The finite horizon singular time-varying H_∞ control problem with dynamic measurement feedback

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Abstract

This paper is concerned with the finite horizon version of the H_∞ problem with measurement feedback. Given a finite-dimensional linear, time-varying system, together with a positive real number γ, we obtain necessary and sufficient conditions for the existence of a possibly time-varying dynamic compensator such that the $\mathcal{L}_2([0,t_1])$-induced norm of the closed loop operator is smaller than γ. These conditions are expressed in terms of a pair of quadratic differential inequalities, generalizing the well-known Riccati differential equations introduced recently in the context of finite horizon H_∞ control.

Keywords : Finite horizon, H_∞ control, Quadratic differential matrix inequality, Riccati differential equation.

1 Introduction

After the publication of [25], H_∞ control has received an overwhelming amount of attention ([4],[5], [7], [10], [11], [14], [19]). However, all of these papers discuss the "standard" H_∞ problem: minimize the $\mathcal{L}_2([0,\infty))$-induced operator norm of the closed loop operator over all internally stabilizing feedback controllers.

Recently a number of generalizations have appeared. One of these is the minimization of the $\mathcal{L}_2$-induced operator norm over a *finite* horizon ([12], [21]). As in the infinite horizon H_∞ problem, difficulties arise in case that the direct feedthrough matrices do not satisfy certain assumptions (the so-called singular case). This paper will use the techniques of [19], [20] to tackle this problem for the finite-horizon case.

The following problem will be considered: given a finite-dimensional system on a bounded time-interval $[0, t_1]$, together with a positive real number γ, find necessary and sufficient conditions for the existence of a dynamic compensator such that the $\mathcal{L}_2([0,t_1])$-induced norm of the resulting closed loop operator is smaller than γ. In [21] and [12] such conditions were formulated in terms of the existence of solutions to certain Riccati differential equations. Of course, in order to guarantee the existence of these Riccati differential equations, certain coefficient matrices of the system under consideration should have *full rank* (the *regular* case). The present paper adresses the problem formulated above *without* these full rank assumptions. We find necessary and sufficient conditions in terms of a pair of quadratic matrix differential inequalities. However, in order to establish these conditions we will have to impose certain, weaker, assumptions on the coefficient matrices under consideration. In two important cases these assumptions are always satisfied:

- if the system is time-invariant (i.e., all coefficient matrices are constant, independent of time).
- if the problem is regular in the sense as explained above.

Thus, our result completely solves the finite-horizon H_∞ problem for time-invariant systems. On the other hand our result is a generalization of the results from [21] and [12] on the regular problem (for time-varying systems).

The outline of the paper is as follows. In section 2 we will formulate our problem and present our main result. In section 3 we will show that if there exists a controller which makes the $\mathcal{L}_2([0,t_1])$-induced operator norm of the closed loop operator less than 1 then there exist matrices functions P and Q satisfying a pair of quadratic matrix differential inequalities, two corresponding rank conditions and two boundary conditions. In section 4 we will introduce a system transformation with an interesting property: a controller "works" for this new system if and only if the same controller "works" for the original system. Using this transformation we will show that another necessary condition for the existence of the desired controller is that P and Q satisfy a coupling condition: $I - PQ$ is invertible for all t. In section 5 we will apply a second transformation, dual to the first, which will show that the necessary conditions derived are also sufficient. This will be done by showing that the system we obtained by our two transformations satisfies the following condition: for all $\varepsilon > 0$ there exists a controller which makes the $\mathcal{L}_2([0,t_1])$-induced norm of the closed loop operator less than ε. We will close the paper with a couple of concluding remarks. Three appendices contain parts of the proof which fall outside the general line of the proof

2 Problem formulation and main results

We consider the linear, time varying, finite-dimensional system:

$$\Sigma : \begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + E(t)w(t), \\ y(t) &= C_1(t)x(t) + D_1(t)w(t), \\ z(t) &= C_2(t)x(t) + D_2(t)u(t), \end{cases} \tag{2.1}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^m$ the control input, $w \in \mathcal{R}^l$ the unknown disturbance, $y \in \mathcal{R}^p$ the measured output and $z \in \mathcal{R}^q$ the unknown output to be controlled. A, B, E, C_1, C_2, D_1, and D_2 are matrix functions of appropriate dimensions. Given an a priori fixed finite time-interval $[0, t_1]$ we would like to minimize the effect of the disturbance w on the output z by finding an appropriate control input u. We restrict the control inputs to be generated by dynamic output feedback. More precisely, we seek possibly time-varying dynamic compensators Σ_F of the form:

$$\Sigma_F : \begin{cases} \dot{p}(t) &= K(t)p(t) + L(t)y(t), \\ u(t) &= M(t)p(t) + N(t)y(t). \end{cases} \tag{2.2}$$

Given a compensator of the form (2.2), the closed loop system $\Sigma \times \Sigma_F$ with initial conditions $x(0) = 0$ and $p(0) = 0$ defines a convolution operator mapping w to z. This operator will be called the closed loop operator and will be denoted by G_{cl}. Our goal is to minimize the $\mathcal{L}_2([0, t_1])$-induced operator norm of G_{cl}, i.e. we seek a controller of the form (2.2) such that

$$\|G_{cl}\|_\infty := \sup_{w \neq 0} \left\{ \frac{\|G_{cl}w\|_2}{\|w\|_2} \;\middle|\; w \in \mathcal{L}_2([0, t_1]) \right\} \tag{2.3}$$

is minimized over all feedbacks Σ_F of the form (2.2). The norm $\|.\|_2$ is the standard norm on $\mathcal{L}_2([0, t_1])$ and is defined by:

$$\|f\|_2 := \left(\int_0^{t_1} \|f(t)\|^2 \, dt \right)^{1/2} \tag{2.4}$$

where $\|.\|$ denotes the Euclidian norm. Obviously, the closed loop system $\Sigma \times \Sigma_F$ is time-varying. Moreover we work over a finite horizon. Therefore, the $\mathcal{L}_2([0, t_1])$-induced operator norm (2.3) differs from the commonly used H_∞ norm. (Recall that the latter norm is equal to the $\mathcal{L}_2([0, \infty))$-induced operator norm in a time-invariant context.) However, the above problem formulation is the most natural formulation for the finite-horizon time-varying case. Hence we will sometimes refer to (2.3) as an H_∞ norm.

In this paper we will derive necessary and sufficient conditions for the existence of a dynamic feedback law (2.2) which makes the resulting $\mathcal{L}_2([0,t_1])$-induced norm of the closed loop operator G_{cl} strictly less than some a priori given bound γ. By a search procedure one can then, in principle, obtain the infimum of these operator norms over all controllers of the form (2.2). It should be noted however that this infimum is not always attained. The problem whether or not the infimum is attained will not be discussed in this paper.

A central role in our study of the above problem will be played by the *quadratic differential matrix inequality*. For any $\gamma > 0$ and for any differentiable matrix function P on $[0,t_1]$ we define the following matrix function:

$$F_\gamma(P)(t) := \begin{pmatrix} \dot{P} + A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + \gamma^{-2}PEE^{\mathrm{T}}P & PB + C_2^{\mathrm{T}}D_2 \\ B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{pmatrix}(t).$$

If $F_\gamma(P)(t) \geq 0 \ \ \forall\, t \in [0,t_1]$, we will say that P is a solution of the quadratic differential matrix inequality $F_\gamma(P) \geq 0$ at γ. We denote $F_\gamma(P)$ by $F(P)$ if $\gamma = 1$.

We also define a dual version of this quadratic matrix inequality. For any $\gamma > 0$ and for any differentiable matrix function Q on $[0,t_1]$ we define the following matrix function:

$$G_\gamma(Q)(t) := \begin{pmatrix} -\dot{Q} + AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^{-2}QC_2^{\mathrm{T}}C_2Q & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix}(t).$$

If $G_\gamma(Q)(t) \geq 0 \ \forall\, t \in [0,t_1]$, we will say that Q is a solution of the dual quadratic differential matrix inequality $G_\gamma(Q) \geq 0$ at γ. We again denote $G_\gamma(Q)$ by $G(Q)$ if $\gamma = 1$. The difference in sign of $\dot{P}$ and $\dot{Q}$ in these expressions stems from the fact that dualization includes time-reversal (see also lemma 3.6).

Finally, if the system (2.1) is time-invariant, we define the following transfer matrices:

$$G(s) := C_2\,(sI - A)^{-1}\,B + D_2, \tag{2.5}$$

$$H(s) := C_1\,(sI - A)^{-1}\,E + D_1. \tag{2.6}$$

We will denote the rank of a matrix over the field $\mathcal{K}$ by $\mathrm{rank}_{\mathcal{K}}$. $\mathcal{R}(s)$ denotes the field of all real rational functions. We are now in the position to formulate our main result:

Theorem 2.1 : *Assume that (2.1) is time-invariant. Let $\gamma > 0$ be given. Then the following two statements are equivalent:*

(*i*) *There exists a time-varying, dynamic compensator Σ_F of the form (2.2) such that the closed loop operator G_{cl} of $\Sigma \times \Sigma_F$ has $\mathcal{L}_2([0,t_1])$-induced operator norm less than γ, i.e. $\|G_{cl}\|_\infty < \gamma$.*

(*ii*) *There exist differentiable matrix functions P, Q satisfying the following conditions:*

(*a*) $F_\gamma(P)(t) \geq 0 \quad \forall t \in [0, t_1]$ *and* $P(t_1) = 0$.

(*b*) $rank_{\mathcal{R}} F_\gamma(P)(t) = rank_{\mathcal{R}(s)}\, G(s) \quad \forall\, t \in [0, t_1]$.

(*c*) $G_\gamma(Q)(t) \geq 0 \quad \forall t \in [0, t_1]$ *and* $Q(0) = 0$.

(*d*) $rank_{\mathcal{R}} G_\gamma(Q)(t) = rank_{\mathcal{R}(s)}\, H(s) \quad \forall\, t \in [0, t_1]$.

(*e*) $\gamma^2 I - P(t)Q(t)$ *is invertible for all* $t \in [0, t_1]$. □

Remarks :

(*i*) Since P and Q satisfy (a)-(d) it can be shown that $P(t) \geq 0$ and $Q(t) \geq 0$. Therefore the matrix $P(t)Q(t)$ has only real and non-negative eigenvalues. Since $P(t_1)Q(t_1) = 0$ and since we have continuity with respect to t it can be shown that (e) is equivalent with $\rho(P(t)Q(t)) < \gamma^2$ for all $t \in [0, t_1]$, where ρ denotes the spectral radius.

(*ii*) The construction of a dynamic compensator Σ_F satisfying the condition in theorem 2.1 (i) can be done according to the method as described in section 5. It turns out that it is always possible to find a compensator of the same dynamic order as the original plant.

(*iii*) By corollary A.5 we know that a solution $P(t)$ of the quadratic matrix inequality $F_\gamma(P) \geq 0$ satisfying the end condition $P(t_1) = 0$ and rank condition (b) is unique. By dualizing corollary A.5 it can also be shown that a solution $Q(t)$ of the dual quadratic matrix inequality $G_\gamma(Q) \geq 0$ satisfying the initial condition $Q(0) = 0$ and rank condition (d) is unique.

(*iv*) We will prove this theorem only for the case $\gamma = 1$. The general result can then be easily obtained by scaling.

We will look more closely to the previous result for a special case:

State feedback: $C_1 = I, \;\; D_1 = 0$.
In this case we have $y = x$, i.e. we know the state of the system. The first matrix inequality $F_\gamma(P) \geq 0$ does not depend on C_1 or D_1 and the same is true for rank condition (b) so we can't expect a simplification there. However $G_\gamma(Q)$ does get a special form:

$$G_\gamma(Q)(t) = \begin{pmatrix} -\dot{Q} + AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^{-2} QC_2^{\mathrm{T}} C_2 Q & Q \\ Q & 0 \end{pmatrix} (t) \tag{2.7}$$

Using this special form it can be easily seen that $G_\gamma(Q)(t) \geq 0$ for all $t \in [0, t_1]$ if and only if $Q(t) = 0$ for all $t \in [0, t_1]$. In order to verify the rank condition we should investigate the rank of the transfer matrix $H(s)$. We have $H(s) = (sI - A)^{-1}E$ so

$$\mathrm{rank}_{\mathcal{R}(s)} H(s) = \mathrm{rank}_{\mathcal{R}} E. \tag{2.8}$$

By using equation (2.8) it can be easily checked that $Q = 0$ indeed satisfies rank condition (d). Hence we find that in this case theorem 2.1 reduces to:

Corollary 2.2 : *Assume that the system (2.1) is time-invariant. Let $\gamma > 0$. Assume $C_1 = I$ and $D_1 = 0$. Then the following two statements are equivalent:*

(*i*) *There exists a time-varying, dynamic compensator Σ_F of the form (2.2) such that the closed loop operator G_{cl} of $\Sigma \times \Sigma_F$ has $\mathcal{L}_2([0, t_1])$-induced operator norm less than γ, i.e. $\|G_{cl}\|_\infty < \gamma$.*

(*ii*) *There exists a differentiable matrix function P satisfying the following conditions:*

(*a*) $F_\gamma(P)(t) \geq 0 \quad \forall t \in [0, t_1]$ *and* $P(t_1) = 0$.

(*b*) $\text{rank}_{\mathcal{R}} F_\gamma(P)(t) = \text{rank}_{\mathcal{R}(s)} G(s) \quad \forall\, t \in [0, t_1]$. □

Remark : If part (ii) is satisfied then it can in fact be shown that there exists a static, time-varying state feedback $u(t) = F(t)x(t)$ satisfying part (i).

At this point we want to note that in previous papers ([12, 21]) on the finite-horizon H_∞ problem it is assumed that the matrices D_1 and D_2 are surjective and injective, respectively. However in [12] and [21] the system (2.1) is allowed to be *time-varying*, whereas in the present paper, up to now, we have restricted (2.1) to be time-invariant. Thus the following question arises: is it possible to obtain a result similar to theorem 2.1 for time-varying systems? We were indeed able to establish such a result, albeit under certain restrictive assumptions on the "singular part" of the time-varying system (2.1). These assumptions will be presented in section 3. However: it will turn out that for two important cases these assumptions are always satisfied, namely if either

(*i*) $D_1(t)$ is injective and $D_2(t)$ is surjective for all $t \in [0, t_1]$

or

(*ii*) the system (2.1) is time invariant

Therefore instead of proving theorem 2.1 directly, we will formulate and prove our more general result for time-varying systems. Although not completely general, this result will then still have as a special case both the main results from [12] and [21] as well as our theorem 2.1.
In the formulation of our more general result we need the following two functions:

$$g_t \;:=\; \text{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A(t) & -B(t) \\ C_2(t) & D_2(t) \end{pmatrix} - n \tag{2.9}$$

$$h_t \;:=\; \text{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A(t) & -E(t) \\ C_1(t) & D_1(t) \end{pmatrix} - n \tag{2.10}$$

($t \in [0,t_1]$). Note that in the time-invariant case g_t is equal to the rank of the transfer matrix $G(s)$ as a matrix with entries in the field of rational functions. The same is true with respect to h_t and the transfer matrix $H(s)$. We have the following result:

Theorem 2.3 : *Let $\gamma > 0$. Consider the system (2.1) and assume that the coefficient matrices are differentiable functions of t. Assume that assumptions 3.3 and 3.9 are satisfied. Then the following two statements are equivalent:*

(*i*) *There exists a time-varying, dynamic compensator Σ_F of the form (2.2) such that the closed loop operator G_{cl} of $\Sigma \times \Sigma_F$ has $\mathcal{L}_2([0,t_1])$-induced operator norm less than γ, i.e. $\|G_{cl}\|_\infty < \gamma$.*

(*ii*) *There exist differentiable matrix functions P, Q satisfying the following conditions:*

(*a*) $F_\gamma(P)(t) \geq 0 \quad \forall t \in [0,t_1]$ *and* $P(t_1) = 0$.
(*b*) $rank_{\mathcal{R}} F_\gamma(P)(t) = g_t \quad \forall\, t \in [0,t_1]$.
(*c*) $G_\gamma(Q) \geq 0 \quad \forall t \in [0,t_1]$ *and* $Q(0) = 0$.
(*d*) $rank_{\mathcal{R}} G_\gamma(Q)(t) = h_t \quad \forall\, t \in [0,t_1]$.
(*e*) $\gamma^2 I - P(t)Q(t)$ *is invertible for all* $t \in [0,t_1]$. □

Remarks :

(*i*) For time invariant systems assumptions 3.3 and 3.9 will turn out to be automatically satisfied. Therefore theorem 2.1 is in fact a special case of theorem 2.3.

(*ii*) It will be shown (see corollary A.5) that P and Q are uniquely defined by (a)-(d). Moreover (see lemma A.3) g_t turns out to be independent of t. It can be shown that for any L such that $F_\gamma(L) \geq 0$, the rank of $F_\gamma(L)(t)$ is always larger than or equal to g_t. Therefore (a) and (b) can be stated more loosely as: P is a rank-minimizing solution of the quadratic differential inequality $F_\gamma(P) \geq 0$ satisfying the end condition $P(t_1) = 0$. The conditions on Q can be reformulated in a similar way.

(*iii*) This theorem will only be proven for $\gamma = 1$. The general result can then be obtained via scaling.

As noted before, from the previous theorem we can also reobtain the results of [12, 21]. Again we assume that our coefficient matrices are differentiable functions of t. We find:

Regular time-varying case: $D_1(t)$ surjective and $D_2(t)$ injective for all $t \in [0,t_1]$:
It will turn out that in this case assumptions 3.3 and 3.9 are satisfied. It can be shown in the same way as in [19] that P satisfies $F_\gamma(P) \geq 0$ together with rank condition (b) if and only if P satisfies the Riccati differential equation:

$$-\dot{P} = A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + \gamma^{-2}PEE^{\mathrm{T}}P - (PB + C_2^{\mathrm{T}}D_2)\left(D_2^{\mathrm{T}}D_2\right)^{-1}(B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2) \quad (2.11)$$

Dually, Q satisfies the dual matrix differential inequality $G_\gamma(Q) \geq 0$ together with rank condition (d) if and only if Q satisfies the dual Riccati equation:

$$\dot{Q} = AQ + QA^T + EE^T + \gamma^{-2}QC_2^TC_2Q - \left(QC_1^T + ED_1^T\right)\left(D_1D_1^T\right)^{-1}\left(C_1Q + D_1E^T\right) \qquad (2.12)$$

We thus obtain the following result:

Corollary 2.4 : *Let $\gamma > 0$. Consider the system (2.1) and assume that the coefficient matrices are differentiable functions of t. Assume $D_1(t)$ is surjective and $D_2(t)$ is injective for all $t \in [0, t_1]$. Then the following two statements are equivalent:*

- *(i) There exists a time-varying, dynamic compensator Σ_F of the form (2.2) such that the closed loop operator G_{cl} of $\Sigma \times \Sigma_F$ has $\mathcal{L}_2([0,t_1])$-induced operator norm less than γ, i.e. $\|G_{cl}\|_\infty < \gamma$.*
- *(ii) There exist differentiable matrix functions P, Q satisfying the following conditions:*
  - *(a) P satisfies (2.11) and $P(t_1) = 0$.*
  - *(b) Q satisfies (2.12) and $Q(0) = 0$.*
  - *(c) $\gamma^2 I - P(t)Q(t)$ is invertible for all $t \in [0, t_1]$.* □

These are exactly the conditions derived in [12]. A proof that in this case assumptions 3.3 and 3.9 are indeed satisfied will be given further on in this paper.

3 Necessary conditions for the existence of the desired dynamic feedback

In this section we will deal with time-varying systems. It will be shown that under the assumptions 3.3 and 3.9 statement (i) of theorem 2.3 implies that there exist differentiable matrix functions P and Q satisfying (a)-(d) of statement (ii) of theorem 2.3. Throughout this section we will assume that $\gamma = 1$.
We will start by stating the assumptions we have to make. We first need a definition.

Definition 3.1 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times m}$ be arbitrary constant matrices. Then the strongly controllable subspace $\mathcal{T}(A,B,C,D)$ associated with the quadruple (A,B,C,D) is defined as the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ for which there exists a matrix $G \in \mathcal{R}^{n\times p}$ such that:*

$$(A + GC)\mathcal{T} \subset \mathcal{T}, \qquad (3.1)$$

$$\text{Im}\,(B + GD) \subset \mathcal{T}, \qquad (3.2)$$

The quadruple (A,B,C,D) is called strongly controllable if $\mathcal{T}(A,B,C,D) = \mathcal{R}^n$ □

In order to calculate this subspace the following lemma (see [16, 22]) is available.

Lemma 3.2 : *Let (A, B, C, D) be as in the previous definition. Then $\mathcal{T}(A, B, C, D)$ is equal to the limit of the following sequence of subspaces:*

$$\mathcal{T}_0 := 0, \qquad \mathcal{T}_{i+1} := \{x \in \mathcal{R}^n \mid \exists\, \tilde{x} \in \mathcal{T}_i,\ u \in \mathcal{R}^m \text{ such that } x = A\tilde{x} + Bu \text{ and } C\tilde{x} + Du = 0\} \tag{3.3}$$

$\{\mathcal{T}_i\}_{i=0}^{\infty}$ is a non-decreasing sequence of subspaces that attains its limit in a finite number of steps. □

We shall now formulate the assumptions to be imposed on our time-varying system (2.1).

Assumption 3.3

(*i*) *The subspace $B(t)$ ker $D_2(t)$ is independent of t.*

(*ii*) *The strongly controllable subspace $\mathcal{T}(A(t), B(t), C_2(t), D_2(t))$ associated with the quadruple $(A(t), B(t), C_2(t), D_2(t))$ is independent of t. It will be denoted by $\mathcal{T}(\Sigma)$.*

(*iii*) *The subspace $\mathcal{T}(\Sigma) \cap C_2^{-1}(t)$ im $D_2(t)$ is independent of t. It will be denoted by $\mathcal{W}(\Sigma)$.*

(*iv*) *$\text{rank}_{\mathcal{R}} \begin{pmatrix} B(t) \\ D_2(t) \end{pmatrix}$ is independent of t.*

(*v*) *There exists a differentiable matrix function F_0 such that*

(*a*) *$D_2^T(t)\,[C_2(t) + D_2(t)F_0(t)] = 0$ for all t*

(*b*) *$(A(t) + B(t)F_0(t))\,|_{\mathcal{W}(\Sigma)}$ is independent of t.* □

Remarks : It is easily seen that assumption 3.3 is trivially satisfied if the system (2.1) is time invariant.
Assumption 3.3 is also satisfied if ker $D_2(t) = \{0\}$ for all t. This can be seen by noting that this implies that $\mathcal{T}(A(t), B(t), C_2(t), D_2(t)) = \{0\}$. This special case is called the regular case.

Assume now that condition (i) of theorem 2.3 is satisfied with $\gamma = 1$. Denote by $z_{u,w}$ the output z we get if we apply functions u and w to the system (2.1) with initial condition $x(0) = 0$. This implies that for all $w \in \mathcal{L}_2^l([0, t_1]), w \neq 0$ we have

$$\inf_{u \in \mathcal{L}_2([0,t_1])} \{\ \|z_{u,w}\|_2 - \|w\|_2\ \} < 0. \tag{3.4}$$

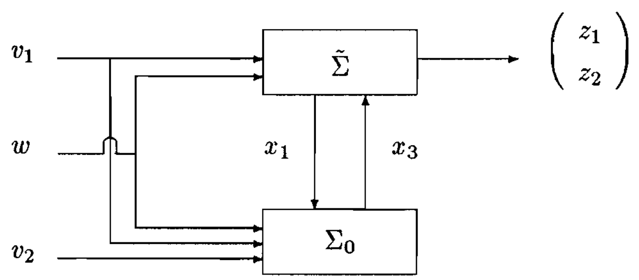
In the above infimization problem $u \in \mathcal{L}_2^m$ is completely arbitrary Hence the problem does not change if we apply a preliminary state feedback F_0 defined by assumption 3.3 part (v). Due to assumption 3.3, we can write our system with respect to the bases as described in appendix A. With respect to this decomposition our system has the form:

$$\dot{x}_1 = A_{11}x_1 + \begin{pmatrix} B_{11} & A_{13} \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix} + E_1 w, \tag{3.5}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} B_{21} & A_{21} \\ B_{31} & A_{31} \end{pmatrix} \begin{pmatrix} v_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} w, \tag{3.6}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ C_{21} \end{pmatrix} x_1 + \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & C_{23} \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix}. \tag{3.7}$$

where the coefficient matrices are differentiable functions of t. As already suggested by the way we arranged these equations we can decompose our system as follows:



(3.8)

In the picture (3.8), $\tilde{\Sigma}$ is the system given by the equations (3.5) and (3.7). It has inputs v_1, w and x_3; state x_1 and outputs z_1, z_2. The system Σ_0 is given by equation (3.6). It has inputs v_1, v_2, w and x_1; state x_2, x_3 and output x_3. It can easily be seen that (3.4) implies that for all $w \in \mathcal{L}_2^l([0, t_1]), w \neq 0$ we have:

$$\inf_{v_1, v_2 \in \mathcal{L}_2([0,t_1])} \{ \|z_{v_1,v_2,w}\|_2 - \|w\|_2 \} < 0, \tag{3.9}$$

where $z_{v_1,v_2,w}$ denotes the output of the system $\tilde{\Sigma}$ after applying the inputs v_1, v_2 and w to the interconnection of $\tilde{\Sigma}$ and Σ_0 as desribed in (3.8).
If we now investigate our decomposition of the original system, it is easily seen that this implies that for all $w \in \mathcal{L}_2^l([0, t_1]), w \neq 0$ we have:

$$\inf_{v_1, x_3 \in \mathcal{L}_2([0,t_1])} \{ \|\tilde{z}_{v_1,x_3,w}\|_2 - \|w\|_2 \} < 0, \tag{3.10}$$

where $\tilde{z}_{v_1,x_3,w}$ denotes the output of the system $\tilde{\Sigma}$ after applying the "inputs" v_1, x_3 and w to that system. On the other hand we have the following lemma:

Lemma 3.4 : *Let $\tilde{\Sigma}$ be defined by equations (3.5) and (3.7). If (3.10) is satisfied for all $w \in \mathcal{L}_2([0,t_1])$ then there exists a matrix function P_1 that satisfies the Riccati differential equation $R(P_1)(t) = 0$, $t \in [0,t_1]$ with end condition $P(t_1) = 0$. Here, $R(P_1)$ is defined by (A.9).* □

Proof : By lemma A.3, C_{23} is injective for all $t \in [0,t_1]$ and, by construction, $\hat{D}_2$ is invertible for all $t \in [0,t_1]$. Therefore the direct feedthrough matrix from control input to output of the system $\tilde{\Sigma}$ is injective for all $t \in [0,t_1]$. We can now apply the results of [12] to the system $\tilde{\Sigma}$. By [12, theorem 2.3] or [21, theorem 5.1] there exists P_1 such that $R(P_1)(t) = 0$ for all $t \in [0,t_1]$ and $P_1(t_1) = 0$. ■

Combining the latter lemma with lemma A.4 we can derive the following corollary.

Corollary 3.5 : *Let the system (2.1) be given and assume assumption 3.3 is satisfied. Assume that the condition in part (i) of theorem 2.3 is satisfied. In that case there exists a differentiable matrix function P satisfying the conditions of theorem 2.3 (a) and (b).*

In order to obtain the existence of a matrix Q satisfying conditions (c) and (d) in the statement of theorem 2.3, we first have to discuss the concept of dualization. Let the system Σ be given by (2.1). We define the dual system Σ' by

$$\Sigma' : \begin{cases} \dot{x}_D(t) = A^T(t_1-t)x_D(t) + C_1^T(t_1-t)u_D(t) + C_2^T(t_1-t)w_D(t), \\ y_D(t) = B^T(t_1-t)x_D(t) + D_2^T(t_1-t)w_D(t), \\ z_D(t) = E^T(t_1-t)x_D(t) + D_1^T(t_1-t)u_D(t), \end{cases} \tag{3.11}$$

Let $G : \mathcal{L}_2^{m+l}([0,t_1]) \to \mathcal{L}_2^{p+q}([0,t_1])$ denote the (open loop) operator from (u,w) to (y,z) defined by the system Σ with $x(0) = 0$. Likewise, let G' be the open loop operator from (u_D, w_D) to (y_D, z_D) associated with Σ'. It can be easily shown that

$$G' = R \circ G^* \circ R \tag{3.12}$$

where G^* is the adjoint of G and where R denotes the time reversal operator $(Rf)(t) = f(t_1 - t)$. Define the dual of the controller Σ_F, as defined by (2.2), in the same way:

$$\Sigma'_F : \begin{cases} \dot{p}_D(t) = K^T(t_1-t)p_D(t) + M^T(t_1-t)y_D(t), \\ u_D(t) = L^T(t_1-t)p_D(t) + N^T(t_1-t)y_D(t), \end{cases} \tag{3.13}$$

If F denotes the operator from y to u, F' the operator from y_D to u_D and F^* is the adjoint of F, then again we have $F' = R \circ F^* \circ R$. Denote the closed loop operator after applying the feedback Σ_F to the system Σ by G_{cl}. Likewise, let $\tilde{G}_{cl}$ denote the closed loop operator of $\Sigma' \times \Sigma'_F$. Then from the above it can be seen that

$$\tilde{G}_{cl} = R \circ G^*_{cl} \circ R \tag{3.14}$$

Since the norms of G_{cl} and G^*_{cl} are equal and since, trivially, R is an isometry, we can conclude that $\|\tilde{G}_{cl}\|_\infty = \|G_{cl}\|_\infty$. We summarize this result in the following lemma:

Lemma 3.6 : *Consider the system Σ given by (2.1) and let a controller Σ_F of the form (2.2) be given. The closed loop operator of the interconnection $\Sigma \times \Sigma_F$ and the closed loop operator of the interconnection $\Sigma' \times \Sigma'_F$ have the same $\mathcal{L}_2([0,t_1])$-induced operator norm.* □

Since part (i) of theorem 2.3 is satisfied for the system (2.1), by the above result statement (i) of theorem 2.3 is also satisfied for the dual system (3.11). We would like to conclude that this implies that there exists a differentiable matrix function satisfying the statements (ii) (a) and (b) of theorem 2.3 for this new system. However we can only do that if assumption 3.3 is satisfied for Σ'. In the following, we will formulate a set of assumptions for the original system Σ which exactly guarantee that the dual system Σ' satisfies the assumptions 3.3. We first need a definition:

Definition 3.7 *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times m}$ be arbitrary constant matrices. Then the weakly unobservable subspace $\mathcal{V}(A,B,C,D)$ associated with the quadruple (A,B,C,D) is defined as the largest subspace $\mathcal{V}$ of $\mathcal{R}^n$ for which there exists a matrix $F \in \mathcal{R}^{n\times p}$ such that:*

$$(A+BF)\mathcal{V} \subset \mathcal{V}, \tag{3.15}$$
$$(C+DF)\mathcal{V} = \{0\}, \tag{3.16}$$

The quadruple (A,B,C,D) is called strongly observable if $\mathcal{V}(A,B,C,D) = \{0\}$ □

In order to calculate this subspace the following lemma (see [17]) is available. It is the dual version of lemma 3.2:

Lemma 3.8 : *Let (A,B,C,D) be as in the previous definition. Then $\mathcal{V}(A,B,C,D)$ is equal* *to the limit of the following sequence of subspaces:*

$$\mathcal{V}_0 := \mathcal{R}^n, \qquad \mathcal{V}_{i+1} := \{x \in \mathcal{R}^n \mid \exists\, \tilde{u} \in \mathcal{R}^m, \text{ such that } Ax + B\tilde{u} \in \mathcal{V}_i \text{ and } Cx + D\tilde{u} = 0\} \tag{3.17}$$

$\{\mathcal{V}_i\}_{i=0}^{\infty}$ *is a non-increasing sequence of subspaces that attains its limit in a finite number of steps.* □

Assumption 3.9

(*i*) *The subspace* $C_1^{-1}(t)$ *im* $D_1(t)$ *is independent of* t.

(*ii*) *The weakly unobservable subspace* $\mathcal{V}(A(t), E(t), C_1(t), D_1(t))$ *associated with the quadruple* $(A(t), E(t), C_1(t), D_1(t))$ *is independent of* t. *It will be denoted by* $\mathcal{V}(\Sigma)$.

(*iii*) *The subspace* $\mathcal{V}(\Sigma) + E(t)$ *ker* $D_1(t)$ *is independent of* t. *It will be denoted by* $\mathcal{Z}(\Sigma)$.

(*iv*) $\text{rank}_{\mathcal{R}} \begin{pmatrix} C_1(t) & D_1(t) \end{pmatrix}$ *is independent of* t.

(*v*) *There exists a differentiable matrix function* G_0 *such that*

(*a*) $D_1(t)\,(E(t) + G_0(t)D_1(t))^{\mathrm{T}} = 0$ *for all* t

(*b*) $T_{\mathcal{Z}(\Sigma)}\,(A(t) + G_0(t)C_1(t))$ *is independent of* t, *where* $T_{\mathcal{Z}(\Sigma)}$ *denotes the orthogonal projection along* $\mathcal{Z}(\Sigma)$ *onto* $\mathcal{Z}(\Sigma)^{\perp}$. □

Remarks : Note that like assumption 3.3, assumption 3.9 is trivially satisfied if the system (2.1) is time invariant.
Assumption 3.9 is also satisfied if im $D_1(t) = \mathcal{R}^n$ for all t. This can be seen by noting that this implies that $\mathcal{V}(A(t), B(t), C_2(t), D_2(t)) = \mathcal{R}^n$. Together with the assumption ker $D_2(t) = \{0\}$ for all t this special case is called the regular case.

If assumption 3.9 is assumed to hold for the system Σ we can easily check that Σ' satisfies assumption 3.3. Using this we can derive the following lemma.

Lemma 3.10 : *Let the system (2.1) be given and assume assumption 3.9 is satisfied. Assume that the condition in part (i) of theorem 2.3 is satisfied. In that case there exists a differentiable matrix function Q satisfying the conditions of theorem 2.3 (c) and (d).*

Proof : We already know that statement (i) of theorem 2.3 is satisfied for Σ and therefore, by lemma 3.6, statement (i) of theorem 2.3 is also satisfied for the dual system Σ'. Using corollary 3.5 we find that there exists a differentiable matrix function $\tilde{Q}$ which satisfies conditions (a) and (b) of theorem 2.3 for the system Σ'. This immediately implies that Q, defined by $Q(t) := \tilde{Q}(t_1 - t)$, satisfies (c) and (d) of theorem 2.3 for the original system Σ. (Here we used that $\tilde{Q}(t)$ is symmetric for all $t \in [0, t_1]$ which follows from corollary A.5) ■

We can summarize the result of this section in the following corollary which is a combination of corollary 3.5 and lemma 3.10:

Corollary 3.11 : *Consider the system (2.1). Assume that assumptions 3.3 and 3.9 are satisfied. If part (i) of theorem 2.3 is satisfied then there exist differentiable matrix functions P and Q satisfying statements (a)-(d) of part (ii) in theorem 2.3.* □

In order to proof the implication (i) $\Rightarrow$ (ii) in theorem 2.3 it only remains to be shown that $I - P(t)Q(t)$ is invertible for all $t \in [0, t_1]$. This will be done in the next section.

4 A first system transformation

In this section we will complete the proof of the implication (i)$\Rightarrow$(ii) of theorem 2.3. At the same time it will give us the first step of the proof of the reverse implication. Throughout this section we will assume $\gamma = 1$. Starting from the existence of a matrix function P satisfying (a) and (b) of theorem 2.3 we will define a new system Σ_P. It turns out that a compensator Σ_F makes the norm of the closed loop operator less than 1 for the original system Σ if and only if it makes the norm of the closed loop operator less than 1 for this new system Σ_P. Therefore it will be sufficient to investigate this new system, which turns out to have a very nice property. Througout this section we will assume that assumption 3.3 and assumption 3.9 hold for the original system Σ.
In order to define the new system Σ_P we need the following lemma:

Lemma 4.1 : *Assume there exists a differentiable matrix function P satisfying $F(P)(t) \geq 0, \forall t \in [0, t_1]$ and $P(t_1) = 0$ together with rank condition (b) in theorem 2.3. Then there exist differentiable matrix functions $C_{2,P}$ and $D_{2,P}$ such that:*

$$F(P)(t) = \begin{pmatrix} C_{2,P}^{\mathrm{T}}(t) \\ D_{2,P}^{\mathrm{T}}(t) \end{pmatrix} \begin{pmatrix} C_{2,P}(t) & D_{2,P}(t) \end{pmatrix}, \forall t \in [0, t_1]. \tag{4.1}$$

□

Proof : Because assumption 3.3 is assumed to hold, we can choose the bases of appendix A. Let P_1 be the matrix function in statement (ii) of lemma A.4. Then we have $R(P_1)(t) =$

$0, \forall t \in [0, t_1]$. We can write down particular choices for $C_{2,P}$ and $D_{2,P}$ in terms of the coefficient matrices as defined in appendix A:

$$C_{2,P}(t) := \begin{pmatrix} \hat{D}_2 \left(\hat{D}_2^{\mathrm{T}} \hat{D}_2 \right)^{-1} B_{11}^{\mathrm{T}} P_1 + C_{11} & C_{12} & C_{13} \\ C_{23} (C_{23}^{\mathrm{T}} C_{23})^{-1} (A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}) & 0 & 0 \end{pmatrix} (t) \tag{4.2}$$

$$D_{2,P}(t) := \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & 0 \end{pmatrix} (t) = D_2(t), \tag{4.3}$$

($t \in [0, t_1]$). Since all basis transformations are differentiable it is immediate that these functions are differentiable. Using $R(P_1) = 0$ it can be checked straightforwardly that indeed 4.1 is satisfied for these choices of $C_{2,P}$ and $D_{2,P}$. ■

Using this lemma we can now define a new system:

$$\Sigma_P : \begin{cases} \dot{x}_P(t) = A_P x_P(t) + B u_P(t) + E w_P(t), \\ y_P(t) = C_{1,P} x_P(t) + D_1 w_P(t), \\ z_P(t) = C_{2,P} x_P(t) + D_{2,P} u_P(t), \end{cases} \tag{4.4}$$

where we define

$$\begin{aligned} A_P(t) &:= A(t) + E(t) E^{\mathrm{T}}(t) P(t), \\ C_{1,P}(t) &:= C_1(t) + D_1(t) E^{\mathrm{T}}(t) P(t). \end{aligned}$$

We stress that (4.4) is a time-varying system with differentiable coefficient matrices. Note that even if the original system Σ is time-invariant, the system Σ_P is time-varying.
If Σ_F is a controller of the form (2.2), let $G_{cl,P}$ denote the closed loop operator from w_P to z_P obtained by interconnecting Σ_P and Σ_F. Recall that G_{cl} denotes the closed loop operator of $\Sigma \times \Sigma_F$. The crucial observation now is that $\|G_{cl}\|_\infty < 1$ if and only if $\|G_{cl,P}\| < 1$, that is, a controller Σ_F "works" for Σ if and only if the same controller "works" for Σ_P. A proof of this can be based on the following completion-of-the-squares-argument:

Lemma 4.2 : *Assume P satisfies (a) and (b) of theorem 2.3. Assume $x_P(0) = x(0) = 0$, $u_P(t) = u(t)$ for all $t \in [0, t_1]$ and suppose w_P and w are related by $w_P(t) = w(t) - E^{\mathrm{T}}(t) P(t) x(t)$ for all $t \in [0, t_1]$. Then for all $t \in [0, t_1]$ we have*

$$\|z(t)\|^2 - \|w(t)\|^2 = \frac{d}{dt} (x^{\mathrm{T}}(t) P(t) x(t)) + \|z_P(t)\|^2 - \|w_P(t)\|^2. \tag{4.5}$$

Consequently:

$$||z||_2^2 - ||w||_2^2 = ||z_P||_2^2 - ||w_P||_2^2. \tag{4.6}$$

□

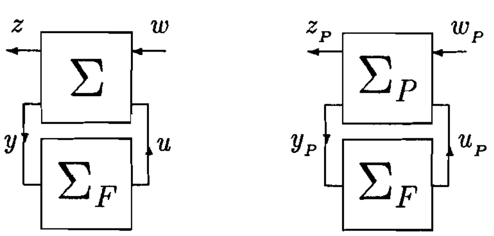
Proof : This can be proven by straightforward calculation, using the factorization (4.1). ■

Theorem 4.3 : *Let P satisfy (a) and (b) of theorem 2.3. Let Σ_F be a compensator of the form (2.2). Then:*

$$||G_{cl}||_\infty < 1 \qquad \Longleftrightarrow \qquad ||G_{cl,P}||_\infty < 1$$

□

Proof : Assume $||G_{cl,P}||_\infty < 1$ and consider the interconnection of Σ and Σ_F and of G_P and Σ_F:



Let $0 \neq w \in \mathcal{L}_2^l([0,t_1])$, let x be the corresponding state trajectory of Σ and define $w_P := w - E^T Px$. Then clearly $y_P = y$ and therefore $u_P = u$. This implies that the equality (4.6) holds. Also, we clearly have

$$||z_P||_2^2 - ||w_P||_2^2 \leq \left(||G_{cl,P}||_\infty^2 - 1\right) ||w_P||_2^2. \tag{4.7}$$

Next, note that the mapping $w_P \to w_P - E^T P x_P$ defines a bounded operator from $\mathcal{L}_2^l([0,t_1])$ to $\mathcal{L}_2^l([0,t_1])$. Hence there exists a constant $\mu > 0$ (independent of w) such that $\mu ||w||_2^2 < ||w_P||_2^2$. Define $\delta > 0$ by $\delta^2 := 1 - ||G_{cl,P}||_\infty^2$. Combining (4.6) and (4.7) then yields

$$\|z\|_2^2 - \|w\|_2^2 \leq -\delta^2\mu\|w\|_2^2.$$

Obviously, this implies that $\|G_{cl}\|_\infty \leq 1 - \delta^2\mu < 1$. The proof that $\|G_{cl}\|_\infty < 1$ implies that $\|G_{cl,P}\|_\infty < 1$ can be given in a similar way. □

We will now prove that condition (e) of theorem 2.3 is satisfied when part (i) of theorem 2.3 is satisfied.

We know that assumption 3.9 is satisfied for the original system Σ. In our transformation from Σ to Σ_P, (A, E, C_1, D_1) is transformed into $(A+EF, E, C_1+D_1F, D_1)$ where $F := E^{\mathrm{T}}P$. It can be easily checked that $\mathcal{V}(\Sigma)$ is invariant under such a feedback transformation. The structure of this transformation can also be used to show that all other assumptions in assumption 3.9 are invariant under the transformation from Σ to Σ_P. This implies that Σ_P satisfies assumption 3.9.

Assume we have a compensator Σ_F such that after applying this feedback law to Σ the resulting closed loop operator has $\mathcal{L}_2([0,t_1])$-induced operator norm less than 1. By applying lemma 3.10 we therefore know that there exists a matrix function Y such that

$$\bar{G}(Y)(t) := \begin{pmatrix} -\dot{Y} + A_PY + YA_P^{\mathrm{T}} + EE^{\mathrm{T}} + YC_{2,P}^{\mathrm{T}}C_{2,P}Y & YC_{1,P}^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_{1,P}Y + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix}(t) \geq 0 \quad (4.8)$$

for all $t \in [0,t_1]$, $Y(0) = 0$ and

$$\operatorname{rank}_{\mathcal{R}}\bar{G}(Y)(t) = \operatorname{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_P(t) & -E(t) \\ C_{1,P}(t) & D_1(t) \end{pmatrix} - n = g_t \quad (4.9)$$

for all $t \in [0,t_1]$. The last equality is a direct consequence of lemma A.6. By the dualized version of corollary A.5, we know that Y is unique on each interval $[0,t_2]$ $(t_2 \leq t_1)$. On the other hand, for any interval $[0,t_2]$ such that $I - P(t)Q(t)$ is invertible on that interval, we have:

$$\begin{pmatrix} I-QP & 0 \\ 0 & I \end{pmatrix} \bar{G}\left((I-QP)^{-1}Q\right) \begin{pmatrix} I-PQ & 0 \\ 0 & I \end{pmatrix} = G(Q)$$

Using this we see that $(I-QP)^{-1}Q$ satisfies both $\bar{G}[(I-QP)^{-1}Q] \geq 0$ as well as the rank condition $\operatorname{rank}_{\mathcal{R}} \bar{G}[(I-QP)^{-1}Q](t) = g_t$ for all t. Therefore on any such interval we find $Y(t) = (I - Q(t)P(t))^{-1}Q(t)$. Clearly, since $Q(0) = 0$, there *exists* $0 < t_2 \leq t_1$ such that $I - Q(t)P(t)$ is invertible on $[0,t_2)$. Assume now that $t_2 > 0$ is the smallest number such that $I - Q(t_2)P(t_2)$ is not invertible. Then on $[0,t_2)$ we have

$$Q(t) = (I - Q(t)P(t))Y(t)$$

and hence, by continuity

$$Q(t_2) = (I - Q(t_2)P(t_2))Y(t_2). \tag{4.10}$$

There exists $x \neq 0$ such that $x^{\mathrm{T}}(I - Q(t_2)P(t_2)) = 0$. By (4.10) this yields $x^{\mathrm{T}}Q(t_2) = 0$ whence $x = 0$, which is a contradiction. We must conclude that $I - Q(t)P(t)$ is invertible for all $t \in [0, t_1]$

This proves the implication (i) $\Rightarrow$ (ii) of theorem 2.3. In the next section we will prove the reverse implication.

5 The transformation into an almost disturbance decoupling problem

In the present section we will give a proof of the implication (ii) $\Rightarrow$ (i) of theorem 2.1. As in the previous sections we set $\gamma = 1$. The main idea is as follows: starting from the original system Σ, for which there exist P and Q satisfying (a)-(e) of theorem 2.3, we shall define a new system $\Sigma_{P,Q}$ which has the following important properties:

(i) Let Σ_F be any compensator. The closed loop operator G_{cl} of the interconnection $\Sigma \times \Sigma_F$ satisfies $\|G_{cl}\|_\infty < 1$ if and only if the closed loop operator $G_{cl,P,Q}$ of $\Sigma_{P,Q} \times \Sigma_F$ satisfies $\|G_{cl,P,Q}\|_\infty < 1$.

(ii) The system $\Sigma_{P,Q}$ is almost disturbance decouplable by dynamic measurement feedback, i.e. for all $\varepsilon > 0$ there exists a compensator Σ_F such that the resulting closed loop operator $G_{cl,P,Q}$ satisfies $\|G_{cl,P,Q}\|_\infty < \varepsilon$.

Property (i) states that a compensator "works" for Σ if and only if the same compensator "works" for $\Sigma_{P,Q}$. On the other hand, property (ii) states that, indeed, there exists a compensator Σ_F that "works" for $\Sigma_{P,Q}$: take any $\varepsilon < 1$ and take a compensator Σ_F such that the resulting closed loop operator $G_{cl,P,Q}$ satisfies $\|G_{cl,P,Q}\|_\infty < \varepsilon$. Then by property (i) the closed loop operator G_{cl} after applying the feedback Σ_F to the original system Σ satisfies $\|G_{cl}\| < 1$. This would clearly establish a proof of the implication (ii) $\Rightarrow$ (i) in theorem 2.3.

We shall now describe how the new system $\Sigma_{P,Q}$ is defined. Assume there exists P and Q satisfying (a)-(e) of theorem 2.3. Apply lemma 4.1 to obtain a differentiable factorization of $F(P)$ and let the system Σ_P be defined by (4.4). Next, consider the dual quadratic differential inequality associated with the system Σ_P: $\bar{G}(Y) \geq 0$, where $\bar{G}$ is defined by (4.8), together with the conditions $Y(0) = 0$ and rank condition (4.9). As was already noted in the previous section, the conditions (a)-(e) of theorem 2.3 assure that there exists an unique solution Y on $[0, t_1]$. (In fact, $Y = (I - QP)^{-1}Q$.) Therefore the dualized version of lemma 4.1 guarantees the existence of a differentiable factorization:

$$\bar{G}(Y)(t) = \begin{pmatrix} E_{P,Q}(t) \\ D_{P,Q}(t) \end{pmatrix} \begin{pmatrix} E_{P,Q}(t) \\ D_{P,Q}(t) \end{pmatrix}^{\mathrm{T}} \tag{5.1}$$

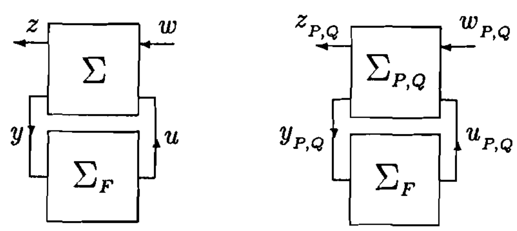
with $E_{P,Q}$ and $D_{P,Q}$ differentiable on $[0, t_1]$. Denote

$$\begin{aligned} A_{P,Q}(t) &:= A_P(t) + Y(t)C^{\mathrm{T}}_{2,P}(t)C_{2,P}(t), \\ B_{P,Q}(t) &:= B(t) + Y(t)C^{\mathrm{T}}_{2,P}(t)D_P(t) \end{aligned}$$

Then, introduce the new system $\Sigma_{P,Q}$ by:

$$\Sigma_{P,Q} : \left\{ \begin{array}{rcl} \dot{x}_{P,Q}(t) &=& A_{P,Q}x_{P,Q}(t) + B_{P,Q}u_{P,Q}(t) + E_{P,Q}w_{P,Q}(t), \\ y_{P,Q}(t) &=& C_{1,P}x_{P,Q}(t) + D_{P,Q}w_{P,Q}(t), \\ z_{P,Q}(t) &=& C_{2,P}x_{P,Q}(t) + D_{2,P}u_{P,Q}(t), \end{array} \right. \tag{5.2}$$

Again $\Sigma_{P,Q}$ is a time-varying system with differentiable coefficient matrices. We note that $\Sigma_{P,Q}$ is obtained by first transforming Σ into Σ_P and by subsequently applying the dual of this transformation to Σ_P. We will now first show that property (i) holds. If Σ_F is a dynamic compensator, let $G_{cl,P,Q}$ denote the closed loop operator from $w_{P,Q}$ to $z_{P,Q}$ in the interconnection of $\Sigma_{P,Q}$ with Σ_F:



Recall that G_{cl} denotes the closed loop operator from w to z in the interconnection of Σ and Σ_F. We have the following:

Theorem 5.1 : *Let Σ_F be a compensator of the form (2.2). Then we have*

$$\|G_{cl}\|_\infty < 1 \quad \Longleftrightarrow \quad \|G_{cl,P,Q}\|_\infty < 1$$

□

Proof : Assume Σ_F yields $\|G_{cl}\|_\infty < 1$. By theorem 4.3, then also $\|G_{cl,P}\|_\infty < 1$, i.e. Σ_F interconnected with Σ_P (given by (4.4)) also yields a closed loop operator with norm less than 1. By lemma 3.6 the dual compensator Σ'_F (given by (3.13)), interconnected with the dual of Σ_P:

$$\Sigma'_P : \begin{cases} \dot{x}_D(t) = A_P^{\mathrm{T}}(t_1 - t)x_D(t) + C_{1,P}^{\mathrm{T}}(t_1 - t)u_D(t) + C_{2,P}^{\mathrm{T}}(t_1 - t)w_D(t), \\ y_D(t) = B^{\mathrm{T}}(t_1 - t)x_D(t) + \qquad D_{2,P}^{\mathrm{T}}(t_1 - t)w_D(t), \\ z_D(t) = E^{\mathrm{T}}(t_1 - t)x_D(t) + D_1^{\mathrm{T}}(t_1 - t)u_D(t), \end{cases} \tag{5.3}$$

yields a closed loop operator $G'_{cl,P}$ (from w_D to z_D) with $\|G'_{cl,P}\|_\infty < 1$. Now, the quadratic differential inequality associated with Σ'_P is the transposed, time-reversed version of the inequality $\bar{G}(Y) \geq 0$ and therefore has a unique solution $\tilde{Y}(t) := Y(t_1 - t)$ such that $\tilde{Y}(t_1) = 0$ and the corresponding rank condition (4.9) holds. By applying theorem 4.3 to the system Σ'_P we may then conclude that the interconnection of Σ'_F with the dual $\Sigma'_{P,Q}$ of $\Sigma_{P,Q}$ yields a closed loop operator with norm less than 1. Again by dualization we then conclude $\|G_{cl,P,Q}\|_\infty < 1$. The converse implication is proven analogously. ■

Property (ii) is stated formally in the following theorem:

Theorem 5.2 : *For all $\varepsilon > 0$ there exists a compensator Σ_F of the form (2.2) such that the resulting closed loop operator $G_{cl,P,Q}$ satisfies $\|G_{cl,P,Q}\|_\infty < \varepsilon$.* □

Proof : For the system $\Sigma_{P,Q}$ for each fixed $t \in [0, t_1]$ we have

$$\begin{aligned} \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_{P,Q}(t) & -B_{P,Q}(t) \\ C_{2,P}(t) & D_{2,P}(t) \end{pmatrix} &= \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_P(t) & -B(t) \\ C_{2,P}(t) & D_{2,P}(t) \end{pmatrix} \\ &= \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A(t) & -B(t) \\ C_2(t) & D_2(t) \end{pmatrix} \\ &= \mathrm{rank}_{\mathcal{R}}\, F(P)(t) + n \\ &= \mathrm{rank}_{\mathcal{R}} \begin{pmatrix} C_{2,P}(t) & D_{2,P}(t) \end{pmatrix} + n \end{aligned} \tag{5.4}$$

The first equality follows by adding in the matrix on the left $YC_{2,P}$ times the second row to the first row. The second equality follows from lemma A.6. The third equality is condition (b) of theorem 2.3 and, finally, the fourth equality follows directly from lemma 4.1.
We also have for each fixed $t \in [0, t_1]$:

$$\begin{aligned} \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_{P,Q}(t) & -E_{P,Q}(t) \\ C_{1,P}(t) & D_{P,Q}(t) \end{pmatrix} &= \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_P(t) & -E(t) \\ C_{2,P}(t) & D_1(t) \end{pmatrix} \\ &= \mathrm{rank}_{\mathcal{R}}\, \bar{G}(Y)(t) + n \\ &= \mathrm{rank}_{\mathcal{R}} \begin{pmatrix} E_{P,Q}(t) \\ D_{P,Q}(t) \end{pmatrix} + n \end{aligned} \tag{5.5}$$

The first equality is implied by the dualized version of lemma A.6. The second equality is obtained from (4.9). The last equality then follows from (5.1).
By equation (5.4) and theorem C.1 we know that for each $\varepsilon > 0$ there exists a differentiable matrix function F such that the time-varying system

$$\Sigma_{cl,1} : \begin{cases} \dot{x}_1 = \left(A_{P,Q} + B_{P,Q}F\right) x_1 + w \\ z = \left(C_{2,P} + D_{2,P}F\right) x_1 \end{cases} \tag{5.6}$$

defines an operator $G_{cl,1}$ (from w to z) with $\mathcal{L}_2([0,t_1])$-induced operator norm less than ε, i.e. $\|G_{cl,1}\|_\infty < \varepsilon$.
By dualizing theorem C.1 we know that equation (5.5) guarantees that for all $\varepsilon > 0$ there exists a differentiable function G such that the system

$$\Sigma_{cl,2} \begin{cases} \dot{x}_2 = \left(A_{P,Q} + GC_{1,P}\right) x_2 + \left(B_{P,Q} + GD_{P,Q}\right) u \\ y = x_2 \end{cases} \tag{5.7}$$

defines an operator $G_{cl,2}$ (from u to y) with $\mathcal{L}_2([0,t_1])$-induced operator norm less than ε, i.e. $\|G_{cl,2}\|_\infty < \varepsilon$.
With each matrix function M we associate the muliplication operator Λ_M which is defined by

$$(\Lambda_M x)(t) := M(t)x(t)$$

It can be easily checked that the $\mathcal{L}_2([0,t_1])$-induced operator norm of Λ_M is given by

$$\|\Lambda_M\|_\infty = \sup_{t\in[0,t_1]} \|M(t)\|,$$

where $\|R\|$ denotes the largest singular value of the matrix R.

Let $\varepsilon > 0$ be given. We will construct a controller Σ_F which makes the $\mathcal{L}_2([0,t_1])$-induced operator norm of the closed loop operator $G_{cl,P,Q}$ less than ε. First choose F such that the norm of $G_{cl,1}$ satisfies

$$\|G_{cl,1}\|_\infty < \frac{\varepsilon}{3\|\Lambda_E\|_\infty + 1}. \tag{5.8}$$

Next choose a G such that the norm of $G_{cl,2}$ satisfies

$$\|G_{cl,2}\|_\infty < \frac{2\varepsilon}{3\|G_{cl,1}\|_\infty \|\Lambda_{B_{P,Q}F}\|_\infty + 3\|\Lambda_{D_P F}\|_\infty + 1} \tag{5.9}$$

The existence of such F and G is guaranteed by the above.
We then apply the following controller to $\Sigma_{P,Q}$:

$$\Sigma_F : \begin{cases} \dot{p} &= A_{P,Q}p + B_{P,Q}u_{P,Q} + G\left(C_{1,P}p - y_{P,Q}\right), \\ u_{P,Q} &= Fp \end{cases} \tag{5.10}$$

The resulting closed loop operator $G_{cl,P,Q}$ then satisfies

$$G_{cl,P,Q} = G_{cl,1}\Lambda_E + G_{cl,1}\Lambda_{B_{P,Q}F}G_{cl,2} - \Lambda_{D_P F}G_{cl,2} \tag{5.11}$$

By inequalities (5.8) and (5.9), equation (5.11) implies that

$$\|G_{cl,P,Q}\|_\infty < \varepsilon$$

Since ε was arbitrary this completes the proof. ■

Remark : For time-invariant systems sufficient conditions under which the system is almost disturbance decouplable by measurement feedback are already known ([22]). These conditions are simply our equalities (5.4) and (5.5). For time-varying systems the surprising fact is that when these equalities are satisfied *for all t* then the almost disturbance decoupling problem with measurement feedback is solvable. This will be proven in appendix C using results from LQ-theory which are given in appendix B.

Theorem 5.1 and theorem 5.2 together give the implication (ii) $\Rightarrow$ (i) of theorem 2.3.

6 Conclusion

In this paper we have studied the finite horizon H_∞ control problem for time-varying systems. Although the techniques we used were not able to tackle this problem in its full generality, still results on two important cases follow from our main results: the time-invariant case and the regular case. One reason for the fact that our techniques failed to solve the general problem formulation is, in our opinion, the fact that the concept of *strongly controllable subspace* does not really have a system-theoretic interpretation for general time-varying systems. One possibility to circumvent this problem would be to generalize the notion of strongly controllable subspace in a context of time-varying systems, in such a way that it does have an intuitive interpretation. However at this moment it is not clear how to do this.

For time-invariant systems an interesting problem for future research would be to investigate what happens if the length t_1 of the horizon $[0, t_1]$ runs off to infinity.

Appendix

A Preliminary basis transformations

In this section we will choose bases in input, output and state space which will give us much more insight in the structure of our problem. Although these decompositions are not used in the formulation of the main steps of the proof of theorem 2.3, the details of our proofs are very much concerned with these decompositions. It will be shown that with respect to these bases the coefficient matrices have a very particular structure. We shall display this structure by writing down the matrices with respect to these bases for the input, state and output spaces.
For details we refer to [19]. In contrast with the latter paper, we will discuss time-varying systems satisfying assumptions 3.3 and 3.9. Our basic tool is the strongly controllable subspace. This subspace has already been defined in definition 3.1.
At this point we will formulate a property of the strongly controllable subspace which will be used in the sequel (see [9, 16]):

Lemma A.1 : *Let $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times m}$ be arbitrary constant matrices. The quadruple (A, B, C, D) is strongly controllable if and only if*

$$rank_{\mathcal{R}} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} = n + rank_{\mathcal{R}} \begin{pmatrix} C & D \end{pmatrix} \tag{A.1}$$

for all $s \in \mathcal{C}$. □

We can now define the bases for the system (2.1) which will be used in the sequel. It is also possible to define a dual version of this decomposition but we will only need the primal one. We first choose a differentiable time-varying basis (i.e. the basis transformation is differentiable) of the control input space $\mathcal{R}^m$. We choose a basis $u_1, u_2, \ldots, u_m$ of $\mathcal{R}^m$ such that $u_1, u_2, \ldots, u_i$ is a basis of ker $D_2(t)$ $(0 \leq i \leq m)$. Note that by combining assumptions 3.3 (i) and (iv) it can be shown that rank $D_2(t)$ is independent of t. The existence of such a basis is then guaranteed by Dolezal's theorem (see [18]).
Next choose an orthonormal differentiable time-varying basis (i.e. the basis transformation is orthonormal for each t) $z_1, z_2, \ldots, z_p$ of the output space $\mathcal{R}^p$ such that $z_1, \ldots, z_j$ is a basis of $\operatorname{im} D_2(t)$ and $z_{j+1}, \ldots, z_p$ is a basis of $(\operatorname{im} D_2(t))^{\perp}$. Because this is an orthonormal basis the corresponding basis transformation does not change the norm $\|z\|$. The existence of such a basis is again guaranteed by Dolezal's theorem.
Finally we choose a *time-invariant* decomposition of the state space $\mathcal{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ such that $\mathcal{X}_2 = \mathcal{W}(\Sigma)$, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma)$ and $\mathcal{X}_1$ is arbitrary. We choose a corresponding time-invariant basis $x_1, x_2, \ldots, x_n$ such that $x_1, \ldots, x_r$ is a basis of $\mathcal{X}_1$, $x_{r+1}, \ldots, x_s$ is a basis

of $\mathcal{X}_2$ and $x_{s+1}, \ldots, x_n$ is a basis of $\mathcal{X}_3$. Note that in the definition of this decomposition we have used assumption 3.3 (ii) and (iii).
With respect to these bases the maps B, C_2 and D_2 have the following form:

$$B(t) = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}, \; C_2(t) = \begin{pmatrix} \hat{C}_1(t) \\ \hat{C}_2(t) \end{pmatrix}, \; D_2(t) = \begin{pmatrix} \hat{D}_2(t) & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.2}$$

where $\hat{D}_2(t)$ is invertible for all t. Let F_0 be such that assumption 3.3 part (v) is satisfied. Then we find that

$$C_2(t) + D_2(t)F_0(t) = \begin{pmatrix} 0 \\ \hat{C}_2(t) \end{pmatrix}. \tag{A.3}$$

Note that this implies that $C_2^{-1}(t)$ im $D_2(t) = \ker \hat{C}_2(t)$. We have the following properties, which were proven in [19] for each fixed t:

Lemma A.2 : *Assume assumption 3.3 is satisfied. Let F_0 satisfy part (v). For each $t \in [0, t_1]$ $\mathcal{T}(\Sigma)$ is the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ satisfying*

(i) $(A(t) + B(t)F_0(t))(\mathcal{T} \cap C_2^{-1}(t)\mathrm{im}D_2(t)) \subseteq \mathcal{T}$,

(ii) im $B_2(t) \subseteq \mathcal{T}$ □

By applying this lemma we find that the matrices $A(t) + B(t)F_0(t)$, $B(t)$, $C_2(t) + D_2(t)F_0(t)$ and $D_2(t)$ with respect to these bases have the following form:

$$\begin{cases} A(t) + B(t)F_0(t) = \begin{pmatrix} A_{11}(t) & 0 & A_{13}(t) \\ A_{21}(t) & A_{22} & A_{23}(t) \\ A_{31}(t) & A_{32} & A_{33}(t) \end{pmatrix}, & B(t) = \begin{pmatrix} B_{11}(t) & 0 \\ B_{21}(t) & B_{22}(t) \\ B_{31}(t) & B_{32}(t) \end{pmatrix}, \\ C_2(t) + D_2(t)F_0(t) = \begin{pmatrix} 0 & 0 & 0 \\ C_{21}(t) & 0 & C_{23}(t) \end{pmatrix}, & D_2(t) = \begin{pmatrix} \hat{D}_2(t) & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{A.4}$$

Note that by assumption 3.3 part (v) A_{22} and A_{32} are independent of t. We decompose the matrices $\hat{C}_1(t)$ and $E(t)$ correspondingly:

$$\hat{C}_1(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \end{pmatrix}, \quad E(t) = \begin{pmatrix} E_1(t) \\ E_2(t) \\ E_3(t) \end{pmatrix}. \tag{A.5}$$

Due to the fact that we only used differentiable basis transformations and since all coefficient matrices are differentiable functions of t, all the above submatrices are differentiable functions of t. These matrices turn out to have some nice structural properties, which were proven in [19]. In the following let g_t be given by (2.9):

Lemma A.3 : *The following properties hold:*

(*i*) *$C_{23}(t)$ is injective for all $t \in [0, t_1]$,*

(*ii*) *For each fixed $t \in [0, t_1]$ the quadruple*

$$\left[\begin{pmatrix} A_{22} & A_{23}(t) \\ A_{32} & A_{33}(t) \end{pmatrix}, \begin{pmatrix} B_{22}(t) \\ B_{32}(t) \end{pmatrix}, \begin{pmatrix} 0 & I \end{pmatrix}, 0 \right] \tag{A.6}$$

is strongly controllable,

(*iii*) *For each fixed $t \in [0, t_1]$ we have*

$$g_t = \text{rank} \begin{pmatrix} C_{23}(t) & 0 \\ 0 & \hat{D}_2(t) \end{pmatrix}. \tag{A.7}$$

Since $C_{23}(t)$ is injective and $\hat{D}_2(t)$ is invertible for all $t \in [0, t_1]$, we know that g_t is independent of time. □

We need the following result which connects the conditions of theorem 2.3 to the matrices as defined in (A.4).

Lemma A.4 : *Let $\gamma = 1$ and let P be a differentiable matrix function. Then the following conditions are equivalent:*

(*i*) *P is a solution of the quadratic matrix inequality $F(P) \geq 0$ on $[0, t_1]$, satisfying the rank condition (b) of theorem 2.3 and the end condition $P(t_1) = 0$.*

(*ii*) *There exists a P_1 such that, with respect to the decomposition of $\mathcal{R}^n$ introduced above, P has the form*

$$P(t) = \begin{pmatrix} P_1(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.8}$$

with P_1 a solution of the Riccati differential equation $R(P_1) = 0$ on $[0, t_1]$ with end condition $P_1(t_1) = 0$. Here

$$R(P_1) := \dot{P}_1 + P_1 A_{11} + A_{11}^{\mathrm{T}} P_1 + C_{21}^{\mathrm{T}} C_{21} + P_1 \left(E_1 E_1^{\mathrm{T}} - B_{11} \left(\hat{D}_2^{\mathrm{T}} \hat{D}_2 \right)^{-1} B_{11}^{\mathrm{T}} \right) P_1$$
$$- \left(P_1 A_{13} + C_{21}^{\mathrm{T}} C_{23} \right) \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right). \tag{A.9}$$

□

Proof : (i) $\Rightarrow$ (ii) : Define

$$M(t) := \begin{pmatrix} \dot{P} + (A+BF_0)^{\mathrm{T}}P + P(A+BF_0) + (C+DF_0)^{\mathrm{T}}(C+DF_0) + PEE^{\mathrm{T}}P & PB \\ B^{\mathrm{T}}P & D^{\mathrm{T}}D \end{pmatrix}(t)$$

Since P is a solution of the quadratic matrix differential inequality we have

$$M(t) = \begin{pmatrix} I & F_0^{\mathrm{T}}(t) \\ 0 & I \end{pmatrix} F(P)(t) \begin{pmatrix} I & 0 \\ F_0(t) & I \end{pmatrix} \geq 0$$

Define the following subspace of $\mathcal{R}^n$:

$$\mathcal{P} := \bigcap_{\tau \in [0,t_1]} \ker P(\tau)$$

We will show that $\mathcal{P} \cap \mathcal{T}(\Sigma)$ satisfies (i) and (ii) of lemma A.2 for each $t \in [0,t_1]$ and hence $\mathcal{P} \supset \mathcal{T}(\Sigma)$. Let $t \in [0,t_1]$ be given. Assume $D_2(t)x = 0$. Then

$$\begin{pmatrix} 0 & x^{\mathrm{T}} \end{pmatrix} M(t) \begin{pmatrix} 0 \\ x \end{pmatrix} = 0.$$

Since $M(t) \geq 0$ this implies that

$$0 = M(t) \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} P(t)B(t)x \\ 0 \end{pmatrix}.$$

Therefore $P(t)B(t)x = 0$. This implies $B(t)\ker D_2(t) \subset \mathcal{P}$ for all $t \in [0,t_1]$ since, by assumption 3.3 part (i), $B(t)\ker D_2(t)$ is independent of t. We already know that $B(t)\ker D_2(t) \subset \mathcal{T}(\Sigma)$ and hence $\mathcal{P} \cap \mathcal{T}(\Sigma) \supset B(t)\ker D_2(t)$.
Next, let $x \in \mathcal{P} \cap \mathcal{T}(\Sigma) \cap C_2^{-1}(t)\mathrm{im}\, D_2(t)$. Note that, by assumption 3.3 part (iii), this subspace is independent of t. Then $P(t)x = 0$ for all $t \in [0,t_1]$ and hence $\dot{P}(t)x = 0$ for all $t \in [0,t_1]$. We thus find that

$$\begin{pmatrix} x^{\mathrm{T}} & 0 \end{pmatrix} M(t) \begin{pmatrix} x \\ 0 \end{pmatrix} = 0.$$

for all $t \in [0,t_1]$. Since $M(t) \geq 0$ this implies that

$$0 = M(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} P(t)\left(A(t)+B(t)F_0(t)\right)x \\ 0 \end{pmatrix}$$

for all $t \in [0, t_1]$. Since $x \in \mathcal{W}(\Sigma)$, by assumption 3.3 part (v), we know that $(A + BF_0)(t)x$ is independent of t. Hence $(A + BF_0)(t)x \in \mathcal{P}$. By lemma A.2 we also know that $(A + BF_0)(t)x \in \mathcal{T}(\Sigma)$. This implies that (i) and (ii) of lemma A.2 are satisfied for $\mathcal{P} \cap \mathcal{T}(\Sigma)$ and hence $\mathcal{P} \supset \mathcal{P} \cap \mathcal{T}(\Sigma) \supset \mathcal{T}(\Sigma)$.

Since P^{T} satisfies $F(P^{\mathrm{T}}) \geq 0$ we also have $\ker P^{\mathrm{T}}(t) \supset \mathcal{T}(\Sigma)$ for all $t \in [0, t_1]$. Therefore P can be written in the form (A.8) for some matrix function P_1. Write all matrices in the form (A.4). Note that rank $M(t)$ = rank $F(P)(t)$ for all $t \in [0, t_1]$. Write out $M(t)$ with respect to the decomposition introduced above. By combining condition (b) of theorem 2.3 with lemma A.3 part (iii) we find that the rank of $M(t)$ is equal to the rank of the submatrix

$$\begin{pmatrix} C_{23}^{\mathrm{T}}(t)C_{23}(t) & 0 \\ 0 & \hat{D}_2^{\mathrm{T}}(t)\hat{D}_2(t) \end{pmatrix}$$

for all $t \in [0, t_1]$. Therefore the Schur complement of this submatrix is equal to zero which exactly implies that $R(P_1)(t) = 0 \ \forall t \in [0, t_1]$. The end condition $P_1(t_1) = 0$ is trivially satisfied since $P(t_1) = 0$.

(ii) $\Rightarrow$ (i) : By reversing the arguments in the proof of (i) $\Rightarrow$ (ii) we find that P as given by (A.8) satisfies $F(P) \geq 0$, the rank condition (b) of theorem 2.3 and $P(t_1) = 0$. ■

Corollary A.5 : *Let g_t be defined by (2.9). If there exists a matrix function P such that $F(P)(t) \geq 0 \ \forall t \in [0, t_1]$ and*

- *(i) rank $F(P)(t) = g_t, \forall t \in [0, t_1]$*
- *(ii) $P(t_1) = 0$.*

then these conditions define P uniquely on each interval $[t_2, t_1]$ ($0 \leq t_2 < t_1$). Moreover, P is symmetric for each $t \in [0, t_1]$. □

Proof : Uniqueness immediately follows from the fact that if the Riccati differential equation (A.9 has a solution P_1 on $[t_2, t_1]$ satisfying the end condition $P_1(t_1) = 0$ then it is unique. The fact that P is symmetric then follows from the fact that both P and P^{T} satisfy the conditions. ■

The following lemma was proven in [19]:

Lemma A.6 : *Let P satisfy condition (i) of lemma A.4 and let P_1 be defined by condition (ii) of the same lemma. Let $t \in [0, t_1]$ and $s_0 \in \mathcal{C}$. Then we have:*

$$rank_{\mathcal{R}} \begin{pmatrix} s_0 I - A_P(t) & -B(t) \\ C_{2,P}(t) & D_{2,P}(t) \end{pmatrix} = rank_{\mathcal{R}(s)} \begin{pmatrix} sI - A(t) & -B(t) \\ C_2(t) & D_2(t) \end{pmatrix}$$

if and only if s_0 is <u>not</u> an eigenvalue of the matrix

$$Z(t) := \left[A_{11} + E_1 E_1^{\mathrm{T}} P_1 - B_{11} \left(\hat{D}_2^{\mathrm{T}} \hat{D}_2 \right)^{-1} B_{11}^{\mathrm{T}} P_1 - A_{13} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right) \right] (t)$$

for this fixed t.

B Some facts about the finite horizon singular LQ-problem

In appendix C we shall discuss some facts concerning the finite horizon almost disturbance decoupling problem. Before we can do this we need some results on the finite horizon LQ-problem. This will be the subject of the present appendix. To a large extent this appendix is a recapitulation of known results ([1, 3]) but molded into the form in which we need it.

Assume we have the following system

$$\Sigma_{lq} : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) , & x(0) = x_0 \\ z(t) = C(t)x(t) + D(t)u(t) \end{cases} \tag{B.1}$$

together with the cost funcional

$$\mathcal{J}(x_0, u) := \int_0^{t_1} \|z(t)\|^2 dt \tag{B.2}$$

Assume all coefficient matrices are differentiable functions of t and assume assumption 3.3 is satisfied (where C_2 is replaced by C and D_2 by D). Denote the strongly controllable subspace associated with the quadruple $(A(t), B(t), C(t), D(t))$ (which, by assumption, is independent of t) by $\mathcal{T}(\Sigma_{lq})$. Moreover denote by $\mathcal{W}(\Sigma_{lq})$ the subspace $\mathcal{T}(\Sigma_{lq}) \cap C^{-1}(t)$ im $D(t)$ which again by assumption is independent of t.
For $\varepsilon > 0$ consider the Riccati differential equation:

$$\begin{cases} \dot{P} + A^{\mathrm{T}} P + PA + C^{\mathrm{T}} C - (PB + C^{\mathrm{T}} D)(D^{\mathrm{T}} D + \varepsilon I)^{-1} (B^{\mathrm{T}} P + D^{\mathrm{T}} C) = 0 \\ P(t_1) = 0, \end{cases} \tag{B.3}$$

where of course all coefficient matrices depend on t. It is well known that (B.3) has a unique solution on $[0, t_1]$ which is positive semidefinite on $[0, t_1]$. Denote this solution by $P_\varepsilon(t)$. It is well known that for each $\tau \in [0, t_1]$

$$x_0^{\mathrm{T}} P_\varepsilon(\tau) x_0 = \min_u \left\{ \int_\tau^{t_1} \|z_{u,x_0}(t)\|^2 + \varepsilon \|u(t)\|^2 dt \right\}, \tag{B.4}$$

where z_{u,x_0} denotes the output of Σ_{lq} with initial condition $x(0) = x_0$ and input u. Define

$$\hat{P}(\tau) := \lim_{\varepsilon \downarrow 0} P_\varepsilon(\tau), \quad \tau \in [0, t_1]. \tag{B.5}$$

Using the above definitions we can derive the following important lemma.

Lemma B.1 : *For $t \in [0, t_1]$ define*

$$Z_0(t) := \begin{pmatrix} \hat{P}(t) + \int_0^t A^{\mathrm{T}} \hat{P} + \hat{P} A + C^{\mathrm{T}} C d\tau & \int_0^t \hat{P} B + C^{\mathrm{T}} D d\tau \\ \int_0^t B^{\mathrm{T}} \hat{P} + D^{\mathrm{T}} C d\tau & \int_0^t D^{\mathrm{T}} D d\tau \end{pmatrix} \tag{B.6}$$

Then $Z_0(t)$ is non-decreasing on $[0, t_1]$ (i.e. $Z_0(t_2) \leq Z_0(t_3)$ if $t_2 \leq t_3$). □

Proof : Since P_ε satisfies (B.3) we immediately obtain that for all $\varepsilon > 0$:

$$\begin{pmatrix} \dot{P}_\varepsilon + A^{\mathrm{T}} P_\varepsilon + P_\varepsilon A + C^{\mathrm{T}} C & P_\varepsilon B + C^{\mathrm{T}} D \\ B^{\mathrm{T}} P_\varepsilon + D^{\mathrm{T}} C & D^{\mathrm{T}} D + \varepsilon I \end{pmatrix} (\tau) \geq 0 \quad \forall \tau \in [0, t_1] \tag{B.7}$$

Define

$$Z_\varepsilon(t) := \begin{pmatrix} P_\varepsilon(t) + \int_0^t A^{\mathrm{T}} P_\varepsilon + P_\varepsilon A + C^{\mathrm{T}} C d\tau & \int_0^t P_\varepsilon B + C^{\mathrm{T}} D d\tau \\ \int_0^t B^{\mathrm{T}} P_\varepsilon + D^{\mathrm{T}} C d\tau & \int_0^t D^{\mathrm{T}} D d\tau \end{pmatrix} \tag{B.8}$$

Using (B.7) we find that $Z_\varepsilon(t)$ is non-decreasing on $[0, t_1]$. The lemma then follows by applying the dominated convergence theorem. ■

We define

$$\begin{aligned}\mathcal{J}^*_{\varepsilon,\tau}(x_0) &:= \min_u \left\{ \int_\tau^{t_1} \|z_{u,x_0}(t)\|^2 + \varepsilon\|u(t)\|^2 dt \right\} \quad (\varepsilon > 0), \\ \mathcal{J}^*_{0,\tau}(x_0) &:= \inf_u \left\{ \int_\tau^{t_1} \|z_{u,x_0}(t)\|^2 dt \right\}.\end{aligned}$$

Lemma B.2 : *For each $\tau \in [0, t_1]$ and $x_0 \in \mathcal{R}^n$ we have*

$$\lim_{\varepsilon \downarrow 0} \mathcal{J}^*_{\varepsilon,\tau}(x_0) = \mathcal{J}^*_{0,\tau}(x_0) \tag{B.9}$$

Moreover:

$$\mathcal{J}^*_{0,\tau}(x_0) = x_0^{\mathrm{T}} \hat{P}(\tau) x_0. \tag{B.10}$$

□

Proof : Obviously $\mathcal{J}^*_{0,\tau}(x_0) \leq \mathcal{J}^*_{\varepsilon,\tau}(x_0)$ for all $\varepsilon > 0$. Let $\varepsilon_1 > 0$. Choose u such that

$$\int_\tau^{t_1} \|z_{u,x_0}(t)\|^2 dt \leq \mathcal{J}^*_{0,\tau}(x_0) + \varepsilon_1.$$

Using this we find

$$\int_\tau^{t_1} \|z_{u,x_0}(t)\|^2 + \varepsilon\|u(t)\|^2 dt \leq \mathcal{J}^*_{0,\tau}(x_0) + \varepsilon_1 + \varepsilon \int_\tau^{t_1} \|u(t)\|^2 dt.$$

Taking ε sufficiently small this yields $\mathcal{J}^*_{\varepsilon,\tau}(x_0) \leq \mathcal{J}^*_{0,\tau}(x_0) + 2\varepsilon_1$. Since ε_1 was arbitrary we find (B.9). Using the definition of $\hat{P}$, (B.10) is then an easy corollary. ■

We can now formulate and proof the result we will need in the appendix about the almost disturbance decoupling problem.

Theorem B.3 : *Let assumption 3.3 be satisfied. Then for all $t \in [0, t_1]$ we have:*

$$\mathcal{T}(\Sigma_{lq}) \subset \mathit{ker}\ \hat{P}(t).$$

□

Proof : The proof is strongly reminiscent of a part of the proof of lemma A.4. It is however complicated by the fact that we do not know whether $\hat{P}$ is differentiable. Let F_0 be any matrix function satisfying assumption 3.3 part (v). Then we have:

$$\mathcal{W}(\Sigma_{lq}) = \mathcal{T}(\Sigma_{lq}) \cap \ker\ [C(t) + D(t)F_0(t)] .$$

Define the following subspace:

$$\hat{\mathcal{P}} = \bigcap_{t\in[0,t_1]} \ker \hat{P}(t)$$

We are going to show that $\hat{\mathcal{P}} \cap \mathcal{T}(\Sigma_{lq})$ satisfies the conditions of lemma A.2 and hence $\hat{\mathcal{P}} \supset \mathcal{T}(\Sigma_{lq})$, which is exactly what we have to prove.
Recall that the dimension of ker $D(t)$ is independent of t. Hence by Dolezal's theorem there exists a differentiable time-varying basis of the input space such that in this new basis ker D is independent of t. We will use this basis. Define the following matrix function:

$$M_0(t) := \begin{pmatrix} \hat{P}(t) + \int_0^t (A+BF_0)^{\mathrm{T}}\hat{P} + \hat{P}(A+BF_0) + (C+DF_0)^{\mathrm{T}}(C+DF_0)d\tau & \int_0^t \hat{P}Bd\tau \\ \int_0^t B^{\mathrm{T}}\hat{P}d\tau & \int_0^t D^{\mathrm{T}}Dd\tau \end{pmatrix}(t)$$

We know that $M_0(t)$ is non-decreasing by lemma B.1 (Optimal costs are invariant under state feedback). Let $s_0 \leq s_1$, $s_0, s_1 \in [0,t_1]$ be arbitrary. We have $M_0(s_1) - M_0(s_0) \geq 0$. Hence

$$\ker \int_{s_0}^{s_1} \hat{P}(\tau)B(\tau)d\tau \subset \ker \int_{s_0}^{s_1} D^{\mathrm{T}}(\tau)D(\tau)d\tau = \ker D(0) \tag{B.11}$$

since, in our new basis of the input space, ker $D(t)$ is independent of t. It can be easily proven that $\hat{P}$ is a continuous function of t by using the fact that the optimal cost is a continuous function of the initial time. Since (B.11) is true for all $s_0, s_1 \in [0,t_1]$ and because $\hat{P}$ and B are continuous functions of t we find ker $\hat{P}(t) \subset B(t)$ ker $D(t)$. Hence, since by assumption $B(t)$ ker $D(t)$ is independent of t, we find $\hat{\mathcal{P}} \subset B(t)$ ker $D(t)$ for all $t \in [0,t_1]$.
Next we show that $\hat{\mathcal{P}} \cap \mathcal{T}(\Sigma_{lq})$ satisfies the second condition of lemma A.2. we shall prove that for any $t \in [0,t_1]$:

$$(A(t) + B(t)F_0(t)) \left[\hat{\mathcal{P}} \cap \mathcal{W}(\Sigma_{lq})\right] \subset \hat{\mathcal{P}} \cap \mathcal{T}(\Sigma_{lq}). \tag{B.12}$$

Let $x \in \hat{\mathcal{P}} \cap \mathcal{W}(\Sigma_{lq})$. We know that $\mathcal{W}(\Sigma_{lq}) \subset \ker\ (C(t) + D(t)F_0(t))$. Hence for all $s_0 \leq s_1$, $s_0, s_1 \in [0,t_1]$:

$$\begin{pmatrix} x^{\mathrm{T}} & 0 \end{pmatrix} (M_0(s_1) - M_0(s_0)) \begin{pmatrix} x \\ 0 \end{pmatrix} = 0.$$

Therefore

$$(M_0(s_1) - M_0(s_0)) \begin{pmatrix} x \\ 0 \end{pmatrix} = \int_{s_0}^{s_1} \hat{P}(A + BF_0)x d\tau = 0$$

for all $s_0 \leq s_1$, $s_0, s_1 \in [0, t_1]$. Since all matrix functions are continuous we find $(A(t) + B(t)F_0(t))x \in \ker \hat{P}(t)$ for all $t \in [0, t_1]$. Since by assumption 3.3 part (v) $(A(t)+B(t)F_0(t))x$ is independent of t and since $\mathcal{T}(\Sigma_{lq})$ satisfies the conditions of lemma A.2 we thus find (B.12). By lemma A.2 we may then conclude that $\mathcal{T}(\Sigma_{lq}) \subset \hat{\mathcal{P}} \cap \mathcal{T}(\Sigma_{lq}) \subset \hat{\mathcal{P}}$. ■

Corollary B.4 *Define the system:*

$$\Sigma_{di} : \left\{ \begin{array}{rclllll} \dot{x}(t) & = & A(t)x(t) & + & B(t)u(t) & + & E(t)w(t), \\ z(t) & = & C(t)x(t) & + & D(t)u(t) & & \end{array} \right. \tag{B.13}$$

and assume A, B, C and D satisfy assumption 3.3 (with $C = C_2$ and $D = D_2$). Assume that $\mathcal{T}(\Sigma_{lq}) = \mathcal{R}^n$. For all $\varepsilon > 0$ define

$$F_\varepsilon(t) := (D^{\mathrm{T}}(t)D(t) + \varepsilon I)^{-1} \left[B^{\mathrm{T}}(t)P_\varepsilon(t) + D^{\mathrm{T}}(t)C(t) \right].$$

Denote the closed loop operator from w to z after applying the feedback $u(t) = F_\varepsilon(t)x(t)$ by $G_{\varepsilon,cl}$ and let $\|G_{\varepsilon,cl}\|_\infty$ denote its $\mathcal{L}_2([0, t_1])$-induced operator norm. Then we have $\|G_{\varepsilon,cl}\|_\infty \to 0$ as $\varepsilon \downarrow 0$. □

Proof : Since $\mathcal{T}(\Sigma_{lq}) = \mathcal{R}^n$ we know that $\hat{P}(\tau) = 0$. for all $\tau \in [0, t_1]$. Therefore by lemma B.2 we know that $x_0^{\mathrm{T}} P_\varepsilon(\tau) x_0 \to 0$ as $\varepsilon \downarrow 0$ for all τ and x_0. Define $K_\varepsilon(t, \tau) := [C(t) + D(t)F_\varepsilon(t)]\Phi_\varepsilon(t, \tau)$ where $\Phi_\varepsilon(t, \tau)$ denotes the closed loop transition matrix after applying the feedback law $u = F_\varepsilon x$. For initial state $x(\tau) = x_0$ and disturbance 0 together with the feedback $u = F_\varepsilon x$ the resulting output of the system is then given by $y(t) = K_\varepsilon(t, \tau)x_0$ ($t \in [\tau, t_1]$). Since $x_0^{\mathrm{T}} P_\varepsilon(\tau) x_0 \to 0$ as $\varepsilon \downarrow 0$ and since the optimal feedback at time t is independent of the initial time τ of this optimization we know that for all $\tau \in [0, t_1]$

$$\int_\tau^{t_1} K_\varepsilon^{\mathrm{T}}(t, \tau) K_\varepsilon(t, \tau) dt \to 0$$

as $\varepsilon \downarrow 0$. This implies that

$$\int_{\tau}^{t_1} \|K_{\varepsilon}(t,\tau)\|^2 dt \ \to 0 \tag{B.14}$$

Here $\|M\|$ denotes the largest singular value of the matrix M. Now we let w be any disturbance and let $x(0) = 0$. In that case the output is given by

$$z(t) = \int_0^t K_{\varepsilon}(t,\tau) w(\tau)\, d\tau$$

Hence using Cauchy-Schwarz we find

$$\begin{aligned} \|z\|_2^2 &\le \int_0^{t_1} \int_0^t \|K_{\varepsilon}(t,\tau)\|^2 d\tau dt \cdot \|w\|_2^2, \\ &= \int_0^{t_1} \int_{\tau}^{t_1} \|K_{\varepsilon}(t,\tau)\|^2 dt d\tau \cdot \|w\|_2^2. \end{aligned} \tag{B.15}$$

Using (B.14) and the dominated convergence theorem we find that

$$\int_0^{t_1} \int_{\tau}^{t_1} \|K_{\varepsilon}(t,\tau)\|^2 dt d\tau \to 0$$

as $\varepsilon \downarrow 0$. Hence by (B.15) the $\mathcal{L}_2([0,t_1])$-induced operator norm converges to zero as $\varepsilon \downarrow 0$. ■

C The almost disturbance decoupling problem with state feedback

Assume we have the following system

$$\Sigma_{lq} : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u + E(t)w(t), \\ z(t) = C(t)x(t) + D(t)u \end{cases} \tag{C.1}$$

Given a state feedback control law $u(t) = F(t)x(t)$, let G_F denote the closed loop operator from w to z obtained by applying this control law to Σ_{lq}. The following lemma gives sufficient conditions such that $\|G_F\|_\infty$ can be made arbitrarily small, i.e. such that Σ_{lq} is almost disturbance decouplable by state feedback.

Theorem C.1 : *Assume A, B, C, D are differentiable functions of t satisfying assumption 3.3 (with C_2 replaced by C and D_2 by D). Moreover, assume*

$$rank_{\mathcal{R}(s)} \begin{pmatrix} sI - A(t) & -B(t) \\ C(t) & D(t) \end{pmatrix} = n + rank_{\mathcal{R}} \begin{pmatrix} C(t) & D(t) \end{pmatrix} \qquad \forall t \in [0, t_1] \tag{C.2}$$

Then for all $\varepsilon > 0$, there exists a differentiable matrix function F such that $\|G_F\|_\infty < \varepsilon$.

Proof : We apply the decomposition of appendix A where C_2 is equal to C and D_2 is equal to D. Decompose w correspondingly in w_1, w_2 and w_3. By lemma A.3, the quadruple given by (A.6) is strongly controllable for each fixed t and C_{23} is injective for all $t \in [0, t_1]$. Assumption (C.2) can be rewritten in these new bases. Applying lemma A.1 to the quadruple (A.6) we find, for each fixed $t \in [0, t_1]$,

$$\text{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_{11}(t) & -A_{13}(t) \\ C_{21}(t) & C_{23}(t) \end{pmatrix} = \dim \mathcal{X}_1 + \text{rank}_{\mathcal{R}} \begin{pmatrix} C_{21}(t) & C_{23}(t) \end{pmatrix} \tag{C.3}$$

for all $t \in [0, t_1]$. Therefore by [9] we have:

$$\mathcal{X}_1 = \mathcal{V}(A_{11}(t), A_{13}(t), C_{21}(t), C_{23}(t)) + \mathcal{T}(A_{11}(t), A_{13}(t), C_{21}(t), C_{23}(t))$$

for all $t \in [0, t_1]$. Since $C_{23}(t)$ is injective for all $t \in [0, t_1]$ we have $\mathcal{T}(A_{11}(t)$, $A_{13}(t)$, $C_{21}(t)$, $C_{23}(t))= \{0\}$ and hence

$$\mathcal{X}_1 = \mathcal{V}(A_{11}(t), A_{13}(t), C_{21}(t), C_{23}(t))$$

By definition 3.7 this implies that for each $t \in [0, t_1]$ there exists an R such that

$$C_{21}(t) + C_{23}(t)R(t) = 0 \tag{C.4}$$

We take a particular choice for R:

$$R(t) := -\left(C_{23}^{\mathrm{T}}(t)C_{23}(t)\right)^{-1} C_{23}^{\mathrm{T}}(t)C_{21}(t) \tag{C.5}$$

for all $t \in [0, t_1]$. Note that if there exists at least one R satisfying (C.4) then R given by (C.5) will also satisfy (C.4). Since C_{21} and C_{23} are differentiable functions of t, R is also differentiable. Define

$$q_3 := x_3 - Rx_1$$

for all $t \in [0, t_1]$. After applying the preliminary feedback $u = F_0 x + (v_1^{\mathrm{T}} \quad v_2^{\mathrm{T}})^{\mathrm{T}}$ we can rewrite the system equations (B.1) as:

$$
\begin{aligned}
\dot{x}_1 &= \tilde{A}_{11} x_1 + \begin{pmatrix} 0 & A_{13} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + B_{11} v_1 + E_1 w \\
\begin{pmatrix} \dot{x}_2 \\ \dot{q}_3 \end{pmatrix} &= \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{21} \\ \tilde{B}_{31} \end{pmatrix} v_1 + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} E_2 \\ \tilde{E}_3 \end{pmatrix} w \\
z &= \begin{pmatrix} 0 & 0 \\ 0 & C_{23} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} \hat{D}_2 \\ 0 \end{pmatrix} v_1
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{A}_{11} &:= A_{11} + A_{13} R, \\
\tilde{A}_{21} &:= A_{21} + A_{23} R, \\
\tilde{A}_{31} &:= A_{31} + A_{33} R - \dot{R} - R\left[A_{11} + A_{13} R\right], \\
\tilde{A}_{33} &:= A_{33} - R A_{13}, \\
\tilde{B}_{31} &:= B_{31} - R B_{11}, \\
\tilde{E}_3 &:= E_3 - R E_1,
\end{aligned}
$$

Recall that, by assumption 3.3, A_{22} and A_{32} are independent of t. It will be convenient to use the following notation

$$
\begin{aligned}
\bar{A}(t) &:= \begin{pmatrix} A_{22} & A_{23}(t) \\ A_{32} & \tilde{A}_{33}(t) \end{pmatrix} \in \mathcal{R}^{\bar{n}}, \\
\bar{B}(t) &:= \begin{pmatrix} B_{22}(t) \\ B_{32}(t) \end{pmatrix}, \\
\bar{C}(t) &:= \begin{pmatrix} 0 & I \end{pmatrix}, \\
\bar{D}(t) &:= 0, \\
\bar{E}(t) &:= \begin{pmatrix} E_2(t) & \tilde{A}_{21}(t) \\ \tilde{E}_3(t) & \tilde{A}_{31}(t) \end{pmatrix},
\end{aligned}
$$

where $\bar{n} = \dim \mathcal{T}(\Sigma_{lq})$. Since the quadruple (A.6) is strongly controllable for each fixed t it can be easily shown using lemma A.1 that the quadruple ($\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$, $\bar{D}(t)$) is strongly controllable for each fixed t. Therefore for all $t \in [0, t_1]$ we have

$$
\mathcal{T}\left(\bar{A}(t), \bar{B}(t), \bar{C}(t), \bar{D}(t)\right) = \mathcal{R}^{\bar{n}} \tag{C.6}
$$

Moreover we have

(i) $\bar{B}(t)$ ker $\bar{D}(t) = B(t)$ ker $D(t)$ is independent of t.

(ii) rank $\begin{pmatrix} \bar{B}(t) \\ \bar{D}(t) \end{pmatrix}$ = dim $B(t)$ ker $D(t)$ is independent of t.

(iii) $\mathcal{T}$ ($\bar{A}(t)$, $\bar{B}(t)$, $\bar{C}(t)$, $\bar{D}(t)$) $\cap$ $\bar{C}^{-1}(t)$ im $\bar{D}(t) = \mathcal{R}^{\bar{n}} \cap$ ker $(0 \quad I)$ is independent of t.

(iv) $F_0 := 0$ is such that $\bar{D}[\bar{C}(t) + \bar{D}(t)F_0] = 0$ and $[\bar{A}(t) + \bar{B}(t)F_0]|_{\ker(0\ I)}$ is independent of t.

We can rewrite the system equations in the following form:

$$\begin{aligned} \dot{p} &= \bar{A}p + \bar{B}v_2 + \begin{pmatrix} B_{21} \\ B_{31} \end{pmatrix} v_1 + \bar{E}\begin{pmatrix} w \\ x_1 \end{pmatrix} \\ q_3 &= \bar{C}p + \bar{D}v_2, \end{aligned} \tag{C.7}$$

combined with the system:

$$\begin{aligned} \dot{x}_1 &= \tilde{A}_{11}x_1 + A_{13}q_3 + B_{11}v_1 + w_1 \\ z_1 &= C_{23}q_3 \\ z_2 &= \hat{D}_2 v_1 \end{aligned} \tag{C.8}$$

Hence we can apply the results of apppendix B. By (C.6) and corollary B.4 we know that for each $\delta > 0$ there exists differentiable matrix functions F_2, F_3 such that by setting

$$\begin{aligned} v_1 &= 0 \\ v_2 &= F_2x_2 + F_3q_3 \end{aligned}$$

we have

$$\|q_3\|_2 \le \delta\,(\|w\|_2 + \|x_1\|_2)$$

Now, choose $\varepsilon > 0$. We know $\|z_2\|_2 \le M_2\|q_3\|_2$, with $M := \|C_{23}\| > 0$. By (C.8) there exist $M_1, M_2 \ge 0$ such that

$$\|x_1\|_2 \le M_1\|q_3\|_2 + M_2\|w\|_2 \tag{C.9}$$

Choose $v_1 = 0$ and $v_2 = F_2x_2 + F_3q_3$ such that

$$\|q_3\|_2 \leq \frac{\varepsilon}{M + MM_2 + \varepsilon M_1} (\|x_1\|_2 + \|w\|_2) \tag{C.10}$$

Using (C.9) and (C.10) it can be shown that

$$\|z\|_2 = \|z_2\| \leq M\|q_3\|_2 \leq \varepsilon\|w\|_2$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. ■

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