. A claim that comes in, has a probability of $f(u)du$ to be of a size between u and $u + du$ ($\mathbb{P}(u \leq Y_i \leq u + du) \approx f(u)du$). This also holds for premium, where it has a probability of $g(v)dv$ to be of a size between v and $v + dv$. If a claim comes in, we jump to the level $\phi(x - u)$, else if a premium comes in, we jump to $\phi(x + v)$. Combining this with the probability of the events and the size of the event, we get:

$$\phi(x) = \frac{\lambda^-}{\lambda^- + \lambda^+} \int_{u=0}^{x} \phi(x - u) f(u) du + \frac{\lambda^+}{\lambda^- + \lambda^+} \int_{v=0}^{\infty} \phi(x + v) g(v) dv.$$

This results in the integral equation of Theorem 5. □

Corollary 5.1 (Cramér-Lundberg lower-bound). *If R is a positive root of the adjustment equation*

$$\lambda^+(\mathbb{E}[e^{-RC_i}] - 1) + \lambda^-(\mathbb{E}[e^{RY_i}] - 1) = 0,$$

then the lower-bound for $\phi(x)$ equals $\phi(x) \geq 1 - e^{-Rx}$.

The paper has proven this statement by the use of martingales. This was a somewhat more difficult approach to proof the corollary. Therefore, an own approach is taken to proof the lower-bound.

Proof. Define $\phi_k(x)$, for $k = 0, 1, \dots$ and $-\infty < x < \infty$ as the probability that $X(t) \geq 0$ up to and including the kth event. For $k \to \infty$ we have that $\lim_{k\to\infty} \phi_k(x) = \phi(x)\ \forall x$. Therefore, it suffices to prove that $\forall k \in \mathbb{N} : \phi_k(x) \geq 1 - e^{-Rx}$. This will be done by induction.
Base case $k = 0$: If $x > 0$, then $\phi_0(x) = 1 \geq 1 - e^{-Rx}$. If $x \leq 0$, then $\phi_0(x) = 0 \geq 1 - e^{-Rx}$. So the lower-bound holds when $k = 0$.
Induction hypothesis: Suppose the lower-bound holds for $k - 1$, that is, $\phi_{k-1}(x) \geq 1 - e^{-Rx}$.
Induction step: Then for k we get that with probability $\frac{\lambda^-}{\lambda^- + \lambda^+}$ that the event is a claim. This claim has probability $f(v)dv$ that the claim size equals v. Analogously, with probability $\frac{\lambda^+}{\lambda^- + \lambda^+}$ we have that the event is a premium and with probability $g(u)du$ that this premium is of size u. Therefore, we get:

$$\phi_k(x) = \frac{\lambda^-}{\lambda^- + \lambda^+} \int_0^\infty \phi_{k-1}(x - v) f(v) dv + \frac{\lambda^+}{\lambda^- + \lambda^+} \int_0^\infty \phi_{k-1}(x + u) g(u) du.$$

Applying the Induction hypothesis, the following is obtained:

$$\begin{aligned}
\phi_k(x) &\geq \frac{\lambda^-}{\lambda^- + \lambda^+} \int_0^\infty (1 - e^{-R(x-v)}) f(v) dv + \frac{\lambda^+}{\lambda^- + \lambda^+} \int_0^\infty (1 - e^{-R(x+u)}) g(u) du \\
&= \frac{\lambda^-}{\lambda^- + \lambda^+} \int_0^\infty f(v) dv - \frac{\lambda^-}{\lambda^- + \lambda^+} e^{-Rx} \int_0^\infty e^{Rv} f(v) dv \\
&+ \frac{\lambda^+}{\lambda^- + \lambda^+} \int_0^\infty g(u) du - \frac{\lambda^+}{\lambda^- + \lambda^+} e^{-Rx} \int_0^\infty e^{-Ru} g(u) du \\
&= \frac{\lambda^-}{\lambda^- + \lambda^+} + \frac{\lambda^+}{\lambda^- + \lambda^+} - e^{-Rx} \left(\frac{\lambda^- \mathbb{E}[e^{RY_i}] + \lambda^+ \mathbb{E}[e^{-RC_i}]}{\lambda^- + \lambda^+} \right)
\end{aligned}$$

Using that R is the solution of the adjustment equation, we get that the fraction after e^{-Rx} equals 1. Therefore, it becomes $\phi_k(x) \geq 1 - e^{-Rx}$. Hence, through induction, it is shown that for all $k \in \mathbb{N}$ the lower-bound holds. With this result, the Cramér-Lundberg lower-bound also holds for $\phi(x)$. □

In some cases, when the distribution functions of the premiums and claims are known, it is possible to derive exact formulas for $\phi(x)$ as in the classic Cramér-Lundberg model.

Theorem 6.

1. *If $\mathbb{P}(C_i = 1) = \mathbb{P}(Y_i = 1) = 1$, then the exact formula for $\phi(x)$ equals $\phi(x) = \phi(\lfloor x \rfloor) = 1 - (\frac{\lambda^-}{\lambda^+})^{\lfloor x \rfloor + 1}$.*
2. *If $G(v) = 1 - e^{-bv}$, $F(u) = 1 - e^{-au}$ with $a, b > 0$, then $\phi(x)$ equals*

$$\phi(x) = 1 - \frac{(a + b)\lambda^-}{a(\lambda^- + \lambda^+)} \exp\left(\frac{(\lambda^- b - \lambda^+ a)x}{\lambda^- + \lambda^+}\right).$$

Proof. 1. First, the safety loading condition is checked. Because $\mathbb{P}(c_i = 1) = \mathbb{P}(y_i = 1) = 1$, the safety loading condition becomes $\lambda^+ > \lambda^-$. Second, the formula for $\phi(x)$ is derived.

Secondly, we assume that the initial capital x is an integer. A premium of size 1 comes in with probability $\frac{\lambda^+}{\lambda^-+\lambda^+}$ and a claim of size 1 comes in with probability $\frac{\lambda^-}{\lambda^-+\lambda^+}$. Then, the integral equation of Theorem 5 becomes

$$\phi(x) = \frac{\lambda^+}{\lambda^- + \lambda^+}\phi(x+1) + \frac{\lambda^-}{\lambda^- + \lambda^+}\phi(x-1).$$

This results in the following recurrence relation: $\phi(x+1) = \frac{\lambda^-+\lambda^+}{\lambda^+}\phi(x) - \frac{\lambda^-}{\lambda^+}\phi(x-1)$. This recurrence relation can be solved by solving the auxiliary equation $w^2 - \frac{\lambda^-+\lambda^+}{\lambda^+}w + \frac{\lambda^-}{\lambda^+} = 0$, then using the formula: $\phi(x) = C_1(w_1)^x + C_2(w_2)^x$, where w_1 and w_2 are solutions of the auxiliary equation. The solutions of the auxiliary equation are $w_1 = 1$ and $w_2 = \frac{\lambda^-}{\lambda^+}$. Therefore, the formula for $\phi(x)$ becomes $\phi(x) = C_1 + C_2(\frac{\lambda^-}{\lambda^+})^x$. The constants C_1 and C_2 can be found using the conditions of $\phi(x)$. From $\phi(\infty) = 1$ it follows that $C_1 = 1$ and from $\phi(-1) = 0$ it follows that $1 + C_2(\frac{\lambda^+}{\lambda^-}) = 0$, so $C_2 = -\frac{\lambda^-}{\lambda^+}$. Hence, $\phi(x) = 1 - (\frac{\lambda^-}{\lambda^+})^{x+1}$.
Then for non-integer initial capital x, the relation $\phi(x) = \phi(\lfloor x \rfloor)$ follows directly from the fact that the claim sizes and premium sizes are one. Combining this gives the first part of Theorem 6.

2. First, the safety loading condition becomes $\frac{\lambda^+}{b} > \frac{\lambda^-}{a}$. The integral equation for $\phi(x)$ has the form of

$$(\lambda^- + \lambda^+)\phi(x) = \lambda^+ \int_0^\infty \phi(x+v)be^{-bv}dv + \lambda^- \int_0^x \phi(x-u)ae^{-au}du. \tag{3.2}$$

Using the change of variables with $v_1 = v + x$ and $u_1 = x - u$ leads to the equation

$$(\lambda^- + \lambda^+)\phi(x) = \lambda^+ \int_x^\infty \phi(v_1)be^{-b(v_1-x)}dv_1 + \lambda^- \int_0^x \phi(u_1)ae^{-a(x-u_1)}du_1.$$

Using Leibniz integral rule for differentiation on the change-of-variables equation, we get the following derivatives:

$$\begin{aligned}\frac{d}{dx}(\int_0^\infty \phi(x+v)be^{-bv}dv) &= \frac{d}{dx}(\int_x^\infty \phi(v_1)be^{-b(v_1-x)}dv_1) \\ &= -b\phi(x) + b\int_x^\infty \phi(v_1)be^{-b(v_1-x)}dv_1 \\ &= -b\phi(x) + b\int_0^\infty \phi(x+v)be^{-bv}dv,\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\int_0^x \phi(x-u)ae^{-au}du) &= \frac{d}{dx}(\int_0^x \phi(u_1)ae^{-a(x-u_1)}du_1) \\ &= a\phi(x) - a\int_0^x \phi(u_1)ae^{-a(x-u_1)}du_1 \\ &= a\phi(x) - a\int_0^x \phi(x-u)ae^{-au}du.\end{aligned}$$

If (3.2) is differentiated on both sides, it becomes

$$(\lambda^-+\lambda^+)\phi'(x) = -\lambda^+b\phi(x)+b\lambda^+\int_0^\infty \phi(x+v)be^{-bv}dv+a\lambda^-\phi(x)-a\lambda^-\int_0^x \phi(x-u)ae^{-au}du \Rightarrow$$

$$(\lambda^-+\lambda^+)\phi'(x)+(\lambda^+b-\lambda^-a)\phi(x) = b\lambda^+\int_0^\infty \phi(x+v)be^{-bv}dv-a\lambda^-\int_0^x \phi(x-u)ae^{-au}du. \tag{3.3}$$

Taking the derivative for a second time, we get

$$(\lambda^- + \lambda^+)\phi''(x) + (\lambda^+ b - \lambda^- a)\phi'(x) = -b^2\lambda^+\phi(x) + b^2\lambda^+ \int_0^\infty \phi(x+v)be^{-bv}dv$$
$$- a^2\lambda^-\phi(x) + a^2\lambda^- \int_0^x \phi(x-u)ae^{-au}du.$$

This results in

$$(\lambda^- + \lambda^+)\phi''(x) + (\lambda^+ b - \lambda^- a)\phi'(x) + (a^2\lambda^- + b^2\lambda^+)\phi(x) \\ = b^2\lambda^+ \int_0^\infty \phi(x+v)be^{-bv}dv + a^2\lambda^- \int_0^x \phi(x-u)ae^{-au}du. \tag{3.4}$$

The right side of (3.4) can be obtained by multiplying the right side of (3.2) with ab and summed with right side of (3.3) multiplied by $(b-a)$. The same is done for the left sides of both equations. Therefore, we get that the left side of (3.4) should equal to the following

$$(\lambda^- + \lambda^+)\phi''(x) + (\lambda^+ b - \lambda^- a)\phi'(x) + (a^2\lambda^- + b^2\lambda^+)\phi(x) \\ = (b-a)((\lambda^- + \lambda^+)\phi'(x) + (\lambda^+ b - \lambda^- a)\phi(x)) + ab(\lambda^- + \lambda^+)\phi(x).$$

This results in $\phi''(x)(\lambda^- + \lambda^+) + \phi'(x)(\lambda^+ a - \lambda^- b) = 0$. Hence, we get a second order differential equation in the form of $\phi''(x) = Q\phi'(x)$ with $Q = \frac{\lambda^- b - \lambda^+ a}{\lambda^- + \lambda^+}$. Solving this differential equation gives us $\phi(x) = C_1 + C_2 e^{Qx}$. Solving this for the constants C_1, C_2 with the conditions of $\phi(\infty) = 1$ gives us $C_1 = 1$. The condition $(\lambda^- + \lambda^+)\phi(0) = \lambda^+ \int_0^\infty \phi(v)be^{-bv}dv$ gives us the following

$$\begin{aligned}
(\lambda^- + \lambda^+)\phi(0) &= \lambda^+ \int_0^\infty \phi(v)be^{-bv}dv \\
(\lambda^- + \lambda^+)(1 + C_2) &= \lambda^+ b \int_0^\infty (1 + C_2 e^{Qv})e^{-bv}dv \\
(\lambda^- + \lambda^+)(1 + C_2) &= \lambda^+ b(\int_0^\infty e^{-bv}dv + C_2 \int_0^\infty e^{(-b+Q)v}dv) \\
(\lambda^- + \lambda^+)(1 + C_2) &= \lambda^+ - C_2 \frac{\lambda^+ b}{Q-b} \\
C_2 &= -\frac{\lambda^-}{\lambda^- + \lambda^+ + \frac{\lambda^+ b}{Q-b}}
\end{aligned}$$

Hence, $C_2 = -\frac{(a+b)\lambda^-}{a(\lambda^- + \lambda^+)}$. Therefore, we have that $\phi(x)$ equals

$$\phi(x) = 1 - \frac{(a+b)\lambda^-}{a(\lambda^- + \lambda^+)} e^{Qx} \text{ with } Q = \frac{\lambda^- b - \lambda^+ a}{\lambda^- + \lambda^+}.$$

This is the result of the second part of Theorem 6. □

At last, the paper also states two theorems for the finite time interval non-ruin probability. The theorems will be stated and the corresponding proofs will be explained, but the proof will not be given in detail.

Theorem 7. *The non-ruin probability on a finite time interval $\phi(x,t)$ satisfies the integral differential equation*

$$\frac{\partial \phi(x,t)}{\partial t} + (\lambda^- + \lambda^+)\phi(x,t) = \lambda^+ \int_0^\infty \phi(x+v,t)g(v)dv + \lambda^- \int_0^x \phi(x-u,t)f(u)du. \tag{3.5}$$

The proof for this follows the same idea as the proof from Theorem 5. There will be looked at an infinitesimal time period Δt. From here, the relation between $\phi(x,t)$ and $\phi(x,t-\Delta t)$ with the integrals is stated. The formula is divided by Δt and then the limit of $\Delta t \to 0$ is applied. Hence, the integral differential equation follows.

Corollary 7.1. *In the case of $G(v) = 1 - e^{-bv}, F(u) = 1 - e^{-au}(a, b > 0)$, then (3.5) can be reduced to a partial differential equation and $\int_0^\infty (1 - \phi(x,t))e^{-ts} ds = A(s)e^{\alpha(s)x}$, where*

$$\alpha(s) = \frac{\lambda^- b - \lambda^+ a + s(b-a)}{2(\lambda^- + \lambda^+ + s)} - \frac{\sqrt{[\lambda^- b - \lambda^+ a + s(b-a)]^2 + 4ab(\lambda^- + \lambda^+ + s)s}}{2(\lambda^- + \lambda^+ + s)},$$

$$A(s) = \frac{\lambda^-}{s(s + \lambda^- + \lambda^+ - \frac{\lambda^+ b}{b - \alpha(s)})}.$$

The proof for the Corollary follows the same idea as the proof for Theorem 6 part 2. The distributions $F(u)$ and $G(v)$ are filled in in the integrals of Theorem 7.1. After that step, the second derivative with regard to x of (3.5) is taken on both sides. This leads to a partial differential equation of $\phi(x,t)$ without integrals. To solve this differential equation, the auxiliary function $W(s,x) = \int_0^\infty (1 - \phi(x,t))e^{-ts} dt$ is introduced. If we take the first and second derivative w.r.t. x of $W(s,x)$, we get a relation between the partial differential equation and the auxiliary function. To obtain a usual differential equation only depending on $W(s,x)$, the partial differential equation of $\phi(x,t)$ is multiplied with e^{-ts} and integrated by t from 0 to ∞. By doing this, a usual differential equation of the auxiliary function $W(s,x)$ is obtained. This is called the adjustment equation, which has two real roots with opposite signs. $\alpha(s)$ stated in Corollary 7.1 is the negative root of the adjustment equation. The adjustment equation is solved by using the property that $\phi(x,t)$ is bounded. Hence, we know that $W(s,x)$ is also bounded. Using this property, a solution for the usual differential equation equals $W(x,s) = A(s)e^{\alpha(s)x}$. The parameter $A(s)$ can be found with filling in $x = 0$ in (3.5), hence we get the parameter stated in Corollary 7.1. Combined with the auxiliary function $W(s,x) = \int_0^\infty (1 - \phi(x,t))e^{-ts} dt$, we obtain the result from Corollary 7.1.

4 Model with linear and stochastic premium

For further analysis, we will use a model that combines linear premium, stochastic premium process and a stochastic claim process which is independent of the premium process. This is an interesting model, because most insurance companies get a fixed amount per client per year or month. Furthermore, the claims come in randomly, and it is possible for an insurance company to get extra premium for clients when they want to change the contract, or a raise in premium is put in place because of a claim.

4.1 Model description

The model describes the capital of an insurance company at time t, $X(t)$, with a combination of the classical Cramér-Lundberg model and the model described in Chapter 3. The Poisson process, $N^-(t)$, with intensity λ^- describes the claims coming in at the company during time $(0, t]$. This process is independent of the Poisson process $N^+(t)$, which describes the stochastic premium coming in during time $(0, t]$. The claim sizes, Y_i, $i \in \{1, 2, ...\}$ are non-negative, i.i.d random variables with distribution function $F(u)$, which are also independent of $N^-(t)$. The aggregate claim amount at time t is equal to $R(t) = \sum_{i=1}^{N^-(t)} Y_i$. The stochastic premium sizes, C_i, $i \in \{1, 2, ...\}$ are non-negative, i.i.d random variables with distribution function $G(v)$, which are also independent of $N^+(t)$ and $R(t)$. The linear premium per time unit t equals to c, therefore the linear premium at time t equals $c \cdot t$. Thus, the aggregated premium amount at time t equals $Q(t) = c \cdot t + \sum_{i=1}^{N^+(t)} C_i$. When started with initial capital, $X(0) = x$, then we obtain the following relation for $X(t)$

$$X(t) = x + Q(t) - R(t), \text{ or equivalently } X(t) = x + ct + \sum_{i=1}^{N^+(t)} C_i - \sum_{i=1}^{N^-(t)} Y_i. \qquad (4.1)$$

An insurance company wants to increase their capital, which gives us the *safety loading condition* $\mathbb{E}[\text{premium per time unit}] > \mathbb{E}[\text{claim amount per time unit}]$, or equivalently, $c + \lambda^+\mathbb{E}[C_i] > \lambda^-\mathbb{E}[Y_i]$. The mathematical process of Equation (4.1) will be analysed in the next sections. In particular, the non-ruin probability will be analysed. This is the probability that $X(t) \geq 0$ for all t. A company would like to have this probability as high as possible, therefore they would like to have some measures to know the exact value of the probability. In some cases, this can be known. The non-ruin probability has a direct relation with the ruin probability that is stated in Chapter 2. Because ruin is the complement of non-ruin, we get the relation that $\psi(x) = 1 - \phi(x)$. Therefore it is not necessary to compute both probabilities separate. If a function or upper-/lower-bound is known for one of the two, the other probability follows directly from this relation.

For further analysis, the non-ruin probability will be analysed. One can find the similar results for the ruin probability when using the relation stated above.

4.2 Non-ruin probability analysis

For the new model, one would also like to obtain similar results as for the previous models. The same analysis will be given for this model in the chapter and some comparisons will be made with the previous models. As the analysis has shown, one need only to change the adjustment equation to have the same lower-bound for the non-ruin probability. The adjustment equation is given in Theorem 8.

Theorem 8. *If R is a positive root of the adjustment equation*

$$\lambda^+\mathbb{E}[e^{-RC_i}] + \lambda^-\mathbb{E}[e^{RY_i}] = \lambda + cR,$$

then the lower-bound for $\phi(x)$ equals $\phi(x) \geq 1 - e^{-Rx}$.

The method to prove this theorem will be the same as the proof of Corollary 5.1.

Proof. Define $\phi_k(x)$ for $k = 0, 1, \ldots$ and $-\infty < x < \infty$ as the probability that $X(t) \geq 0$ up to and including the k^{th} event. There is no difference in the event being a claim or a premium coming in. For $k \to \infty$ it holds that $\phi_k(x)$ decreases to its limit $\phi(x)$ for all x. So, it is sufficient to show that $\phi_k(x) \geq 1 - e^{-Rx}$ for all k. This will be done using the induction method.
For $k = 0$, it holds that $\phi_0(x) = 1 \geq 1 - e^{-Rx}$ if $x > 0$ and for $x \leq 0$ it holds that $\phi_0(x) = 0 \geq 1 - e^{-Rx}$. Thus, for the base case of the induction, the inequality holds. Now, assume that for $k-1$ the inequality also holds. That is, $\phi_{k-1}(x) \geq 1 - e^{-Rx}$.
For $k > 0$, we are splitting up the event 'ruin at or before the k^{th} event' as regards of the time and size of the first event. It will be assumed that the first event occurs between time t and $t + dt$. This event has a probability of $\lambda e^{-\lambda t}dt$ to occur. Also, we have with probability $\frac{\lambda^+}{\lambda}$ that the event is a premium and with probability $\frac{\lambda^-}{\lambda}$ this event is a claim.
The size of the event depends on whether the event is a claim or premium. If it is a claim, it will have a probability of $f(v)dv$ to be between the size v and $v + dv$. When it is a premium, the size of the event wil have a probability of $g(u)du$ to be between u and $u + du$. Using these probabilities, we can write out $\phi_k(x)$ in terms of $\phi_{k-1}(x)$.

$$\phi_k(x) = \int_{t=0}^{\infty} \Big(\frac{\lambda^+}{\lambda}\int_{u=0}^{\infty} \phi_{k-1}(x+ct+u)g(u)du + \frac{\lambda^-}{\lambda}\int_{v=0}^{x+ct} \phi_{k-1}(x+ct-v)f(v)dv\Big)\lambda e^{-\lambda t}dt.$$

Using the induction hypothesis, we get the following inequality

$$\begin{aligned}\phi_k(x) &\geq \int_{t=0}^{\infty} \Big(\frac{\lambda^+}{\lambda}\int_{u=0}^{\infty} (1 - e^{-R(x+ct+u)})g(u)du + \frac{\lambda^-}{\lambda}\int_{v=0}^{\infty} (1 - e^{-R(x+ct-v)})f(v)dv\Big)\lambda e^{-\lambda t}dt \\ &= \int_{t=0}^{\infty} \Big(\frac{\lambda^+}{\lambda}\int_{u=0}^{\infty} g(u)du - \frac{\lambda^+}{\lambda}\int_{u=0}^{\infty} e^{-R(x+ct+u)}g(u)du \\ &\quad + \frac{\lambda^-}{\lambda}\int_{v=0}^{\infty} f(v)dv - \frac{\lambda^-}{\lambda}\int_{v=0}^{\infty} e^{-R(x+ct-v)}f(v)dv\Big)\lambda e^{-\lambda t}dt.\end{aligned}$$

To get to the next step, it is used that the integral of a density function of a continuous random variable equals to one. In other words, the integrals $\frac{\lambda^+}{\lambda}\int_{u=0}^{\infty} g(u)du$ and $\frac{\lambda^-}{\lambda}\int_{v=0}^{\infty} f(v)dv$ summed up equal $\frac{\lambda^+}{\lambda} + \frac{\lambda^-}{\lambda} = 1$. Furthermore, the exponential functions are written out of the integrals to get the following inequality

$$\begin{aligned}\phi_k(x) \geq& \int_{t=0}^{\infty} \Big(1 - \frac{\lambda^+}{\lambda} e^{-R(x+ct)} \int_{u=0}^{\infty} e^{-Ru} g(u) du - \frac{\lambda^-}{\lambda} e^{-R(x+ct)} \int_{v=0}^{\infty} e^{Rv} f(v) dv \Big) \lambda e^{-\lambda t} dt \\ =& \int_{t=0}^{\infty} \Big(1 - \frac{\lambda^+}{\lambda} e^{-R(x+ct)} \mathbb{E}[e^{-RC_i}] - \frac{\lambda^-}{\lambda} e^{-R(x+ct)} \mathbb{E}[e^{RY_i}] \Big) \lambda e^{-\lambda t} dt \\ =& \int_{t=0}^{\infty} \lambda e^{-\lambda t} dt - \big(\lambda^+ \mathbb{E}[e^{-RC_i}] e^{-Rx} + \lambda^- \mathbb{E}[e^{RY_i}] e^{-Rx}\big) \int_{t=0}^{\infty} e^{-(\lambda + cR)t} dt \\ =& 1 - \big(\lambda^+ \mathbb{E}[e^{-RC_i}] e^{-Rx} + \lambda^- \mathbb{E}[e^{RY_i}] e^{-Rx}\big) \frac{1}{\lambda + cR} \\ =& 1 - e^{-Rx} \frac{\lambda^+ \mathbb{E}[e^{-RC_i}] + \lambda^- \mathbb{E}[e^{RY_i}]}{\lambda + cR}.\end{aligned}$$

If R is a positive root of the adjustment equation, the fraction after e^{-Rx} will equal one. Therefore, we have the inequality $\phi_k(x) \geq 1 - e^{-Rx}$ for all k. This suffices to prove that $\phi(x) \geq 1 - e^{-Rx}$ for all x. □

Not only would we like to have a lower-bound for the non-ruin probability, an integral equation like in the previous analysis would be interesting to look at. Theorem 9 will state this integral equation the non-ruin probability need to satisfy.

Theorem 9. *The non-ruin probability $\phi(x)$ satisfies the following integral equation*

$$\phi(x) = \lambda^+ \int_{t=0}^{\infty} \int_{u=0}^{\infty} \phi(x + ct + u) g(u) e^{-\lambda t} du dt + \lambda^- \int_{t=0}^{\infty} \int_{v=0}^{x+ct} \phi(x + ct - v) f(v) e^{-\lambda t} dv dt.$$

Proof. The equation above is also written in the proof of Theorem 8 only in terms of $\phi_k(x)$ and $\phi_{k-1}(x)$. This states that $\phi_k(x)$ satisfies the following equation:

$$\phi_k(x) = \int_{t=0}^{\infty} \Big(\frac{\lambda^+}{\lambda} \int_{u=0}^{\infty} \phi_{k-1}(x+ct+u) g(u) du + \frac{\lambda^-}{\lambda} \int_{v=0}^{x+ct} \phi_{k-1}(x+ct-v) f(v) dv \Big) \lambda e^{-\lambda t} dt.$$

When taking the limit of $k \to \infty$, on both sides, $\phi_k(x)$ and $\phi_{k-1}(x)$ both decrease to their limit function $\phi(x)$. The integral and limit are interchangeable, therefore the same integral equation still holds for $\phi(x)$. □

4.3 Exponential premium and claim sizes

For the new model, one would also like to have an exact formula for $\phi(x)$ when the claim and premium are exponentially distributed and independent of each other. To derive that formula, the same technique is used as the proof of Theorem 6-(2). Therefore, we will start with the integral equation of Theorem 9, of which we will take the derivative twice to obtain a differential equation. From that differential equation, the exact formula can be derived.

If the claims and stochastic premium are exponentially distributed, their distribution functions are $F(v) = 1 - e^{-av}$ and $G(u) = 1 - e^{-bu}$, respectively with $a, b > 0$. The probability density functions are $f(v) = ae^{-av}$ and $g(u) = be^{-bu}$. The safety loading condition in this case becomes $c + \frac{\lambda^+}{b} > \frac{\lambda^-}{a}$, or equivalently, $c + \frac{\lambda^+}{b} - \frac{\lambda^-}{a} > 0$. When we take the integral equation of Theorem 9,

$$\phi(x) = \lambda^+ \int_{t=0}^{\infty} \int_{u=0}^{\infty} \phi(x+ct+u) g(u) e^{-\lambda t} du dt + \lambda^- \int_{t=0}^{\infty} \int_{v=0}^{x+ct} \phi(x+ct-v) f(v) e^{-\lambda t} dv dt. \quad (4.2)$$

and fill in the probability density functions, we obtain the following equation.

$$\phi(x) = \lambda^+ \int_{t=0}^{\infty} \int_{u=0}^{\infty} \phi(x+ct+u)be^{-bu}e^{-\lambda t}dudt + \lambda^- \int_{t=0}^{\infty} \int_{v=0}^{x+ct} \phi(x+ct-v)ae^{-av}e^{-\lambda t}dvdt. \tag{4.3}$$

For ease of notation, we will call the first double integral of Equation (4.3) I_1 and the second double integral will be called I_2. Therefore we have

$$\phi(x) = \lambda^+ I_1 + \lambda^- I_2. \tag{4.4}$$

Using change of variables $y = x + ct + u$ and $z = x + ct - v$, Equation (4.3) becomes:

$$\phi(x) = \lambda^+ \int_{t=0}^{\infty} \int_{y=x+ct}^{\infty} \phi(y)be^{-b(y-x-ct)}e^{-\lambda t}dydt + \lambda^- \int_{t=0}^{\infty} \int_{z=0}^{x+ct} \phi(z)ae^{-a(x+ct-z)}e^{-\lambda t}dzdt \tag{4.5}$$

Differentiating Equation (4.5) on both sides with respect to x using Leibniz integral rule, we obtain

$$\begin{aligned}
\phi'(x) &= \lambda^+ \int_{t=0}^{\infty} \frac{d}{dx}\Big(\int_{y=x+ct}^{\infty} \phi(y)be^{-b(y-x-ct)}e^{-\lambda t}dy\Big)dt \\
&+ \lambda^- \int_{t=0}^{\infty} \frac{d}{dx}\Big(\int_{z=0}^{x+ct} \phi(z)ae^{-a(x+ct-z)}e^{-\lambda t}dz\Big)dt \\
&= -b\lambda^+ \int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt + b\lambda^+ \int_{t=0}^{\infty}\int_{y=x+ct}^{\infty} \phi(y)be^{-b(y-x-ct)}e^{-\lambda t}dydt \\
&+ a\lambda^- \int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt - a\lambda^- \int_{t=0}^{\infty}\int_{z=0}^{x+ct} \phi(z)ae^{-a(x+ct-z)}e^{-\lambda t}dzdt \\
&= (a\lambda^- - b\lambda^+) \int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt + b\lambda^+ \int_{t=0}^{\infty}\int_{y=x+ct}^{\infty} \phi(y)be^{-b(y-x-ct)}e^{-\lambda t}dydt \\
&- a\lambda^- \int_{t=0}^{\infty}\int_{z=0}^{x+ct} \phi(z)ae^{-a(x+ct-z)}e^{-\lambda t}dzdt.
\end{aligned}$$

So, when changing the variables y and z back, we obtain the equation

$$\phi'(x) = (a\lambda^- - b\lambda^+) \int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt + b\lambda^+ I_1 - a\lambda^- I_2. \tag{4.6}$$

Using change of variables $w = x + ct$ for the integral, we get

$$\phi'(x) = (a\lambda^- - b\lambda^+) \int_{w=x}^{\infty} \frac{1}{c}\phi(w)e^{-\lambda\frac{w-x}{c}}dw + b\lambda^+ I_1 - a\lambda^- I_2. \tag{4.7}$$

Differentiating Equation (4.7) on both sides we obtain

$$\begin{aligned}
\phi''(x) &= (a\lambda^- - b\lambda^+)\frac{d}{dx}\Big(\int_{w=x}^{\infty} \frac{1}{c}\phi(w)e^{-\lambda\frac{w-x}{c}}dw\Big) \\
&+ b\lambda^+\Big(- b\int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt + b\int_{t=0}^{\infty}\int_{y=x+ct}^{\infty} \phi(y)be^{-b(y-x-ct)}e^{-\lambda t}dydt\Big) \\
&- a\lambda^-\Big(a\int_{t=0}^{\infty} \phi(x+ct)e^{-\lambda t}dt - a\int_{t=0}^{\infty}\int_{z=0}^{x+ct} \phi(z)ae^{-a(x+ct-z)}e^{-\lambda t}dzdt\Big) \\
&= (a\lambda^- - b\lambda^+)\Big(\frac{-1}{c}\phi(x) + \frac{\lambda}{c}\int_{w=x}^{\infty} \frac{1}{c}\phi(w)e^{-\lambda\frac{w-x}{c}}dw\Big) - (b^2\lambda^+ + a^2\lambda^-)\int_{t=0}^{\infty}\phi(x+ct)e^{-\lambda t}dt \\
&+ b^2\lambda^+ I_1 + a^2\lambda^- I_2.
\end{aligned}$$

If we call $\int_{t=0}^{\infty}\phi(x+ct)e^{-\lambda t}dt = I_3$ and change the variable w back to t in the integral, we obtain for the second derivative the equation

$$\phi''(x) = -\frac{1}{c}(a\lambda^- - b\lambda^+)\phi(x) + \Big(\frac{\lambda}{c}(a\lambda^- - b\lambda^+) - b^2\lambda^+ - a^2\lambda^-\Big)I_3 + b^2\lambda^+ I_1 + a^2\lambda^- I_2. \quad (4.8)$$

Therefore, we get the following equations for $\phi(x), \phi'(x)$ and $\phi''(x)$

$$\phi(x) = \lambda^+ I_1 + \lambda^- I_2 \quad (4.9)$$

$$\phi'(x) = (a\lambda^- - b\lambda^+)I_3 + b\lambda^+ I_1 - a\lambda^- I_2, \quad (4.10)$$

$$\phi''(x) + \frac{1}{c}(a\lambda^- - b\lambda^+)\phi(x) = \Big(\frac{\lambda}{c}(a\lambda^- - b\lambda^+) - b^2\lambda^+ - a^2\lambda^-\Big)I_3 + b^2\lambda^+ I_1 + a^2\lambda^- I_2. \quad (4.11)$$

If we multiply Equation (4.9) with ab and multiply Equation (4.10) with $(b-a)$ and sum this together, we obtain

$$ab\phi(x) + (b-a)\phi'(x) - ab\lambda I_3 = (-b^2\lambda^+ - a^2\lambda^-)I_3 + b^2\lambda^+ I_1 + a^2\lambda^- I_2. \quad (4.12)$$

Therefore, we can replace a part of the right side of Equation (4.11) with the left side of Equation (4.12) and we obtain the equation

$$\phi''(x) + \frac{1}{c}(a\lambda^- - b\lambda^+)\phi(x) = \frac{\lambda}{c}(a\lambda^- - b\lambda^+)I_3 + ab\phi(x) + (b-a)\phi'(x) - ab\lambda I_3$$

All in all, we get the equation

$$\phi''(x) + f\phi'(x) + g\phi(x) = h\int_{t=0}^{\infty}\phi(x+ct)e^{-\lambda t}dt, \quad (4.13)$$

where $f = -(b-a), g = \frac{1}{c}(a\lambda^- - b\lambda^+) - ab$ and $h = \frac{\lambda}{c}(a\lambda^- - b\lambda^+) - ab\lambda$. So, we can see that $h = \lambda g$ and with the change of variables $w = x + ct$ Equation (4.13) becomes

$$\phi''(x)+f\phi'(x)+g\phi(x) = \lambda g\int_{w=x}^{\infty}\frac{1}{c}\phi(w)e^{-\lambda(\frac{w-x}{c})}dw, \text{ where } f = a-b, g = \frac{1}{c}(a\lambda^- - b\lambda^+) - ab.$$

Differentiating again on both sides gives us

$$\phi'''(x) + f\phi''(x) + g\phi'(x) = -\frac{\lambda g}{c}\phi(x) + \lambda g\cdot\frac{\lambda}{c}\int_{w=x}^{\infty}\frac{1}{c}\phi(w)e^{-\lambda(\frac{w-x}{c})}dw. \quad (4.14)$$

Seeing that $\lambda g\int_{w=x}^{\infty}\frac{1}{c}\phi(w)e^{-\lambda(\frac{w-x}{c})}dw$ equals to the left side of Equation (4.13) we obtain:

$$\phi'''(x) + f\phi''(x) + g\phi'(x) = -\frac{\lambda g}{c}\phi(x) + \frac{\lambda}{c}(\phi''(x) + f\phi'(x) + g\phi(x)) \Longrightarrow \quad (4.15)$$

$$\phi'''(x) + (f - \frac{\lambda}{c})\phi''(x) + (g - \frac{\lambda f}{c})\phi'(x) = 0. \quad (4.16)$$

Let's call $h(x) = \phi'(x)$, then we have a second order differential equation in the form of

$$h''(x) + (f - \frac{\lambda}{c})h'(x) + (g - \frac{\lambda f}{c})h(x) = 0. \quad (4.17)$$

This can be solved using the auxiliary equation, where we substitute $h(x) = e^{kx}$ in Equation (4.17):

$$k^2e^{kx} + k(f - \frac{\lambda}{c})e^{kx} + (g - \frac{\lambda f}{c})e^{kx} = 0 \Rightarrow \qquad k^2 + (f - \frac{\lambda}{c})k + g - \frac{\lambda f}{c} = 0$$

This is a quadratic equation, where we find the solution for k to be

$$k_{1,2} = \frac{\lambda - fc \pm \sqrt{(fc-\lambda)^2 - 4c(gc-\lambda f)}}{2c} \tag{4.18}$$

To see if the solutions for k are real numbers, we need to use the safety condition $c + \frac{\lambda^+}{b} - \frac{\lambda^-}{a} > 0$. Now, if we fill in the parameters for f and g, we get for the term $-4c(gc - \lambda f)$ the following

$$\begin{aligned} -4c(gc-\lambda f) &= -4c(-a\lambda^+ + b\lambda^- - abc) \\ &= 4ac\lambda^+ - 4bc\lambda^- + 4abc^2. \end{aligned}$$

When divided by $4abc(> 0)$, we obtain for the term $-4c(gc - \lambda f)$ the following: $\frac{\lambda^+}{b} - \frac{\lambda^-}{a} + c$, which following the safety condition is larger than zero. Therefore, the term $-4c(gc - \lambda f)$ is positive and combining it with that a square always is positive, we have that the root in the solutions is real.
Furthermore, we have that $\sqrt{(fc-\lambda)^2 - 4c(gc-\lambda f)} > \sqrt{(fc-\lambda)^2} = |fc-\lambda|$. Now, we have two cases, namely $\lambda - fc > 0$ and $\lambda - fc < 0$. Define k_1 as the positive root solution, it holds that $k_1 > \frac{\lambda - fc + |fc-\lambda|}{2c} \geq 0$. Define k_2 as the negative root solution, then it holds that $k_2 < \frac{\lambda - fc - |fc-\lambda|}{2c} \leq 0$. This holds in both cases. Therefore, we have that $k_1 > 0$ and $k_2 < 0$.
Hence, we have that the solution for the differential equation of $h(x)$ becomes $h(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}$, with $C_1, C_2 \in \mathbb{R}$ and $k_1 > 0, k_2 < 0$.
Recall that $h(x) = \phi'(x)$, hence we obtain the solution for $\phi(x)$ by integrating the solution of $h(x)$:

$$\phi(x) = \int h(x) = \int C_1 e^{k_1 x} + C_2 e^{k_2 x} = C_0 + D_1 e^{k_1 x} + D_2 e^{k_2 x} \text{ where } C_0, D_1, D_2 \in \mathbb{R}. \tag{4.19}$$

The next step is to solve for the constants C_0, D_1 and D_2 with the conditions

$$\lim_{x\to\infty} \phi(x) = 1 \tag{4.20}$$

and

$$\phi(0) = \lambda^+ \int_{t=0}^{\infty} \int_{u=0}^{\infty} \phi(ct+u) b e^{-bu} e^{-\lambda t} du dt + \lambda^- \int_{t=0}^{\infty} \int_{v=0}^{ct} \phi(ct-v) a e^{-av} e^{-\lambda t} dv dt. \tag{4.21}$$

The first condition (4.20) gives us the following relation

$$\begin{aligned} &\lim_{x\to\infty} (C_0 + D_1 e^{k_1 x} + D_2 e^{k_2 x}) = 1 \\ &C_0 + D_1 \lim_{x\to\infty} e^{k_1 x} = 1 \end{aligned}$$

This gives us that $D_1 = 0$ and $C_0 = 1$, because if $D_1 \neq 0$ the limit of $e^{k_1 x}$ goes to infinity from the positivity of k_1. Hence, we obtain $\phi(x) = 1 + D_2 e^{k_2 x}$. Filling this in for $\phi(x)$ in condition (4.21) gives us

$$1 + D_2 = \lambda^+ \int_0^{\infty} \int_0^{\infty} (1 + D_2 e^{k_2(ct+u)}) b e^{-bu} e^{-\lambda t} du dt + \lambda^- \int_0^{\infty} \int_0^{ct} (1 + D_2 e^{k_2(ct-v)}) a e^{-av} e^{-\lambda t} dv dt. \tag{4.22}$$

The double integrals after working them out become the following

$$\begin{aligned}\int_0^\infty \int_0^\infty (1 + D_2 e^{k_2(ct+u)}) b e^{-bu} e^{-\lambda t} du dt &= \int_0^\infty \int_0^\infty b e^{-bu} e^{-\lambda t} du dt + \\ &D_2 \int_0^\infty \int_0^\infty b e^{(k_2-b)u} e^{(k_2 c - \lambda)t} du dt \\ &= \int_0^\infty e^{-\lambda t} dt - D_2 \int_0^\infty \frac{b}{k_2 - b} e^{(k_2 c - \lambda)t} dt \\ &= \frac{1}{\lambda} + D_2 \frac{b}{(k_2 - b)(k_2 c - \lambda)};\end{aligned}$$

$$\begin{aligned}\int_0^\infty \int_0^{ct} (1 + D_2 e^{k_2(ct-v)}) a e^{-av} e^{-\lambda t} dv dt &= \int_0^\infty \int_0^{ct} a e^{-av} e^{-\lambda t} dv dt + \\ &D_2 \int_0^\infty \int_0^{ct} a e^{-(a+k_2)v} e^{(k_2 c - \lambda)t} dv dt \\ &= -\int_0^\infty e^{-(\lambda+ac)t} dt + \int_0^\infty e^{-\lambda t} dt - \frac{D_2 a}{a + k_2} \int_0^\infty e^{-(\lambda+ac)t} dt + \\ &\frac{D_2 a}{a + k_2} \int_0^\infty e^{(k_2 c - \lambda)t} dt \\ &= \frac{-1}{\lambda + ac} + \frac{1}{\lambda} - \frac{aD_2}{(a + k_2)(\lambda + ac)} - \frac{aD_2}{(a + k_2)(k_2 c - \lambda)}.\end{aligned}$$

Combining this and filling it in in Equation (4.22) we obtain

$$1 + D_2 = \lambda^+ \left(\frac{1}{\lambda} + D_2 \frac{b}{(k_2 - b)(k_2 c - \lambda)}\right) + \lambda^- \left(\frac{-1}{\lambda + ac} + \frac{1}{\lambda} - \frac{aD_2}{(a + k_2)(\lambda + ac)} - \frac{aD_2}{(a + k_2)(k_2 c - \lambda)}\right)$$

Using that $\lambda^+ + \lambda^- = \lambda$ and writing all D_2 together, we get that D_2 equals

$$D_2 = \frac{\frac{-\lambda^-}{\lambda+ac}}{1 - \frac{b\lambda^+}{(k_2-b)(k_2c-\lambda)} + \frac{a\lambda^-}{(a+k_2)(\lambda+ac)} + \frac{a\lambda^-}{(a+k_2)(k_2c-\lambda)}}. \tag{4.23}$$

After rewriting the fractions, we obtain for D_2:

$$D_2 = \frac{-\lambda^-(k_2 c - \lambda)(k_2 - b)}{(\lambda + ac)(k_2 c - \lambda)(k_2 - b) - abc\lambda - b\lambda^+\lambda + ack_2\lambda^-} \tag{4.24}$$

All in all, the solution for $\phi(x)$ equals $\phi(x) = 1 + D_2 e^{k_2 x}$ where we have defined k_2 as $k_2 = \frac{\lambda - fc - \sqrt{(fc-\lambda)^2 - 4c(gc - \lambda f)}}{2c}$, D_2 from Equation (4.24), $f = a - b$ and $g = \frac{1}{c}(a\lambda^- - b\lambda^+) - ab$. This formula only depends on the parameters $\lambda^+, \lambda^-, a, b$ and c.

4.4 Comparison with previous models

To check if the lower-bound, the integral equation and exact formula for $\phi(x)$ are correct, one can compare the results with the previous models. This can be done by reducing the mathematical model back to the previous models with slight adjustments. At first, the model will be reduced to the Cramér-Lundberg model by setting λ^+ to zero. This way, no stochastic premium comes in at the company and the model only has linear premium. To adjust the model to only stochastic premium like the model from Chapter 3, one can set c to zero. If the theory is correct, the same results would be obtained when reducing the new analysis.

4.4.1 Cramér-Lundberg model

As said before, to reduce the model, λ^+ will be equal to zero in the this subsection. First, we will look if the adjustment equation for Theorem 8 will be the same as in Chapter 2. After that, the integral equation from Theorem 9 will be checked with the equation from Theorem 2.

Adjustment equation and lower-bound

Using the adjustment equation from Theorem 8 we have that $\lambda^+\mathbb{E}[e^{-RC_i}] + \lambda^-\mathbb{E}[e^{RY_i}] = \lambda + cR$, where $\lambda = \lambda^- + \lambda^+$. If $\lambda^+ = 0$, the equation becomes the following:

$$\lambda^-\mathbb{E}[e^{RY_i}] = \lambda^- + cR.$$

In Chapter 2, λ is the rate of the claim process. In this notation, λ^- is the rate of the claim process. Therefore, the adjustment equation is the same as in the Cramér-Lundberg model. The lower-bound for $\psi(x)$ still holds and the upper-bound for $\phi(x)$ also holds.

Integral equation

The integral equation of Theorem 9 reduces to the following equation if $\lambda^+ = 0$:

$$\phi(x) = \lambda^- \int_{t=0}^{\infty}\int_{v=0}^{x+ct} \phi(x+ct-v)f(v)e^{-\lambda^- t}dvdt$$

Comparing it to Theorem 2 for Chapter 2, we did not obtain precisely the same result as in terms of notation. However, noting that $P(y)$ is the cdf of the claim sizes and λ is the rate of the claim process in the first theorem, we have found the same result in the reduced model.

Exponential claims

To reduce the exact formula for $\phi(x)$ from Section 4.3, we have that $\lambda^+ = 0$ and $b \in \mathbb{R}$. When we set $\lambda^+ = 0$, the formula for D_2 and k_2 reduces to

$$\begin{aligned} k_2 &= \frac{\lambda^- - (a-b)c - \sqrt{((a-b)c-\lambda^-)^2 - 4c(a\lambda^- - abc - (a-b)\lambda^-)}}{2c} \\ &= \frac{\lambda^- - (a-b)c - \sqrt{(\lambda^- - (a+b)c)^2}}{2c} \\ &= \frac{\lambda^- - (a-b)c - |\lambda^- - (a+b)c|}{2c}, \end{aligned}$$

and because of the safety loading condition, we know that $c > \frac{\lambda^-}{a} \Rightarrow ac - \lambda^- > 0$. Thus $k_2 = \frac{\lambda^- -(a-b)c-((a+b)c-\lambda^-)}{2c} = \frac{\lambda^-}{c} - a$. Also, reducing it for D_2 gives us

$$D_2 = \frac{\frac{-\lambda^-}{\lambda+ac}}{1 + \frac{a\lambda^-}{(a+k_2)(\lambda+ac)} + \frac{a\lambda^-}{(a+k_2)(k_2c-\lambda)}}$$
$$= \frac{-\frac{\lambda^-}{\lambda^-+ac}}{\frac{(\lambda^-+ac)((\frac{\lambda^-}{c}-a)c-\lambda^-)+ac\lambda^-}{(\lambda^-+ac)((\frac{\lambda^-}{c}-a)c-\lambda^-)}}$$
$$= \frac{ac\lambda^-}{-ac(\lambda^- + ac) + ac\lambda^-} = \frac{-\lambda^-}{ac}.$$

Comparing it to the exact formula of Theorem 4, we have some small issues with notation. Theorem 4 states that $\phi(x) = 1 - \frac{\lambda}{\mu c}e^{-\mu(1-\frac{\lambda}{\mu c})x}$. In our case, we have that $a = \mu$ and $\lambda^- = \lambda$. Furthermore, we have that $\phi(x) = 1 + D_2e^{k_2x}$. Filling in the given D_2 and k_2 we obtain

$$\phi(x) = 1 - \frac{\lambda^-}{ac}e^{(\frac{\lambda^-}{c}-a)x} = 1 - \frac{\lambda^-}{ac}e^{-a(1-\frac{\lambda^-}{ac})x}.$$

Therefore, we obtain the same result as in Theorem 4 when reducing the exact formula from Section 4.3.

4.4.2 Stochastic premium model

To reduce the model to the stochastic premium model from Chapter 3, the linear premium, c, will be equal to zero. The comparison will be the same as the previous sections.

Adjustment equation and lower-bound

If $c = 0$ in the adjustment equation of Theorem 8, the equation will become:

$$\lambda^+\mathbb{E}[e^{-RC_i}] + \lambda^-\mathbb{E}[e^{RY_i}] = \lambda$$

In this equation, $\lambda = \lambda^- + \lambda^+$. This looks quite similar to adjustment equation of Corollary 5.1, but it is not precisely the same. When rearranging the equation we just obtained, we get $\lambda^+(\mathbb{E}[e^{-RC_i}] - 1) + \lambda^-(\mathbb{E}[e^{RY_i}] - 1) = 0$. This is exactly the same as in Corollary 5.1. We still have the same lower-bound for both models.

Integral equation

The integral equation of Theorem 9 will become the following if $c = 0$:

$$\phi(x) = \lambda^+ \int_{t=0}^{\infty}\int_{u=0}^{\infty} \phi(x+u)g(u)e^{-\lambda t}dudt + \lambda^- \int_{t=0}^{\infty}\int_{v=0}^{x} \phi(x-v)f(v)e^{-\lambda t}dvdt.$$

This is not directly the same integral equation of Theorem 5, because of the double integral still. Only, one can see that the integral containing $\phi(x)$ is not dependent of t, so the integral depending on t can be calculated. Using the knowledge of $\int_{t=0}^{\infty} e^{-\alpha t}dt = \frac{1}{\alpha}$, the equation becomes:

$$\phi(x) = \lambda^+/\lambda \int_{u=0}^{\infty} \phi(x+u)g(u)du + \lambda^-/\lambda \int_{v=0}^{x} \phi(x-v)f(v)dv$$

With the knowledge that $\lambda = \lambda^- + \lambda^+$ we have obtained the same integral equation as in Theorem 5.

Exponential claims and premium

To reduce the exact formula for $\phi(x)$ from Section 4.3, we need to take the limit of c to zero. Therefore, we get for k_2 the following limit

$$\lim_{c\to 0+} k_2 = \lim_{c\to 0+} \frac{\lambda - fc - \sqrt{(fc-\lambda)^2 - 4c((\lambda^- a - b\lambda^+) - abc - \lambda f)}}{2c}.$$

Because both numerator and denominator go to zero when $c \to 0$, we can apply L'Hopital's rule and differentiate w.r.t. c for both the numerator and denominator. Call the numerator $f(c)$, then $f'(c)$ equals

$$f'(c) = -f - \frac{1}{2}((fc-\lambda)^2 - 4c((\lambda^- a - b\lambda^+) - abc - \lambda f))^{-\frac{1}{2}}\left(2f(fc-\lambda) - 4((\lambda^- a - b\lambda^+) - abc - \lambda f) + 4abc\right)$$

and when filled in $c = 0$ we have that $f'(0) = -2f + 2\frac{\lambda^- a - b\lambda^+}{\lambda}$. Therefore, the limit of k_2 becomes

$$\lim_{c\to 0+} k_2 = \lim_{c\to 0+} \frac{f'(c)}{2} = \frac{-2f + 2\frac{\lambda^- a - b\lambda^+}{\lambda}}{2} = -f - \frac{\lambda^- a - b\lambda^+}{\lambda} = -(a-b) + \frac{\lambda^- a - b\lambda^+}{\lambda} = \frac{b\lambda^- - a\lambda^+}{\lambda}.$$

This is exactly the same as Q from Theorem 6-(2), with the knowledge that $\lambda = \lambda^+ + \lambda^-$. Given that D_2 equals

$$\frac{-\lambda^-(k_2 c - \lambda)(k_2 - b)}{(\lambda + ac)(k_2 c - \lambda)(k_2 - b) - abc\lambda - b\lambda^+\lambda + ack_2\lambda^-},$$

and that $k_2 \cdot c$ goes to zero, when taking the limit of c to zero, D_2 can be reduced to

$$\begin{aligned}\lim_{c\to 0+} D_2 &= \lim_{c\to 0+} \frac{-\lambda^-(-\lambda)(k_2 - b)}{\lambda(-\lambda)(k_2 - b) - b\lambda^+\lambda} = \frac{\lambda^-\lambda(\frac{b\lambda^- - a\lambda^+}{\lambda} - b)}{-\lambda^2(\frac{b\lambda^- - a\lambda^+}{\lambda} - b) - b\lambda^+\lambda} \\ &= \frac{\lambda^-(b\lambda^- - a\lambda^+ - b\lambda}{-\lambda(b\lambda^- - a\lambda^+ - b\lambda + b\lambda^+)} = \frac{-(a+b)\lambda^+\lambda^-}{a\lambda^+\lambda} = \frac{-(a+b)\lambda^-}{a\lambda}.\end{aligned}$$

Given that $\phi(x) = 1 + D_2 e^{k_2 x}$ and filling in the k_2 and D_2 we have obtained, the formula can be reduced to

$$\phi(x) = 1 - \frac{-(a+b)\lambda^-}{a\lambda} e^{\frac{b\lambda^- - a\lambda^+}{\lambda} x}.$$

Hence, when taking the limit of c to zero of the exact formula for $\phi(x)$ from Section 4.3, we obtain the same exact formula as derived in the proof for Theorem 6-(2) with the notation that $\lambda = \lambda^+ + \lambda^-$.

All in all, the new analysis is correct when reducing the new model back to the previous models.

5 Simulation

Not only would one like to have exact formula's for the ruin probability, one would also like to visualize how the ruin probability behaves when the initial capital increases. For exponential claims and stochastic premium, this can be done with the exact formula derived in the analysis. But this can also be done by making a simulation of the models and calculating the ruin probability from the simulation. This way, an approximation for the ruin probability function can be made for other scenario's, when deriving an exact formula is not possible. A simulation is also a great method to compare the influence of the parameters from the models. The simulation will also estimate the time to ruin, when ruin occurs during a simulation. The time to ruin is the amount of time it takes before the company is ruined. To compare and visualize the ruin probability and the time to ruin of the three models, a simulation is written.

5.1 Description of simulation

First, the risk process of the three models is implemented. The simulation will simulate $X(t)$, where we have both linear and stochastic premium. In this way, when we want to compare the models during the simulation, one only need to change the parameters at the beginning of the algorithm. The simulation was written in the language R. This program was derived from an existing program from the TU/e course "2DF30: Insurance and Credit Risk". This program already simulated the Cramér-Lundberg model and estimated the ruin probability. For our simulation, we only needed to add the stochastic premium process and that was implemented in Algorithm 1.

Algorithm 1 Simulation of $X(t)$, one simulation

Initiate parameters: λ^-, λ^+, mean premium size, mean claim size, time horizon, initial capital, linear premium
Set λ , the rate of the Poisson process to λ^- + λ^+
function SIMCOMPOUNDPOISSON2(N)
    Create inter-arrival times of events
    Sum the inter-arrival times
    Create N randomly distributed claim sizes and premium sizes
    Make an empty vector for the sizes of the events
    Create N random numbers between (0,1) from a random number generator
    for i in 1:N **do**
        if random number[i] < $\frac{\lambda^-}{\lambda}$ **then**
            event size[i] = - claim size[i]
        else
            event size[i] = premium size[i]
    Sum the event sizes
    Make boolean ruin
    Make indicator when ruin occurs
    Create time to ruin as moment when indicator occurs in the arrival times
    Create a list of results containing ruin and time to ruin
    return List of results

Algorithm 1 will only run one simulation of the risk process, with a certain initial capital. This is not enough to simulate the ruin probability. To simulate the ruin probability, we use this algorithm and let it run multiple times. If we keep track of when ruin occurred, we know the fraction of time that ruin occurs in a certain setting. This will equal the ruin probability of $X(t)$ with a given initial capital x. Therefore, to find a function for the ruin probability, we need to run the simulation with multiple initial capital settings. This will be done in Algorithm 2. Here, we find the ruin probability as a function of the initial capital.

Algorithm 2 Simulation of ruin probability

Initiate amount of runs
Initiate N, the amount of events
Create a vector of initial capitals
for number of runs **do**
    for amount of initial capitals **do**
        Set initial capital to certain value
        Run *SimCompoundPoisson2(N)*
        Add boolean ruin to results
Ruin probability = $\frac{results}{\#runs}$

To show the results, multiple plotting functions in R are used. Those will not be explained, because it is a straight forward plotting function. The specifics are not important for this simulation. Because we are dealing with a form of Monte-Carlo simulation, we would also like to obtain some 95% confidence intervals of the ruin probability. This will be explained in Algorithm 3 and the method came from the lecture notes "Stochastic Simulation " [2](page 26).

Algorithm 3 Computation of (1-α)% confidence interval of the ruin probability

Initiate amount of runs, number of events and initial capital
for number of runs **do**
  Set initial capital to certain value
  Run *SimCompoundPoisson2(N)*
  Add boolean ruin to results
Ruin probability = $\hat{p} = \frac{results}{\#runs}$
Set confidence level α
Initiate $z_{\alpha/2}$
Confidence interval = $\hat{p} \pm \frac{z_{\alpha/2} \cdot \sqrt{\hat{p}(1-\hat{p})}}{\sqrt{\#runs}}$

Furthermore, if ruin occurs, one would like to know how long it takes for it happens. This is called the Time to ruin. Algorithm 4 will calculate the time to ruin for a given initial capital, using the function *simCompoundPoisson2(N)*. This will be done for multiple runs, from which we can find the mean Time to ruin. Also, the 95% confidence interval can directly be calculated from this algorithm.

Algorithm 4 Computation of average time to ruin and 95% confidence interval

Initiate amount of runs, the amount of events(N), and value of initial capital
Create empty vector for Time-to-ruin
for number of runs **do**
  Set initial capital to certain value
  Run simCompoundPoisson2(N)
  Time-to-ruin[i] = timetoruin of simCompoundPoisson2(N)
Average time to ruin = mean(Time-to-ruin)
Compute standard deviation of Time-to-ruin
Confidence interval = Average time to ruin $\pm$ 1.96* $\frac{standarddeviation}{\sqrt{\#ruins}}$

5.2 Results

The numerical results from the simulation will be displayed in this section. The results of each model shall be discussed and the given settings will be explained. After the results of each model, a comparison of the three models will be displayed in one figure. This will give a visual comparison between the models.

5.2.1 Accuracy

To test the accuracy of the program, the amount of simulation runs had to be chosen. The amount of simulations runs, N, was varied so that the amount of runs which would suffice could be determined. Decided to look only at one given initial capital, for the results would be too much to process and the simulation would take too much time. The settings for the parameters of the first model were $\lambda = 1$, $c = 2$ and $\mathbb{E}[Y] = 1.5$. The parameters for the second model were $\lambda^+ = \lambda^- = 1$, $\mathbb{E}[C] = 2$ and $\mathbb{E}[Y] = 1.5$ and for the third model the parameters were $\lambda^+ = \lambda^- = c = \mathbb{E}[C] = 1$ and $\mathbb{E}[Y] = 1.5$. The confidence intervals for the ruin probability with the initial capital equal to five are given in Table 5.1.

95% Confidence interval			
$x = 5$	$N = 100$	$N = 1000$	$N = 2000$
First model	[0.257 ; 0.443]	[0.296 ; 0.354]	[0.318 ; 0.360]
Second model	[0.412 ; 0.608]	[0.555 ; 0.617]	[0.557 ; 0.600]
Third model	[0.314 ; 0.506]	[0.392 ; 0.454]	[0.395 ; 0.439]

Table 5.1: 95% Confidence interval of $\psi(x)$

As can be seen in the table, the confidence intervals become smaller when the amount of simulation runs is increased for all models. The results will become more accurateuphwhen we chose $N = 2000$ as amount of runs. Because the simulation needs to be run for more initial capitals, the time that it will run will increase. Therefore, it is chosen not to increase the amount of runs more because of time constraints. The simulation will be run with $N = 2000$ and an extra check for accuracy will be done by plotting the theoretical analysis results against the simulation.

5.2.2 Ruin probability in first model

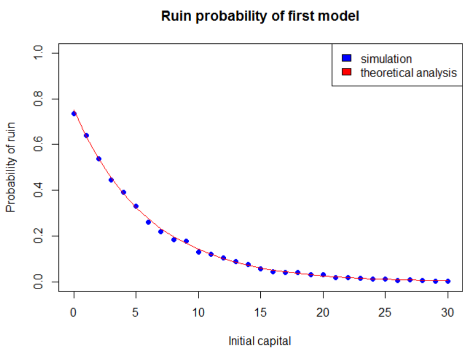

Figure 5.1: Simulation results against the theoretical analysis of the first model

To check if the simulation is accurate, the theoretical formula for $\psi(x)$ found in the analysis is plotted against the simulation. For this figure, it is chosen that the linear premium per time unit $c = 2$, the rate of the claim Poisson process $\lambda = 1$ and the mean claim size $\mathbb{E}[Y] = 1.5$. The results can be found in Figure 5.1.
As can be seen in the figure, the simulation results lie on the theoretical function of $\psi(x)$. The blue points of the simulation sometimes do not exactly lie on the red line of the analysis, but there are almost no outliers. It is fair to say that the simulation is accurate with the theoretical analysis from Section 2.1.
To see how the parameters λ, c and $\mathbb{E}[Y_i]$ influence the ruin probability, multiple simulations were run with different settings for the parameters. Those results can be found in Figure 5.2.

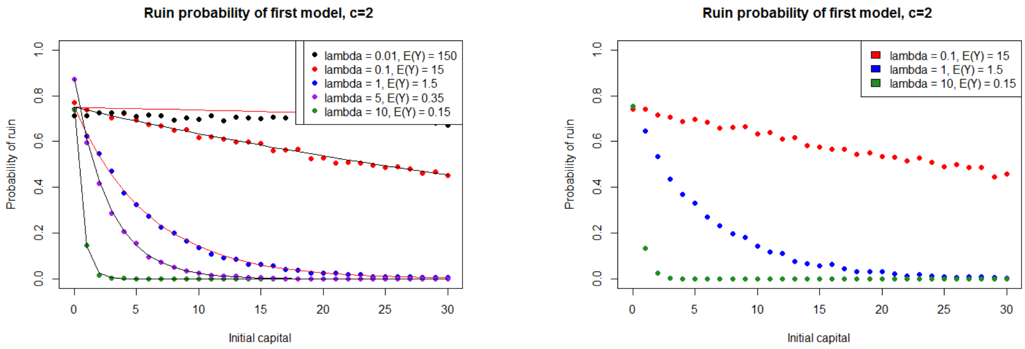

Figure 5.2: Alternating λ and $\mathbb{E}[Y]$

Figure 5.2 is created with the same linear premium per time unit, while alternating the rate of the claim process and the average claim size. To not have a ruin probability of 1, the safety loading condition needs to be satisfied. Therefore, we need to have that $c > \lambda\mathbb{E}[Y]$. From the figure, one can see that the ruin probability decreases when the mean claim size decreases and the rate of the claim process increases. The ruin probability decreases extremely quickly when $\mathbb{E}[Y] = 0.15$ and $\lambda = 10$. Therefore, it is preferable to have a lot of claims coming in, but the claim size to be small. Also can be seen that when $\lambda = 0.1$ or 0.01 the ruin probability stays above 0.4.Thus, in 40% of the cases, the insurance company will be ruined if their initial capital is less than the average claim size. Therefore, when little claims come in, but the claim size is extremely large compared to the linear premium, the initial capital needs to be much larger compared to small claim sizes. Even when the safety loading condition is satisfied, one needs to have more initial capital to cover the large claim sizes.

5.2.3 Ruin probability in second model

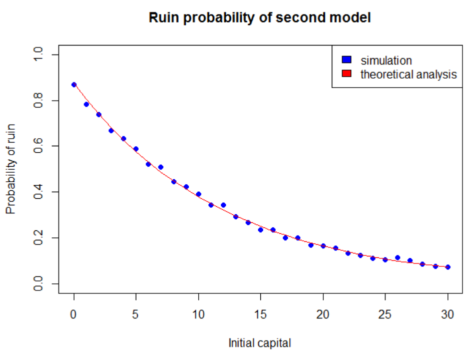

Figure 5.3: Simulation results against the theoretical analysis of the second model

Again, to check the simulation in the second model, the theoretical formula for $\psi(x)$ from the analysis in Section 3 is plotted against the simulation results.
For the simulation, it is chosen to have $\lambda^+ = \lambda^- = 1, \mathbb{E}[C] = 2$ and $\mathbb{E}[Y] = 1.5$. Therefore, the safety loading condition $\lambda^+\mathbb{E}[C] = 2 > 1.5 = \lambda^-\mathbb{E}[Y]$ is satisfied. As can be seen, the numerical results are quite accurate. The results are a bit less accurate as the first model, because there are more stochastic variables to simulate. But, the numerical results lie almost completely on the theoretical analysis line. Therefore, we can say that the simulation is accurate compared to the analysis done in Section 3.
To investigate the influence of the parameters on the ruin probability, multiple options can be checked. We now deal with four parameters, which influence the ruin probability. First, to see the influence of the claim process, the rate λ^- and claim size $\mathbb{E}[Y]$ are altered while having the premium process the same. Secondly, the influence of the premium process is observed by changing the parameters λ^+ and

$\mathbb{E}[C]$. At last, the rate of both Poisson process will equal the same, while they will be altered to check the influence of the rates. The results can be found in Figures 5.4, 5.5 and 5.6.

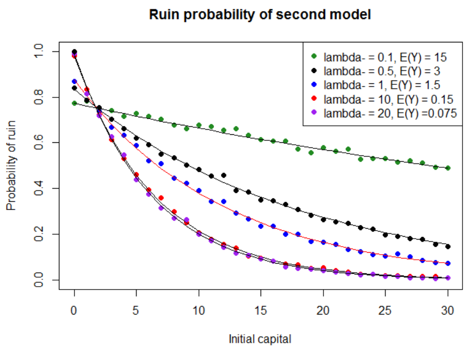

Figure 5.4: Alternating λ^- and $\mathbb{E}[Y]$

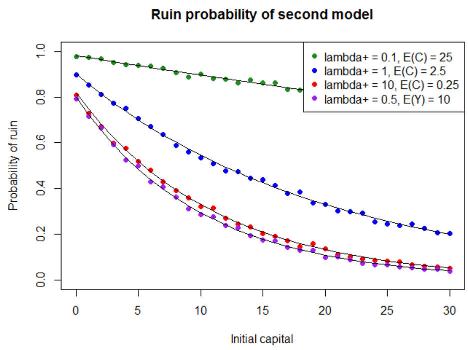

Figure 5.5: Alternating λ^+ and $\mathbb{E}[C]$

Figure 5.4 is obtained by setting λ^+ to one and $\mathbb{E}[C]$ to two. As one sees, the highest ruin probability is obtained when the mean claim sizes are 15, while the rate of the claim process is the lowest. Furthermore, when the rate increases and the mean claim size decreases, the ruin probability becomes lower for all initial capitals. This behavior was also found in the first model. Also, as can be seen by the lines where $\lambda^- = 20$ and $\lambda^- = 10$, it does not matter if the rate doubles and the claim sizes halves. This will give almost the same function for the ruin probability. All in all, compared to Figure 5.2, the ruin probability is greater in the second model, while the safety condition is almost the same.
Figure 5.5 is obtained by setting λ^- to one and $\mathbb{E}[Y]$ to two. Immediately, one can see that the ruin probabilities are higher than in Figure 5.4. Same as in previous figures, when the rate of the premium process is low and the premium size is large, the ruin probability will be the highest for all initial capitals. The ruin probability behaves the same as in Figure 5.4 where it decreases when the rate increases and the premium size decreases. Therefore, it does not matter if claim process changes or the premium process. The ruin probability will decrease when the rate of risk process increases and the mean claim or premium size decreases.

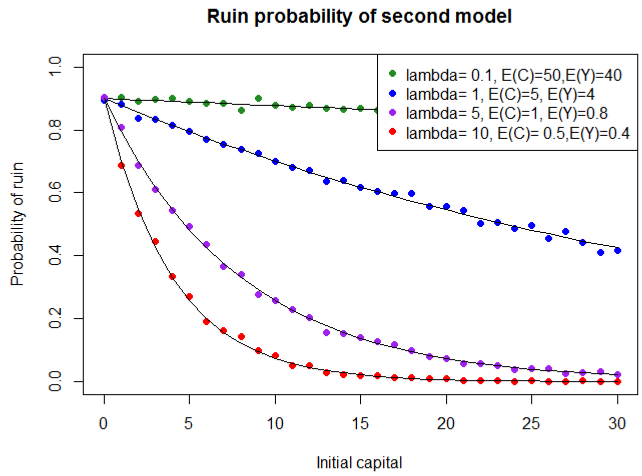

Figure 5.6: Alternating λ^- and λ^+

The safety loading condition in Figure 5.6 is satisfied by $\lambda^+\mathbb{E}[C] = 5 > 4 = \lambda^-\mathbb{E}[Y]$. The rate of both the claim process and the premium process are the same. As seen before, the ruin probability follows the same behavior of decreasing when the mean claim sizes are small. This suggests that the mean claim size is a greater influence for the ruin probability than the rate of the claim process.

5.2.4 Ruin probability in third model

As can be seen in the previous sections, the simulation results are quite accurate compared to the theoretical analysis. Therefore, it was decided to not implement it in the simulation for the third model. As found in the previous simulations of the models and the 95% confidence interval, one can assume that the simulation is reasonable accurate for the third model as well. For the simulation, the safety condition $c + \lambda^+\mathbb{E}[C] > \lambda^-\mathbb{E}[Y]$ needed to be satisfied for useful results. Otherwise, the ruin probability would always equal one. To see the influence of the claim process, it is chosen to set the linear and stochastic premium to a certain value and vary λ^- and $\mathbb{E}[Y]$. It was chosen to set $c = 2, \lambda^+ = 1$ and $\mathbb{E}[C] = 2$. The results of the simulations can be found in Figure 5.7. To see the influence of the stochastic premium on the ruin probability, it was chosen to set the claim process and the linear premium on a certain value. The results can be found in Figure 5.8.

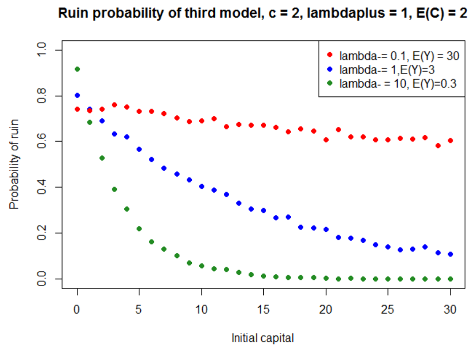

Figure 5.7: Alternating λ^- and $\mathbb{E}[Y]$

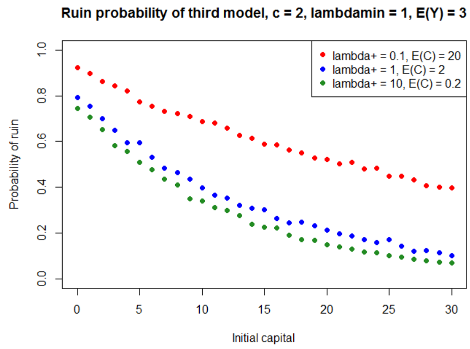

Figure 5.8: Alternating λ^+ and $\mathbb{E}[C]$

From Figure 5.7 one can see that the ruin probability stays above 0.6 when $\lambda^- = 0.1$ and $\mathbb{E}[Y] = 30$ for the initial capitals of 0 to 30. The exponential decrease of the function $\psi(x)$ is almost equal to zero for this situation. Therefore, it is not desirable to have large claim sizes while there are less claims coming in compared to having $\lambda^- = 1$ and $\mathbb{E}[Y] = 3$. A much larger initial capital is needed to pay the claim sizes that are coming in. The same behavior as in the first two models still holds. When the rate of the claim process increases but the claim sizes decreases, the ruin probability is the lowest. Thus it is preferable to have low claim sizes, for a lower initial capital or lower premium per time unit.
As one can see, the ruin probability in Figure 5.8 lies higher than in Figure 5.7 for all situations. Therefore, the stochastic premium process is a great influence on the ruin probability. The ruin probability still decreases when the stochastic premium rate increases, but not as fast as when the rate of the claim process increases. One can say that the expected claim sizes and expected premium sizes are more of influence on the ruin probability than the rate of which those stochastic premium and claims come in at the company.

5.2.5 Comparison of the models

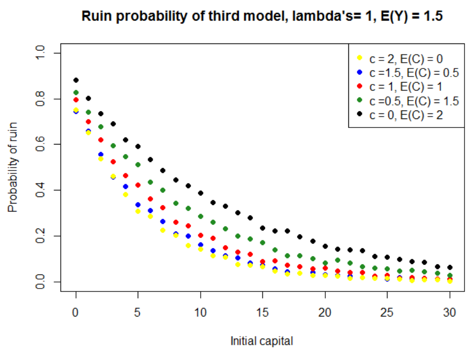

Figure 5.9: Varying percentage of stochastic premium

It is preferable to know which model is the best for the ruin probability, or gives the lowest ruin probability on average when the safety loading condition is the same for all models. This is done, by comparing the models with the simulation. The way the comparison is done, is by setting the rate of the stochastic process at the same value and setting the expected claim size the same for all situations. Then, the amount of stochastic and linear premium is varied, in which the first and second model can be seen as 0% stochastic premium and 100% stochastic premium, respectively. The safety loading condition $c + \lambda^+\mathbb{E}[C] > \lambda^-\mathbb{E}[Y]$ is satisfied because $c + \lambda^+\mathbb{E}[C] = 2 > 1.5 = \lambda^-\mathbb{E}[Y]$. Only the amount of linear and stochastic premium is varied, where we took 0%, 25%, 50%, 75% and 100% of the premium coming in to be stochastic. The results from this simulation can be found in Figure 5.9.

What can be seen from Figure 5.9 that the ruin probability is the lowest when no stochastic premium comes in. The ruin probability is the highest when the premium is completely stochastic. If the premium is a mix of stochastic and linear premium, the ruin probability lies between the first and second model. The blue points lie almost completely at the same level as the yellow points in the figure, which means that having 25% stochastic premium is almost the same as having only linear premium coming in. So, comparing the models, it can be seen that only linear premium is the best in terms of the lowest ruin probability. The second model, having only stochastic premium, is the worst case in terms of ruin probability. It has the highest ruin probability for all initial capitals. The second model has a higher ruin probability than the first model, but is still better than the second model.

5.2.6 Time to ruin

When ruin occurs, one likes to know how long it took to get ruined. This is what is called time to ruin. How this is calculated can be found in Algorithm 4. To compare it for multiple initial capitals, it is chosen to calculate the time to ruin for $x = 0$, $x = 15$ and $x = 30$. The safety loading condition is satisfied for all three models with $2 > 1.5$. The settings for the first model where $c = 2$, $\lambda = 1$ and $\mathbb{E}[Y] = 1.5$. For the second model, we have $\mathbb{E}[C] = 2, \lambda^+ = \lambda^- = 1$ and $\mathbb{E}[Y] = 1.5$ and for the third model we have $\lambda^+ = \lambda^- = 1, \mathbb{E}[C] = c = 1$ and $\mathbb{E}[Y] = 1.5$. The histograms of time to ruin of all models can be found in the Appendix. The results of the simulation can be found in Table 5.2.

	$x = 0$	$x = 15$	$x = 30$
First model	3.01	25.54	46.72
95% CI	*[2.95 ; 3.07]*	*[24.91 ; 26.17]*	*[43.84 ; 49.60]*
Second model	3.57	33.88	64.49
95% CI	*[3.48 ; 3.66]*	*[33.38 ; 34.37]*	*[63.18 ; 65.81]*
Third model	3.21	29.39	55.09
95% CI	*[3.15 ; 3.27]*	*[28.81 ; 29.97]*	*[52.79 ; 57.39]*

Table 5.2: Average time to ruin with different initial capitals

As can be seen from the table, the time to ruin increases when the initial capital increases. This is logical, because there need to be more claims of the same size to let ruin occur. Furthermore, the time to ruin of the second model is the highest for all initial capitals. The first model had the lowest ruin probability, but the time to ruin is quicker for all initial capitals compared to the other models. What also can be seen from the table is that the time to ruin extremely increases when the initial capital goes from 0 to 15. It is almost multiplied by ten for all models, while when we compare the time to ruin of the initial capital from 15 to 30, the difference is less. Here, the factor between the time to ruin is two. From the histograms in the Appendix, we can see that the time to ruin range increases when the initial capital increases. When the initial capital equals zero, the range of the time to ruin is from 0 to 60, while for initial capital 15 and 30 the time to ruin range is (0,150) and (0,300). The histograms also show that in the most cases, the time to ruin still ranges between 0 and 50 for all initial capitals. So, the average time to ruin increases when the initial capital increases, but in most cases the ruin will occur before the 50th time unit.

6 Conclusion

The report tried to answer the questions given in Chapter 1. The aim of the study was to determine the difference between the three models and what influence the extra stochastic factor has on the ruin probability. Another goal was to find exact equations and formulas the ruin probability needs to satisfy. From the theoretical analysis, one can find exact equations for the ruin probability to satisfy for all three models. A lower-bound was found for the non-ruin probability in the analysis of the stochastic premium model and the third model. An upper-bound for the ruin-probability was found in the analysis for the Cramér-Lundberg model. The only difference between the lower- and upper-bound was the adjustment equation needed to be solved for the bound. Not only is the upper/lower bound found for the ruin probabilities, an integral equation for the ruin probability was also found in the three models. The integral equations are quite similar, only the stochastic factor adds an extra integral and the linear premium factor added a double integral to the equation. The method of analysis was the same for all three models, only the analysis became more difficult when working with both linear and stochastic premium. In the case that the claim sizes were exponentially distributed, an exact formula for the non-ruin probability was derived for the models. Those formulas can be found in the respective sections of the report.
As the simulation was a visual representation of the results found in the analysis, it can be seen that the influence of the stochastic factor on the ruin probability was that it made the ruin probability higher for all initial capitals. The stochastic premium made the ruin probability higher, but the most influence on the ruin probability was the size of the expected claim sizes. All in all, the model with only stochastic premium will give the highest ruin probability compared to the model with linear premium and the model with both stochastic and linear premium. For the lowest ruin probability, one would recommend the linear premium model. Adding the stochastic premium process to this model gave a more difficult analysis, however the analysis could be done in the same method as the other two models. The difference between the models in terms of the ruin probability is that there are more parameters to consider as an influence, when the stochastic factor is added to the linear premium.

During the study, there were some difficulties. The simulation can be more precise and if possible a more efficient implementation of the models. One could also look into the behavior of the time to ruin more than is done in this study. Furthermore, there could be more investigating in bigger initial capitals or different parameters for the models. The parameters needed to satisfy the safety loading condition, but more research to data from insurance companies could be done. For further research, one can look at different assumptions for the models. For this report, it is chosen to model the claim and stochastic premium process as a compound Poisson process. As one can do more research on the claim process, maybe a compound Poisson process is not the most accurate in real life. A different process for the claims coming in at the company could be a more realistic approach. Furthermore, if the compound Poisson process can be used as model for the claim process, one could further investigate more densities for the claim and stochastic premium sizes. The report only tries to solve the integral equations when the claim and stochastic premium sizes are deterministic or exponentially distributed. Further research could go into more or different densities, and try to find exact formulas for the ruin probability in those cases. Also the simulation could be more explored with more initial capitals to simulate the ruin probability. There are many parameters in the model, which all can be varied and investigated.

Bibliography

[1] A. V. BOIKOV, the Cramér-Lundberg model with stochastic premium processes, Theory Probab. Appl., 47 (2002), pp. 489–493.

[2] M. BOON, J. V. LEEUWAARDEN, B. MATHIJSEN, J. V. D. POL, AND J. RESING, Stochastic simulation. Lecture Notes of Department of Mathematics and Computer Science of the University of Technology Eindhoven, 1 april 2019. accessed on 2019/10/25.

[3] J. GRANDELL, Aspects of Risk Theory, Springer, 1991.

[4] R. KAAS, M. GOOVAERTS, J. DHAENE, AND M. DENUIT, Modern Actuarial Risk Theory, Springer, 2008.

A Histograms of Time to ruin

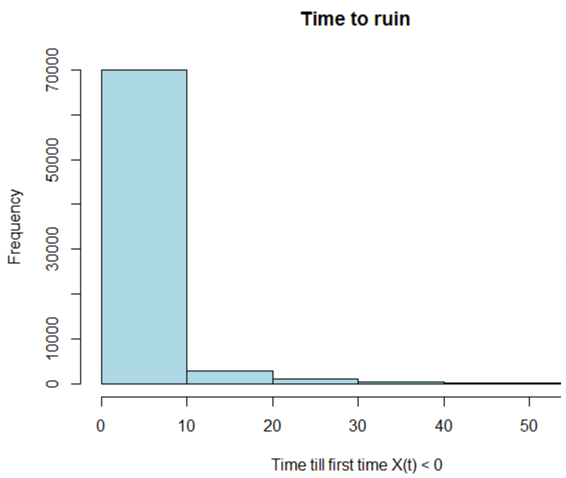

Figure A.1: Model 1, x = 0

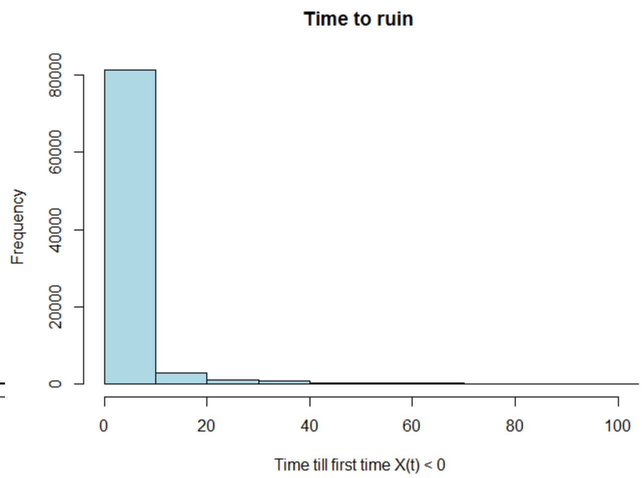

Figure A.2: Model 2, x = 0

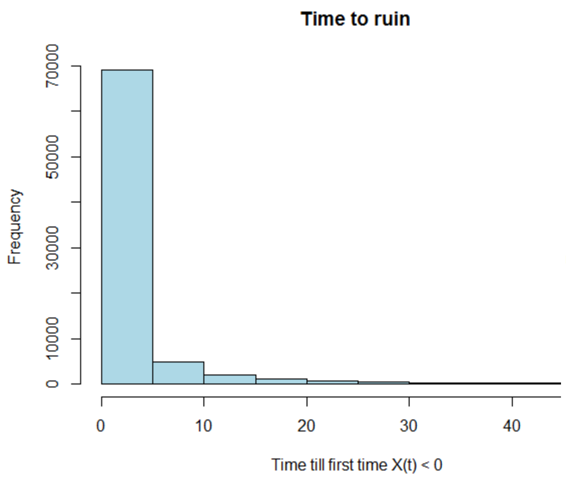

Figure A.3: Model 3, x = 0

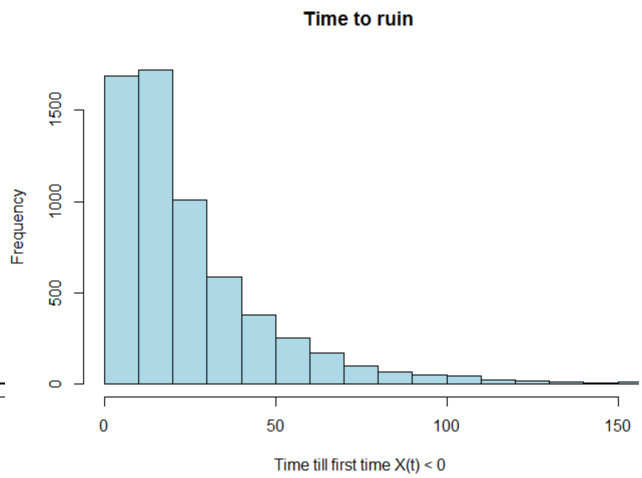

Figure A.4: Model 1, x = 15

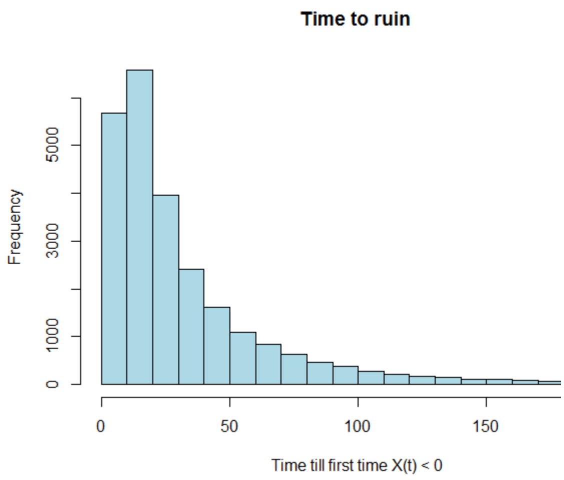

Figure A.5: Model 2, x = 15

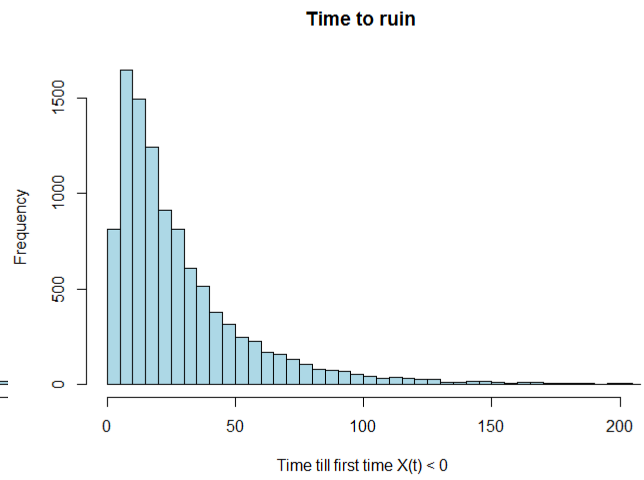

Figure A.6: Model 3, x = 15

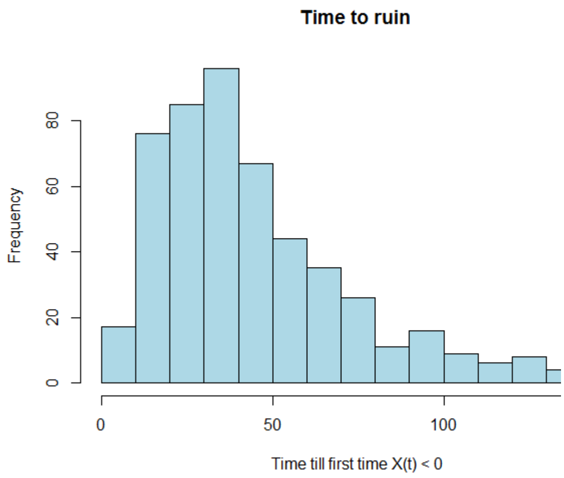

Figure A.7: Model 1, x = 30

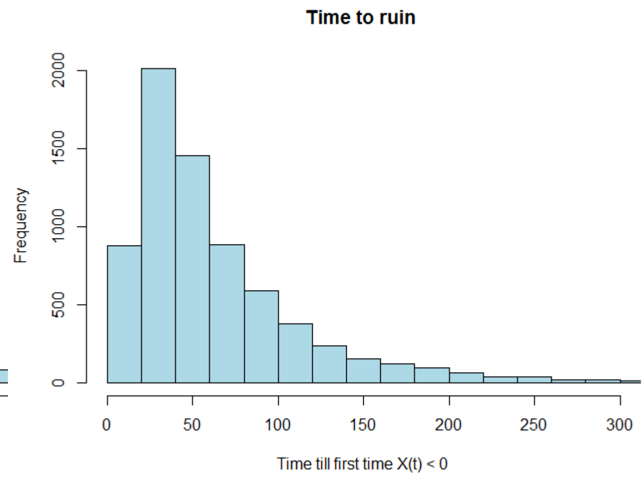

Figure A.8: Model 2, x = 30

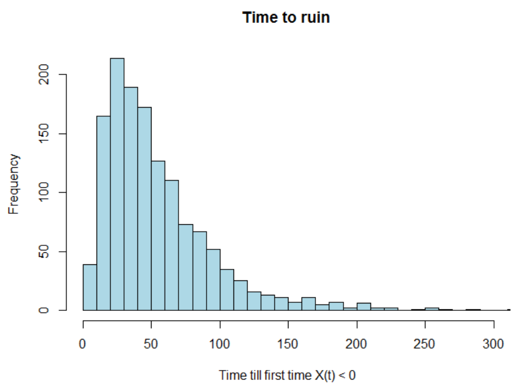

Figure A.9: Model 3, x = 30

B R-code used for report

```r
# --- Compound Poisson Risk model ---#

# --- Parameters --- #
meanClaimSize <- 1.5 #mean claim size
meanPremiumSize <- 1 # mean premium size
T <- 200              # Time horizon
u0 <- 0               # initial capital
c <- 1         # premium per time unit
lambdaplus <-1  #Rate of the premium Poisson process
lambdamin <- 1  #Rate of the claim Poisson process

lambda <- lambdamin + lambdaplus # Rate of the Poisson process

# --- Simulation of X(t), one simulation --- #

# Much more efficient implementation. Instead of simulating up to a time T, we fix N and simulate N events
simCompoundPoisson2 <- function(N) {
  #events Poisson process
  interarrivaltimes <- rexp(N, lambda)
  arrivaltimes <- cumsum(interarrivaltimes)
  #Give a distribution to the claim and premium sizes
  claimsizes <- rexp(N, 1/meanClaimSize)
  premiumsizes <- rexp(N, 1/meanPremiumSize)
  #Create empty vector for the sizes of events (claim and premium)
  amountsizes <- c()
  #number generator to determine claim or premium
  randomnumbers <- runif(N)
  #Distribute a claim or premium to event[i]
  for(i in 1:N){
    if(randomnumbers[i] < (lambdamin/lambda)){
      amountsizes[i] <- - claimsizes[i]
    } else {
      amountsizes[i] <- premiumsizes[i]
    }
  }
  # Compound Poisson processes
  levels <- cumsum(amountsizes)
  # x + ct
  premium <- u0 + arrivaltimes * c
  #create boolean whether X(t) < 0
  ruin <- min(premium + levels) < 0
  #checking when X(t) is smaller then zero, indicate when time of ruin is
  x <- premium + levels
  indexruin <- match(TRUE, x < 0 )
  timetoruin <- arrivaltimes[indexruin]
  # return 1 object containing arrival times (arr), cumulative event sizes (am), ruin (ruin)  and time to ruin (timetoruin)
  ret <- list(arr=arrivaltimes, am = levels, ruin=ruin, timetoruin = timetoruin )
  return(ret)
}

# --- Draw a plot of X(t) --#
simClaimsPlot <- function(simresults) {
  xpoints <- c(0)
  ypoints <- c(u0)
  currentLevel <- 0
  n <- length(simresults$arr) # number of events
   for (i in 1:n) {
    t <- simresults$arr[i]
    xpoints <- c(xpoints, t, t)
    print(u0 + c*t-currentLevel)
    ypoints <- c(ypoints, u0 + c * t + currentLevel)
    currentLevel <- simresults$am[i]
    ypoints <- c(ypoints, u0 + c * t + currentLevel)
  }
  plot(xpoints, ypoints, type='l', col='black', xlab='Time', ylab='Capital', main = "Capital of company over time t", lwd = 1.5)
  abline(h=0, col='darkred')
}
N <- lambda*T
simClaimsPlot(sim)
sim$ruin


# ---- Estimate the probability on ruin --- #
nruns <- 2000
T <- 2000
N <- lambda*T
simus <- 0:30 # initial capital
res <- rep(0, length(simus))
pb <- winProgressBar(min = 0,max = nruns) # create progress bar
for (i in 1:nruns) {
  for (j in 1:length(simus)) {
    u0 <<- simus[j]
    res[j] <- res[j] + simCompoundPoisson2(N)$ruin
  }
  setWinProgressBar(pb, i)
}
close(pb)
res <- res/nruns
```

```r

# --- Plotting the ruin probability MODEL 1 --- #
plot(simus, res, type='p', pch  = 19, col='blue', xlab = "Initial capital", ylab = "Probability of ruin", main = "Ruin probability of
       first model, c=2 ",xlim=c(0, 30), ylim=c(0, 1))
points(simus, res, pch = 19, col = 'red')
points(simus, res, pch = 19, col ='Forest green')
points(simus, res, pch = 19, col = 'black')
points(simus, res, pch = 19, col = 'purple')
legend("topright", legend = c("lambda = 0.01, E(Y) = 150","lambda = 0.1, E(Y) = 15","lambda = 1, E(Y) = 1.5","lambda = 5, E(Y) = 0.35",
       "lambda = 10, E(Y) = 0.15"), col =c("black","red","blue","purple","Forest green"),pch = 19)

# -- Theoretical for exponential(mu) claim sizes- MODEL 1: -- #
psi_theorie_1 <- lambda/(1/meanClaimSize*c)*exp(-(1/meanClaimSize)*(1-lambda/(1/meanClaimSize*c))*simus)
# plot theoretical ruin probability
lines(simus, lambda/(1/meanClaimSize*c)*exp(-(1/meanClaimSize)*(1-lambda/(1/meanClaimSize*c))*simus), type = 'l', col = "black", lwd =
       1.5)

# -- Theoretical for exp(a) for claim and exp(b) for premium - MODEL 2 -- #
a <- 1/meanClaimSize
b <- 1/meanPremiumSize
psi_theorie_2 <- ((a+b)*lambdamin)/(a*lambda)*exp((lambdamin*b - lambdaplus*a)*simus/(lambda))

# --- Plotting the ruin probability MODEL 2 --- #
plot(simus, res, type='p', pch  = 19, col='blue', xlab = "Initial capital", ylab = "Probability of ruin", main = "Ruin probability of
       second model",xlim=c(0, 30), ylim=c(0, 1))
points(simus, res, pch = 19, col = 'red')
points(simus, res, pch = 19, col = 'Forest green')
points(simus, res, pch = 19, col = 'purple')
points(simus, res, pch = 19, col = 'black')
lines(simus, psi_theorie_2, type = 'l', col = "black", lwd = 1.5)
legend("topright", legend = c("lambda= 0.1, E(C)=50,E(Y)=40","lambda= 1, E(C)=5, E(Y)=4", "lambda= 5, E(C)=1, E(Y)=0.8", "lambda= 10, E(
       C)= 0.5,E(Y)=0.4"), col =c("Forest green", "blue","purple","red"), pch =19)

# --- Plotting the ruin probability MODEL 3 --- #
plot(simus, res, type='p', pch  = 19, col='blue', xlab = "Initial capital", ylab = "Probability of ruin", main = "Ruin probability of
       third model, lambda's= 1, E(Y) = 1.5",xlim=c(0, 30), ylim=c(0, 1))
points(simus, res, pch = 19, col = 'red')
points(simus, res, pch = 19, col = 'Forest green')
points(simus, res, pch = 19, col = 'black')
points(simus, res, pch = 19, col = 'yellow')
legend("topright", legend = c("c = 2, E(C) = 0","c =1.5, E(C) = 0.5","c = 1, E(C) = 1", "c =0.5, E(C) = 1.5", "c = 0, E(C) = 2"), col =c
       ("yellow", "blue","red","Forest green", "black"), pch =19)


#-- Estimate the Time to ruin -- #
  # estimate for one 1 initial capital
nruns <- 100000
T <- 2000
N <- lambda*T
simus <- 30
resT <- rep(0,length(nruns))
pb <- winProgressBar(min = 0,max = nruns) # create progress bar
for(i in 1:nruns){
  u0 <- simus
    resT[i] <- simCompoundPoisson2(N)$timetoruin

    setWinProgressBar(pb, i)
}
close(pb)
averageTTR <- mean(na.omit(resT))
#confidence interval for Time to ruin
sdTTR <- sd(na.omit(resT))
CI_resT <- c(averageTTR - (1.96*sdTTR)/sqrt(length(na.omit(resT))), averageTTR + (1.96*sdTTR)/sqrt(length(na.omit(resT))))
hist(resT,main="Time to ruin ",
     xlab="Time till first time X(t) < 0",
     xlim=c(0,200), breaks = 60, col = 'light blue')
averageTTR
CI_resT


#-- Confidence intervals for ruin probability -- #
nruns <- 2000
T <- 2000
N <- lambda*T
simus <- 5 # initial capital
resCI <- 0
pb <- winProgressBar(min = 0,max = nruns) # create progress bar
for (i in 1:nruns) {
    u0 <<- simus
    resCI <- resCI + simCompoundPoisson2(N)$ruin
  setWinProgressBar(pb, i)
}
close(pb)
resCI <- resCI/nruns

alpha <- 0.05
z <- qnorm(1-alpha/2)
CI_psi <- c(resCI - (z*sqrt(resCI*(1-resCI)))/sqrt(nruns),resCI + (z*sqrt(resCI*(1-resCI)))/sqrt(nruns))
CI_psi
```